# Proofs and Refutations for Intuitionistic and Second-Order Logic

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# Outline

PRK: typing

PRK: reduction rules

Properties

Conclusion

# Syntax of propositions (aka types)

In previous work.

Pure propositions $A \quad ::= \quad \alpha \mid A \wedge A \mid A \vee A \mid \neg A$

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$$
\begin{array}{llll}
\text{Propositions} & P & ::= & A^{+} & \text{strong affirmation} \\
 & & \mid & A^{-} & \text{strong denial} \\
 & & \mid & A^{\oplus} & \text{weak affirmation} \\
 & & \mid & A^{\ominus} & \text{weak denial}
\end{array}
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# Syntax of propositions (aka types)

Pure propositions $A \quad ::= \quad \alpha \mid A \wedge A \mid A \vee A \mid \neg A$
$\mid A \to A \mid A \ltimes A \mid \forall \alpha.\, A \mid \exists \alpha.\, A$ In this work.

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Propositions $P \quad ::=$

| | | |
|---|---|---|
| | $A^{+}$ | strong affirmation |
| $\mid$ | $A^{-}$ | strong denial |
| $\mid$ | $A^{\oplus}$ | weak affirmation |
| $\mid$ | $A^{\ominus}$ | weak denial |

# Typing rules of system PRK

- Most rules are derived mechanically from standard typing rules.

## Example: rules for conjunction

Standard rule

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash s : B}{\Gamma \vdash \langle t, s \rangle : (A \wedge B)}\,\mathrm{I}\wedge$$

$$\frac{\Gamma \vdash t : (A_1 \wedge A_2)}{\Gamma \vdash \pi_i(t) : A_i}\,\mathrm{E}\wedge_i$$

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- Elimination rules have strong premises and weak conclusions.

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Standard rule + weak/strong distinction

$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : B^{\oplus}}{\Gamma \vdash \langle t, s \rangle^{+} : (A \wedge B)^{+}} \, \mathrm{I}\wedge^{+}$$

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- Introduction rules have weak premises and strong conclusions.
- Elimination rules have strong premises and weak conclusions.
- Dual pairs of connectives are $(\wedge/\vee)$, $(\rightarrow/\ltimes)$, $(\forall/\exists)$.

## Example: rules for conjunction

Standard rule + weak/strong distinction + affirmation/denial distinction.

$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : B^{\oplus}}{\Gamma \vdash \langle t, s\rangle^{+} : (A \wedge B)^{+}}\,\mathrm{I}\wedge^{+} \qquad \frac{\Gamma \vdash t : A^{\ominus} \qquad \Gamma \vdash s : B^{\ominus}}{\Gamma \vdash \langle t, s\rangle^{-} : (A \wedge B)^{-}}\,\mathrm{I}_{\vee}^{-}$$

$$\frac{\Gamma \vdash t : (A_1 \wedge A_2)^{+}}{\Gamma \vdash {\pi_i}^{+}(t) : {A_i}^{\oplus}}\,\mathrm{E}{\wedge_i}^{+} \qquad \frac{\Gamma \vdash t : (A_1 \vee A_2)^{-}}{\Gamma \vdash {\pi_i}^{-}(t) : {A_i}^{\ominus}}\,\mathrm{E}_{\vee i}^{-}$$

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Typing rules for weak propositions are based on the following informal equations.

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$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : A^{\ominus}}{\Gamma \vdash t \bullet^{+} s : A^{+}} \mathrm{E}_{\circ}^{+} \qquad \frac{\Gamma \vdash t : A^{\ominus} \qquad \Gamma \vdash s : A^{\oplus}}{\Gamma \vdash t \bullet^{-} s : A^{-}} \mathrm{E}_{\circ}^{-}$$

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## Weak introduction and elimination

Typing rules for weak propositions are based on the following informal equations. Note that these equations are mutually recursive.

$$A^{\oplus} \simeq (A^{\ominus} \rightarrow A^{+}) \qquad\qquad A^{\ominus} \simeq (A^{\oplus} \rightarrow A^{-})$$

# Outline

# Reduction rules

## Simplification of intro./elim. pairs (Natural deduction-like)

In previous work:

$$(\bigcirc^{\pm}_{x}.\, t) \bullet^{\pm} s \xrightarrow{\beta^{+}_{\circ} \,/\, \beta^{-}_{\circ}} t\{x := s\}$$

$$\pi^{\pm}_{i}(\langle t_1, t_2\rangle^{\pm}) \xrightarrow{\beta^{+}_{\wedge} \,/\, \beta^{-}_{\vee}} t_i$$

$$\delta^{\pm}(\mathsf{in}^{\pm}_{i}(t))\,[_{x}.s_1][_{x}.s_2] \xrightarrow{\beta^{+}_{\vee} \,/\, \beta^{-}_{\wedge}} s_i\{x := t\}$$

$$\mathsf{M}^{\pm}(\mathsf{N}^{\pm} t) \xrightarrow{\beta^{+}_{\neg} \,/\, \beta^{-}_{\neg}} t$$

New rules:

$$(\lambda^{\pm}_{x}.t) @^{\pm} s \xrightarrow{\beta^{+}_{\rightarrow} \,/\, \beta^{-}_{\ltimes}} t\{x := s\}$$

$$\varrho^{\pm}(t\,;^{\pm} s)[_{x;y}.u] \xrightarrow{\beta^{+}_{\ltimes} \,/\, \beta^{-}_{\rightarrow}} u\{x := t\}\{y := s\}$$

$$(\lambda^{\pm}_{\alpha}.\, t) @^{\pm} A \xrightarrow{\beta^{+}_{\forall} \,/\, \beta^{-}_{\exists}} t\{\alpha := A\}$$

$$\nabla^{\pm}\,\langle A, t\rangle^{\pm}[_{(\alpha,x)}.s] \xrightarrow{\beta^{+}_{\exists} \,/\, \beta^{-}_{\forall}} s\{\alpha := A\}\{x := t\}$$

# Reduction rules

## Simplification of intro./intro. cuts (Sequent calculus-like)

In previous work:

$$\langle t_1, t_2\rangle^+ \bowtie \mathsf{in}_i^-(s) \xrightarrow{\bowtie_\wedge} (t_i \bullet^+ s) \bowtie (s \bullet^- t_i)$$
$$\mathsf{in}_i^+(t) \bowtie \langle s_1, s_2\rangle^- \xrightarrow{\bowtie_\vee} (t \bullet^+ s_i) \bowtie (s_i \bullet^- t)$$
$$(\mathsf{N}^+ t) \bowtie (\mathsf{N}^- s) \xrightarrow{\bowtie_\neg} (s \bullet^+ t) \bowtie (t \bullet^- s)$$

New rules:

$$\lambda_x^+.t \bowtie (s \,;^- u) \xrightarrow{\bowtie_\to} (t\{x := s\} \bullet^+ u) \bowtie (u \bullet^- t\{x := s\})$$
$$(t \,;^+ s) \bowtie \lambda_x^-.u \xrightarrow{\bowtie_\ltimes} (s \bullet^+ u\{x := t\}) \bowtie (u\{x := t\} \bullet^- s)$$
$$(\lambda_\alpha^+.\, t) \bowtie \langle A, s\rangle^- \xrightarrow{\bowtie_\forall} (t\{\alpha := A\} \bullet^+ s) \bowtie (s \bullet^- t\{\alpha := A\})$$
$$\langle A, t\rangle^+ \bowtie (\lambda_\alpha^-.\, s) \xrightarrow{\bowtie_\exists} (t \bullet^+ s\{\alpha := A\}) \bowtie (s\{\alpha := A\} \bullet^- t)$$

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# Main properties

### Theorem (Classical refinement)

The following are equivalent:

- $A_1, \ldots, A_n \vdash B$ holds in classical second-order logic.
- There is a witness of $A_1^{\oplus}, \ldots, A_n^{\oplus} \vdash B^{\oplus}$ in PRK.

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## Theorem (Symmetry)

1. $\Gamma \vdash t : P$ if and only if $\Gamma^{\perp} \vdash t^{\perp} : P^{\perp}$
2. $t \to s$ if and only if $t^{\perp} \to s^{\perp}$

where $-^{\perp}$ flips all the signs and exchanges dual connectives.

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## Theorem (Convergence)

PRK enjoys **subject reduction**, **confluence**, and **strong normalization**.

This provides a computational interpretation for classical logic.

# Strong normalization

In previous work

We proved SN of the $\{\wedge, \vee, \neg\}$ fragment by translating PRK to Mendler's extension of System F with non-strictly positive recursion.

The proof does not extend to second-order PRK.

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In this work

We prove SN of the $\{\wedge, \vee, \neg, \rightarrow, \ltimes, \forall, \exists\}$ fragment by constructing a reducibility model, adapting Mendler's proof.

The interesting part is the interpretation of the mutually recursive equations:

$$A^{\oplus} \simeq (A^{\ominus} \rightarrow A^{+}) \qquad A^{\ominus} \simeq (A^{\oplus} \rightarrow A^{-})$$

respectively as a fixpoint/co-fixpoint in a complete lattice of reducibility candidates.

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A subterm is *useless* if it lies inside the argument of the elimination of a positive weak proposition:

$$\dfrac{\begin{array}{cc} \pi_1 & \pi_2 \\ \vdots & \vdots \\ \Gamma \vdash t : A^{\oplus} & \Gamma \vdash s : A^{\ominus} \end{array}}{\Gamma \vdash t \bullet^{+} s : A^{+}} \mathrm{E}^{+}_{\circ}$$

A term is *intuitionistic* if:

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## Theorem (Intuitionistic refinement)

The following are equivalent:

- $A_1, \ldots, A_n \vdash B$ holds in intuitionistic second-order logic.
- There is an intuitionistic witness of $A_1^{\oplus}, \ldots, A_n^{\oplus} \vdash B^{\oplus}$ in PRK.

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# Conclusion

- We have extended system PRK to second-order logic.
- The good logical and computational properties remain.
- To prove SN we adapted Mendler's reducibility model.
- We identified an intuitionistic subset of PRK.
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## Future work

- Study decidability results for fragments of PRK.
- Can PRK be related with existing classical calculi?
  (Parigot, Barbanera–Berardi, Curien–Herbelin, ...)
- Can PRK be understood through translations to linear logic?
- Can these ideas be extended to a dependently typed setting?