

Proof Terms for Higher-Order Rewriting and Their Equivalence

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First-order proof terms

Proof terms for **first-order** rewriting

A well-known first-order term rewriting system

$$\begin{aligned}\mathbf{add}(\mathbf{zero}, x) &\rightarrow x \\ \mathbf{add}(\mathbf{suc}(x), y) &\rightarrow \mathbf{suc}(\mathbf{add}(x, y))\end{aligned}$$

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$$\begin{aligned} \varrho(x) & : \quad \mathbf{add}(\mathbf{zero}, x) \rightarrow x \\ \vartheta(x, y) & : \quad \mathbf{add}(\mathbf{suc}(x), y) \rightarrow \mathbf{suc}(\mathbf{add}(x, y)) \end{aligned}$$

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Some first-order proof terms

$$\vartheta(\mathbf{zero}, \mathbf{suc}(\mathbf{zero})) : \mathbf{add}(\mathbf{suc}(\mathbf{zero}), \mathbf{suc}(\mathbf{zero})) \rightarrow \mathbf{suc}(\mathbf{add}(\mathbf{zero}, \mathbf{suc}(\mathbf{zero})))$$

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$$\begin{aligned} \vartheta(\mathbf{zero}, \mathbf{suc}(\mathbf{zero})) ; \mathbf{suc}(\varrho(\mathbf{suc}(\mathbf{zero}))) \\ : \mathbf{add}(\mathbf{suc}(\mathbf{zero}), \mathbf{suc}(\mathbf{zero})) \rightarrow \mathbf{suc}(\mathbf{suc}(\mathbf{zero})) \end{aligned}$$

Proof terms for **first-order** rewriting

First-order proof terms (formal syntax)

ρ	$::=$	$\mathbf{c}(\rho_1, \dots, \rho_n)$	congruence	\mathbf{c} is any n -ary function symbol
		$\varrho(\rho_1, \dots, \rho_n)$	rule application	ϱ is any n -ary rule symbol
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Rewriting judgment

$$\frac{\dots \rho_i : s_i \rightarrow t_i \dots}{\mathbf{c}(\rho_1, \dots, \rho_n) : \mathbf{c}(s_1, \dots, s_n) \rightarrow \mathbf{c}(t_1, \dots, t_n)}$$

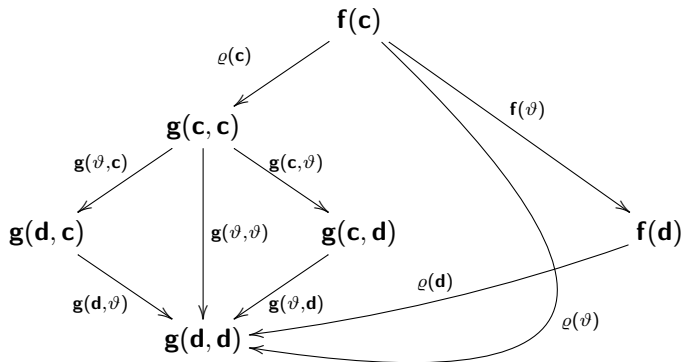
$$\frac{(\varrho(x_1, \dots, x_n) : s \rightarrow t) \in \mathcal{R} \quad \dots \rho_i : s_i \rightarrow t_i \dots}{\varrho(\rho_1, \dots, \rho_n) : s\{x_i \setminus s_i\}_{i \in 1..n} \rightarrow t\{x_i \setminus t_i\}_{i \in 1..n}}$$

$$\frac{\rho : s_1 \rightarrow s_2 \quad \sigma : s_2 \rightarrow s_3}{\rho ; \sigma : s_1 \rightarrow s_3}$$

Permutation equivalence of reductions

(example)

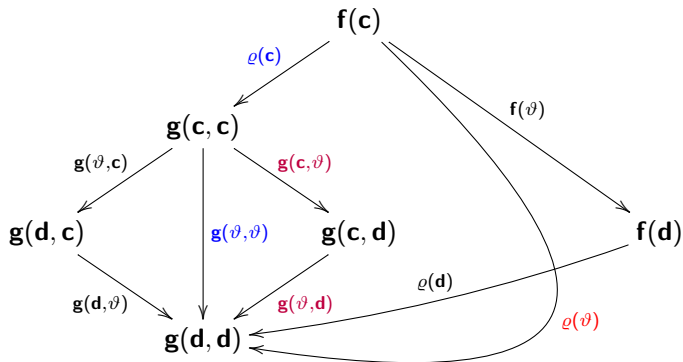
$$\begin{aligned} \varrho(x) &: \mathbf{f}(x) \rightarrow \mathbf{g}(x, x) \\ \vartheta &: \mathbf{c} \rightarrow \mathbf{d} \end{aligned}$$



Permutation equivalence of reductions

(example)

$$\begin{aligned} \varrho(x) &: \mathbf{f}(x) \rightarrow \mathbf{g}(x, x) \\ \vartheta &: \mathbf{c} \rightarrow \mathbf{d} \end{aligned}$$



$$\varrho(\vartheta) \approx \varrho(c) ; \mathbf{g}(\vartheta, \vartheta) \approx \varrho(c) ; (\mathbf{g}(c, \vartheta) ; \mathbf{g}(\vartheta, \mathbf{d}))$$

Permutation equivalence of reductions (important remark)

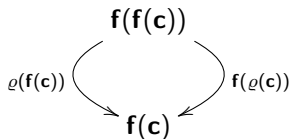
- ▶ If $\rho \approx \sigma$ then ρ and σ have the same source and target:

$$\rho : s \rightarrow t \quad \text{and} \quad \sigma : s \rightarrow t$$

- ▶ But the converse does not hold, for instance, if:

$$\varrho(x) : \mathbf{f}(x) \rightarrow x$$

then:

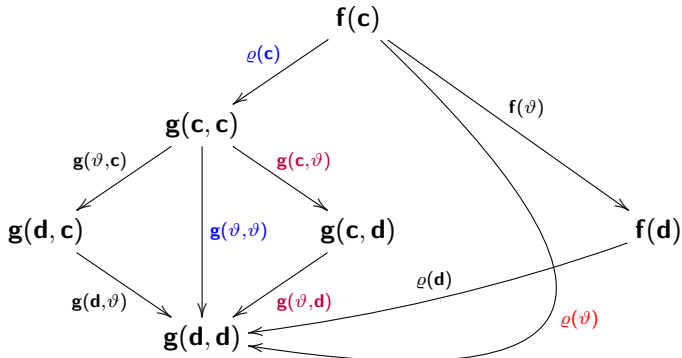


and $\mathbf{f}(\varrho(\mathbf{c})) \not\approx \varrho(\mathbf{f}(\mathbf{c}))$.

Projection of reductions

(example)

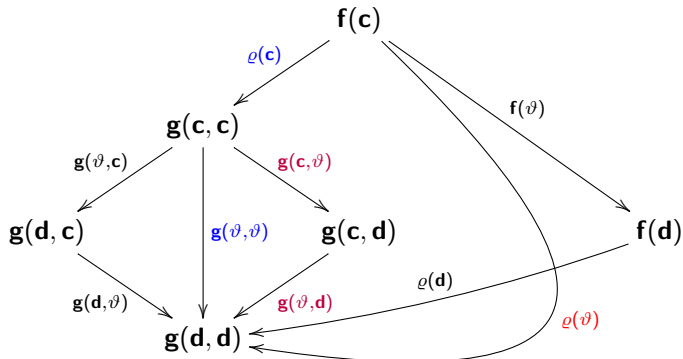
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Projection of reductions

(example)

$$\begin{aligned} \varrho(x) &: \mathbf{f}(x) \rightarrow \mathbf{g}(x, x) \\ \vartheta &: \mathbf{c} \rightarrow \mathbf{d} \end{aligned}$$



$$\begin{aligned} \varrho(\vartheta)/\varrho(\mathbf{c}) &= \mathbf{g}(\vartheta, \vartheta) & \varrho(\mathbf{c})/\varrho(\vartheta) &= \mathbf{g}(\mathbf{d}, \mathbf{d}) \\ \varrho(\vartheta)/(\varrho(\mathbf{c}); \mathbf{g}(\mathbf{c}, \vartheta)) &= \mathbf{g}(\vartheta, \mathbf{d}) \\ \varrho(\vartheta)/(\varrho(\mathbf{c}); (\mathbf{g}(\mathbf{c}, \vartheta); \mathbf{g}(\vartheta, \mathbf{d}))) &= \mathbf{g}(\mathbf{d}, \mathbf{d}) \end{aligned}$$

Equivalent notions of equivalence

Theorem (de Vrijer, van Oostrom)

The following are equivalent:

1. **Permutation equivalence:** $\rho \approx \sigma$.
2. **Projection equivalence:** ρ/σ and σ/ρ are empty.
Here “empty” means that it contains no rule symbols.

Equivalent notions of equivalence

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Basic historical notes

- ▶ Permutation equivalence and projection equivalence had been studied and shown equivalent by Jean-Jacques Lévy (~1978). (But without proof terms).
- ▶ Proof terms were introduced in the work of José Meseguer. (~1992; keyword: “rewriting logic”).
- ▶ Proof terms were extensively studied by Roel de Vrijer and Vincent van Oostrom (~2002) to study notions of equivalence between reductions, including also **standardization equivalence** and **labeling equivalence**. (See e.g. the Terese book, Chapter 8).

Higher-order proof terms

A well-known higher-order rewriting system

$$\mathbf{app} (\mathbf{lam} f) x \rightarrow f x$$

The object language is encoded in **higher-order abstract syntax**:

- ▶ First-order terms become simply-typed λ -terms:

$$\mathbf{app} : \iota \rightarrow \iota \rightarrow \iota \quad \mathbf{lam} : (\iota \rightarrow \iota) \rightarrow \iota \quad f : \iota \rightarrow \iota \quad x : \iota$$

- ▶ Terms are considered up to $\beta\eta$ -equivalence.
- ▶ In HRSs, left-hand sides of rules must be *patterns*.
- ▶ HRSs **strictly generalize** first-order term rewriting systems.
- ▶ We work with **orthogonal** HRSs: left-linear, no critical pairs.
- ▶ Orthogonal HRSs are confluent.
- ▶ HRSs were introduced by Tobias Nipkow (~ 1991).
There are other flavors of HORSs (e.g. Klop's CRSs).

Proof terms for **higher-order** rewriting

Example

$$\beta : \lambda f. \lambda x. \mathbf{app} (\mathbf{lam} f) x \rightarrow \lambda f. \lambda x. f x : (\iota \rightarrow \iota) \rightarrow \iota \rightarrow \iota$$

The reduction step of the object language:

$$\lambda x. (\lambda z. z (z x)) I \rightarrow \lambda x. I (I x)$$

can be encoded as the higher-order proof term:

$$\mathbf{lam} (\lambda x. \beta \underbrace{(\lambda z. \mathbf{app} z (\mathbf{app} z x))}_{\iota \rightarrow \iota} \underbrace{(\mathbf{lam} (\lambda x. x))}_{\iota}) : s \rightarrow t$$

with

$$\begin{aligned} s &= \mathbf{lam} (\lambda x. \mathbf{app} (\mathbf{lam} (\lambda z. \mathbf{app} z (\mathbf{app} z x))) (\mathbf{lam} (\lambda x. x))) \\ t &= \mathbf{lam} (\lambda x. (\lambda z. \mathbf{app} z (\mathbf{app} z x)) (\mathbf{lam} (\lambda x. x))) \\ &=_{\beta\eta} \mathbf{lam} (\lambda x. \mathbf{app} (\mathbf{lam} (\lambda x. x)) (\mathbf{app} (\mathbf{lam} (\lambda x. x)) x)) \end{aligned}$$

Proof terms for **higher-order** rewriting

Higher-order proof terms (formal syntax)

ρ	$::=$	x	variable
		\mathbf{c}	constant
		ϱ	rule symbol
		$\lambda x. \rho$	abstraction
		$\rho_1 \rho_2$	application
		$\rho_1 ; \rho_2$	composition

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Rewriting judgment

$$\frac{}{x : x \rightarrow x} \quad \frac{}{\mathbf{c} : \mathbf{c} \rightarrow \mathbf{c}} \quad \frac{(\varrho : s \rightarrow t) \in \mathcal{R}}{\varrho : s \rightarrow t} \quad \frac{\rho : s \rightarrow t}{\lambda x. \rho : \lambda x. s \rightarrow \lambda x. t}$$

$$\frac{\rho_1 : s_1 \rightarrow t_1 \quad \rho_2 : s_2 \rightarrow t_2}{\rho_1 \rho_2 : s_1 s_2 \rightarrow t_1 t_2} \quad \frac{\rho_1 : s_1 \rightarrow s_2 \quad \rho_2 : s_2 \rightarrow s_3}{\rho_1 ; \rho_2 : s_1 \rightarrow s_3}$$

$$\frac{s =_{\beta\eta} s' \quad \rho : s' \rightarrow t' \quad t' =_{\beta\eta} t}{\rho : s \rightarrow t}$$

A stumbling block

Proof terms for higher-order rewriting were studied by Bruggink (~ 2008).

What does “ $(\lambda x.\rho)\sigma$ ” mean?

$$(\lambda x.\rho)\sigma \stackrel{?}{\approx} \rho\{x\backslash\sigma\}$$

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As noted by Bruggink, this is not sound

Suppose that $\rho : s \rightarrow t$ is such that $s \neq t$. Then:

$$x \quad : \quad x \quad \rightarrow \quad x$$

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$(\lambda x.(x ; x))\rho$:	$(\lambda x.x) s$	\rightarrow	$(\lambda x.x) t$
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But $\rho ; \rho$ is not well-typed, as ρ cannot be composed with itself.

Bruggink sidesteps the problem by allowing compositions (“;”) only at the toplevel.

Permutation equivalence for higher-order proof terms

Definition

$$\begin{aligned}\rho^{\text{src}} ; \rho &\approx \rho \\ \rho ; \rho^{\text{tgt}} &\approx \rho \\ (\rho ; \sigma) ; \tau &\approx \rho ; (\sigma ; \tau) \\ (\lambda x. \rho) ; (\lambda x. \sigma) &\approx \lambda x. (\rho ; \sigma) \\ (\rho_1 \rho_2) ; (\sigma_1 \sigma_2) &\approx (\rho_1 ; \sigma_1) (\rho_2 ; \sigma_2) \\ (\lambda x. s) \rho &\approx s\{x \parallel \rho\} \\ (\lambda x. \rho) s &\approx \rho\{x \setminus s\} \\ \lambda x. \rho x &\approx \rho \quad \text{if } x \notin \text{fv}(\rho)\end{aligned}$$

- ▶ ρ^{src} and ρ^{tgt} denote the source and the target term of ρ .
- ▶ $s\{x \parallel \rho\}$ substitutes a variable in a λ -term for a proof term (yielding a proof term).
- ▶ $\rho\{x \setminus s\}$ substitutes a variable in a proof term for a λ -term (yielding a proof term).

Permutation equivalence for higher-order proof terms

Example

$$\begin{array}{l} \varrho : \lambda z. \mathbf{mu} z \rightarrow \lambda z. z (\mathbf{mu} z) : (\iota \rightarrow \iota) \rightarrow \iota \\ \vartheta : \mathbf{f} \rightarrow \mathbf{g} : \iota \rightarrow \iota \end{array}$$

Then:

$$\varrho \vartheta : \mathbf{mu} \mathbf{f} \rightarrow \mathbf{g} (\mathbf{mu} \mathbf{g})$$

And:

$$\begin{array}{ll} \varrho \vartheta & \\ \approx (\varrho ; (\lambda z. z (\mathbf{mu} z))) \vartheta & \text{as } \lambda z. z (\mathbf{mu} z) \text{ is the target of } \varrho \\ \approx (\varrho ; (\lambda z. z (\mathbf{mu} z))) (\mathbf{f} ; \vartheta) & \text{as } \mathbf{f} \text{ is the source of } \vartheta \\ \approx \varrho \mathbf{f} ; (\lambda z. z (\mathbf{mu} z)) \vartheta & \text{by the application rule} \\ \approx \varrho \mathbf{f} ; \vartheta (\mathbf{mu} \vartheta) & \text{by the term/rewrite } \beta\text{-like rule} \end{array}$$

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Proposition

$$(\lambda x. \rho) \sigma \approx \rho \{x \setminus \sigma^{\text{src}}\} ; \rho^{\text{tgt}} \{x \parallel \sigma\} \approx \rho^{\text{src}} \{x \parallel \sigma\} ; \rho \{x \setminus \sigma^{\text{tgt}}\}$$

Flattening

Definition

We have proposed a *flattening* relation between higher-order proof terms:

$$\begin{aligned}\lambda x.(\rho ; \sigma) &\stackrel{b}{\mapsto} (\lambda x.\rho) ; (\lambda x.\sigma) \\ (\rho ; \sigma) \mu &\stackrel{b}{\mapsto} (\rho \mu^{\text{src}}) ; (\sigma \mu) \\ \mu(\rho ; \sigma) &\stackrel{b}{\mapsto} (\mu \rho) ; (\mu^{\text{tgt}} \sigma) \\ (\rho_1 ; \rho_2)(\sigma_1 ; \sigma_2) &\stackrel{b}{\mapsto} ((\rho_1 ; \rho_2) \sigma_1^{\text{src}}) ; (\rho_2^{\text{tgt}} (\sigma_1 ; \sigma_2)) \\ (\lambda x.\mu) \nu &\stackrel{b}{\mapsto} \mu\{x \setminus \nu\} \\ \lambda x.\mu x &\stackrel{b}{\mapsto} \mu \quad \text{if } x \notin \text{fv}(\mu)\end{aligned}$$

where μ, ν, \dots stand for **multisteps**, that is, multisteps without occurrences of the composition operator “;”.

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where μ, ν, \dots stand for **multisteps**, that is, multisteps without occurrences of the composition operator “;”.

Theorem

Flattening is **confluent** and **strongly normalizing**.

The normal forms are called **flat proof terms**.

Compositions only appear at the toplevel, as in Bruggink’s work.

Flat permutation equivalence

A notion of permutation equivalence **between flat proof terms** can be defined as follows:

$$\begin{aligned}(\rho ; \sigma) ; \tau &\sim \rho ; (\sigma ; \tau) \\ \mu &\sim \mu_1^b ; \mu_2^b \quad \text{if } \mu \Leftrightarrow \mu_1 ; \mu_2\end{aligned}$$

where $\mu \Leftrightarrow \mu_1 ; \mu_2$ is a ternary relation meaning that the multistep μ can be “split” as the composition of the multisteps μ_1 and μ_2 .

Example

If, as before:

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Then, for example:

$$\varrho \vartheta \sim \varrho \mathbf{f} ; \vartheta (\mathbf{mu} \vartheta) \quad \text{since } \varrho \vartheta \Leftrightarrow \varrho \mathbf{f} ; (\lambda z. z (\mathbf{mu} z)) \vartheta$$

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Theorem (Flat permutation equivalence)

$\rho \approx \sigma$ if and only if $\rho^b \sim \sigma^b$.

Projection

A notion of **projection** can be defined for **multisteps** (no composition):

$$\begin{array}{c} \frac{}{x \parallel x \Rightarrow x} \quad \frac{}{\mathbf{c} \parallel \mathbf{c} \Rightarrow \mathbf{c}} \quad \frac{}{\varrho \parallel \varrho \Rightarrow \varrho^{\text{tgt}}} \quad \frac{}{\varrho \parallel \varrho^{\text{src}} \Rightarrow \varrho} \\ \\ \frac{}{\varrho^{\text{src}} \parallel \varrho \Rightarrow \varrho^{\text{tgt}}} \quad \frac{\mu \parallel \nu \Rightarrow \xi}{\lambda x. \mu \parallel \lambda x. \nu \Rightarrow \lambda x. \xi} \quad \frac{\mu_1 \parallel \nu_1 \Rightarrow \xi_1 \quad \mu_2 \parallel \nu_2 \Rightarrow \xi_2}{\mu_1 \mu_2 \parallel \nu_1 \nu_2 \Rightarrow \xi_1 \xi_2} \end{array}$$

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$$\begin{array}{c} \frac{}{x \parallel x \Rightarrow x} \quad \frac{}{\mathbf{c} \parallel \mathbf{c} \Rightarrow \mathbf{c}} \quad \frac{}{\varrho \parallel \varrho \Rightarrow \varrho^{\text{tgt}}} \quad \frac{}{\varrho \parallel \varrho^{\text{src}} \Rightarrow \varrho} \\ \\ \frac{}{\varrho^{\text{src}} \parallel \varrho \Rightarrow \varrho^{\text{tgt}}} \quad \frac{\mu \parallel \nu \Rightarrow \xi}{\lambda x. \mu \parallel \lambda x. \nu \Rightarrow \lambda x. \xi} \quad \frac{\mu_1 \parallel \nu_1 \Rightarrow \xi_1 \quad \mu_2 \parallel \nu_2 \Rightarrow \xi_2}{\mu_1 \mu_2 \parallel \nu_1 \nu_2 \Rightarrow \xi_1 \xi_2} \end{array}$$

This can be extended to **flat** proof terms in a typical way:

$$\begin{array}{l} \mu^b \parallel \nu^b \stackrel{\text{def}}{=} \xi^b \quad \text{if } \mu \parallel \nu \Rightarrow \xi \\ \rho \parallel (\sigma ; \tau) \stackrel{\text{def}}{=} (\rho \parallel \sigma) \parallel \tau \\ (\rho ; \sigma) \parallel \tau \stackrel{\text{def}}{=} (\rho \parallel \tau) ; (\sigma \parallel (\tau \parallel \rho)) \end{array}$$

(The first equation uses pattern matching against LHSs of rewrite rules).

Projection

A notion of **projection** can be defined for **multisteps** (no composition):

$$\begin{array}{c}
 \overline{x \parallel x \Rightarrow x} \quad \overline{\mathbf{c} \parallel \mathbf{c} \Rightarrow \mathbf{c}} \quad \overline{\varrho \parallel \varrho \Rightarrow \varrho^{\text{tgt}}} \quad \overline{\varrho \parallel \varrho^{\text{src}} \Rightarrow \varrho} \\
 \\
 \overline{\varrho^{\text{src}} \parallel \varrho \Rightarrow \varrho^{\text{tgt}}} \quad \frac{\mu \parallel \nu \Rightarrow \xi}{\lambda x. \mu \parallel \lambda x. \nu \Rightarrow \lambda x. \xi} \quad \frac{\mu_1 \parallel \nu_1 \Rightarrow \xi_1 \quad \mu_2 \parallel \nu_2 \Rightarrow \xi_2}{\mu_1 \mu_2 \parallel \nu_1 \nu_2 \Rightarrow \xi_1 \xi_2}
 \end{array}$$

This can be extended to **flat** proof terms in a typical way:

$$\begin{array}{l}
 \mu^b \parallel \nu^b \stackrel{\text{def}}{=} \xi^b \quad \text{if } \mu \parallel \nu \Rightarrow \xi \\
 \rho \parallel (\sigma ; \tau) \stackrel{\text{def}}{=} (\rho \parallel \sigma) \parallel \tau \\
 (\rho ; \sigma) \parallel \tau \stackrel{\text{def}}{=} (\rho \parallel \tau) ; (\sigma \parallel (\tau \parallel \rho))
 \end{array}$$

(The first equation uses pattern matching against LHSs of rewrite rules).

Finally, it can be extended to **arbitrary** proof terms by flattening first:

$$\rho / \sigma \stackrel{\text{def}}{=} \rho^b \parallel \sigma^b$$

Projection equivalence

Theorem (Projection equivalence)

$\rho \approx \sigma$ if and only if ρ/σ and σ/ρ are empty.

Again, “empty” means that it contains no rule symbols.

Future work

- ▶ Formulate a standardization procedure.
- ▶ Study labeling equivalence.
- ▶ Relate with 2-categorical models (Hirschowitz, 2013).