A Constructive Logic with Classical Proofs and Refutations

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Logics and Dynamics of Programming Languages

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Outline

The BHK Interpretation ($\{\wedge, \vee, \neg\}$ fragment)

$$\begin{aligned}(A \wedge B)^+ &\simeq A^+ \times B^+ \\ (A \vee B)^+ &\simeq A^+ \uplus B^+ \\ (\neg A)^+ &\simeq A^+ \Rightarrow \mathbf{0}\end{aligned}$$

$A^+ =$ "proofs of A"

Nelson's Strong Negation

(Nelson, 1949)

$$
\begin{aligned}
(A \wedge B)^+ &\simeq A^+ \times B^+ & \qquad (A \wedge B)^- &\simeq A^- \uplus B^- \\
(A \vee B)^+ &\simeq A^+ \uplus B^+ & \qquad (A \vee B)^- &\simeq A^- \times B^- \\
(\neg A)^+ &\simeq A^- & \qquad (\neg A)^- &\simeq A^+
\end{aligned}
$$

$A^+ =$ "proofs of A" $\qquad$ $A^- =$ "refutations of A"

Starting point: a BHK interpretation for classical logic

Can we recover classical logic by extending Nelson's system as follows?

$$A^{\oplus} \overset{?}{\simeq} A^{-} \Rightarrow A^{+} \qquad A^{\ominus} \overset{?}{\simeq} A^{+} \Rightarrow A^{-}$$

A^{+} = "strong proofs of A"

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$A^{\oplus}$ = "classical proofs of A"

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- From the strictly logical point of view, this gives us classical logic.
- From the point of view of proof normalization, it is not clear how to normalize a cut $A^{\oplus}$ / $A^{\ominus}$.

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- From the strictly logical point of view, this gives us classical logic.
- From the point of view of proof normalization, it is not clear how to normalize a cut $A^{\oplus}$ / $A^{\ominus}$.

Our approach is based on the (mutually recursive!) equations:

$$A^{\oplus} \simeq A^{\ominus} \Rightarrow A^{+} \qquad A^{\ominus} \simeq A^{\oplus} \Rightarrow A^{-}$$

Starting point: a BHK interpretation for classical logic

$$
\begin{array}{rcl@{\qquad}rcl}
(A \wedge B)^+ & \simeq & A^\oplus \times B^\oplus & (A \wedge B)^- & \simeq & A^\ominus \uplus B^\ominus \\
(A \vee B)^+ & \simeq & A^\oplus \uplus B^\oplus & (A \vee B)^- & \simeq & A^\ominus \times B^\ominus \\
(\neg A)^+ & \simeq & A^\ominus & (\neg A)^- & \simeq & A^\oplus \\
A^\oplus & \simeq & A^\ominus \Rightarrow A^+ & A^\ominus & \simeq & A^\oplus \Rightarrow A^-
\end{array}
$$

$A^+ =$ "strong proofs of A"

$A^\oplus =$ "classical proofs of A"

$A^- =$ "strong refutations of A"

$A^\ominus =$ "classical refutations of A"

Outline

Introduction

The Logical System PRK

Propositions as Types: the λ^{PRK}-calculus

Kripke Semantics

Further Extensions

Conclusion

Natural Deduction

Pure propositions	A	$::=$	$\alpha \mid A \wedge A \mid A \vee A \mid \neg A$	
Propositions	P	$::=$	A^+	strong affirmation

Example rules

$$\frac{\Gamma \vdash A^+ \qquad \Gamma \vdash B^+}{\Gamma \vdash (A \wedge B)^+}\ \mathrm{I}\wedge^+$$

$$\frac{\Gamma \vdash (A_1 \wedge A_2)^+}{\Gamma \vdash {A_i}^+}\ \mathrm{E}\wedge_i^+$$

Nelson's strong negation

$$\begin{array}{llll} \text{Pure propositions} & A & ::= & \alpha \mid A \wedge A \mid A \vee A \mid \neg A \\ \text{Propositions} & P & ::= & A^{+} \qquad\qquad\qquad\qquad \text{strong affirmation} \\ & & \mid & \color{red}{A^{-} \qquad\qquad\qquad\qquad \text{strong denial}} \end{array}$$

Example rules

$$\frac{\Gamma \vdash A^{+} \qquad \Gamma \vdash B^{+}}{\Gamma \vdash (A \wedge B)^{+}}\,\mathrm{I}\wedge^{+} \qquad \color{red}{\frac{\Gamma \vdash A^{-} \qquad \Gamma \vdash B^{-}}{\Gamma \vdash (A \vee B)^{-}}\,\mathrm{I}\vee^{-}}$$

$$\frac{\Gamma \vdash (A_1 \wedge A_2)^{+}}{\Gamma \vdash {A_i}^{+}}\,\mathrm{E}\wedge_i^{+} \qquad \color{red}{\frac{\Gamma \vdash (A_1 \vee A_2)^{-}}{\Gamma \vdash {A_i}^{-}}\,\mathrm{E}\vee_i^{-}}$$

System PRK

Pure propositions	A	$::=$	$\alpha \mid A \wedge A \mid A \vee A \mid \neg A$	
Propositions	P	$::=$	A^{+}	strong affirmation
		$\mid$	A^{-}	strong denial
		$\mid$	$A^{\oplus}$	classical affimation
		$\mid$	$A^{\ominus}$	classical denial

Example rules

$$\frac{\Gamma \vdash A^{\oplus} \qquad \Gamma \vdash B^{\oplus}}{\Gamma \vdash (A \wedge B)^{+}} \mathrm{I}\wedge^{+} \qquad \frac{\Gamma \vdash A^{\ominus} \qquad \Gamma \vdash B^{\ominus}}{\Gamma \vdash (A \vee B)^{-}} \mathrm{I}\vee^{-}$$

$$\frac{\Gamma \vdash (A_1 \wedge A_2)^{+}}{\Gamma \vdash A_i^{\oplus}} \mathrm{E}\wedge_i^{+} \qquad \frac{\Gamma \vdash (A_1 \vee A_2)^{-}}{\Gamma \vdash A_i^{\ominus}} \mathrm{E}\vee_i^{-}$$

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\begin{array}{llll}
\text{Pure propositions} & A & ::= & \alpha \mid A \wedge A \mid A \vee A \mid \neg A \\
\\
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& & \mid & A^{-} & \text{strong denial} \\
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\end{array}
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Example rules

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$$

$$
\frac{\Gamma \vdash (A_1 \wedge A_2)^{+}}{\Gamma \vdash A_i{}^{\oplus}}\,\mathrm{E}\wedge_i^{+} \qquad \frac{\Gamma \vdash (A_1 \vee A_2)^{-}}{\Gamma \vdash A_i{}^{\ominus}}\,\mathrm{E}\vee_i^{-}
$$

A strong affirmation A^{+} is canonically proved with an introduction rule.

System PRK – Noteworthy rules

Absurdity

$$\frac{\Gamma \vdash A^{+} \qquad \Gamma \vdash A^{-}}{\Gamma \vdash P}\,\textsc{Abs}$$

Negation

$$\frac{\Gamma \vdash A^{\ominus}}{\Gamma \vdash (\neg A)^{+}}\,\mathrm{I}\neg^{+} \qquad \frac{\Gamma \vdash A^{\oplus}}{\Gamma \vdash (\neg A)^{-}}\,\mathrm{I}\neg^{-}$$

$$\frac{\Gamma \vdash (\neg A)^{+}}{\Gamma \vdash A^{\ominus}}\,\mathrm{E}\neg^{+} \qquad \frac{\Gamma \vdash (\neg A)^{-}}{\Gamma \vdash A^{\oplus}}\,\mathrm{E}\neg^{-}$$

Classical formulas

$$\frac{\Gamma, A^{\ominus} \vdash A^{+}}{\Gamma \vdash A^{\oplus}}\,\mathrm{I}\bigcirc^{+} \qquad \frac{\Gamma, A^{\oplus} \vdash A^{-}}{\Gamma \vdash A^{\ominus}}\,\mathrm{I}\bigcirc^{-}$$

$$\frac{\Gamma \vdash A^{\oplus} \qquad \Gamma \vdash A^{\ominus}}{\Gamma \vdash A^{+}}\,\mathrm{E}\bigcirc^{+} \qquad \frac{\Gamma \vdash A^{\ominus} \qquad \Gamma \vdash A^{\oplus}}{\Gamma \vdash A^{-}}\,\mathrm{E}\bigcirc^{-}$$

System PRK – Noteworthy rules

Absurdity

$$\frac{\Gamma \vdash A^+ \qquad \Gamma \vdash A^-}{\Gamma \vdash P}\ \textsc{Abs}$$

Negation

$$\frac{\Gamma \vdash A^\ominus}{\Gamma \vdash (\neg A)^+}\ \mathrm{I}\neg^+ \qquad \frac{\Gamma \vdash A^\oplus}{\Gamma \vdash (\neg A)^-}\ \mathrm{I}\neg^-$$

$$\frac{\Gamma \vdash (\neg A)^+}{\Gamma \vdash A^\ominus}\ \mathrm{E}\neg^+ \qquad \frac{\Gamma \vdash (\neg A)^-}{\Gamma \vdash A^\oplus}\ \mathrm{E}\neg^-$$

Classical formulas

$$\frac{\Gamma, A^\ominus \vdash A^+}{\Gamma \vdash A^\oplus}\ \mathrm{I}\bigcirc^+ \qquad \frac{\Gamma, A^\oplus \vdash A^-}{\Gamma \vdash A^\ominus}\ \mathrm{I}\bigcirc^-$$

$$\frac{\Gamma \vdash A^\oplus \qquad \Gamma \vdash A^\ominus}{\Gamma \vdash A^+}\ \mathrm{E}\bigcirc^+ \qquad \frac{\Gamma \vdash A^\ominus \qquad \Gamma \vdash A^\oplus}{\Gamma \vdash A^-}\ \mathrm{E}\bigcirc^-$$

A classical affirmation $A^\oplus$ is canonically proved by assuming $A^\ominus$ and proving A^+.

System PRK – Admissible rules

Weakening

$$\frac{\Gamma \vdash P}{\Gamma, Q \vdash P}$$

Cut

$$\frac{\Gamma, P \vdash Q \qquad \Gamma \vdash P}{\Gamma \vdash Q}$$

Substitution

$$\frac{\Gamma \vdash Q}{\Gamma[\alpha := A] \vdash Q[\alpha := A]}$$

General absurdity

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash P^{\sim}}{\Gamma \vdash Q}$$

Contraposition

$$\frac{\Gamma, P \vdash Q \qquad P \text{ classical}}{\Gamma, Q^{\sim} \vdash P^{\sim}}$$

Strengthening

$$\frac{\Gamma, P^{\sim} \vdash P \qquad P \text{ classical}}{\Gamma \vdash P}$$

Where:

$$(A^{+})^{\sim} \stackrel{\text{def}}{=} A^{-} \qquad (A^{-})^{\sim} \stackrel{\text{def}}{=} A^{+} \qquad (A^{\oplus})^{\sim} \stackrel{\text{def}}{=} A^{\ominus} \qquad (A^{\ominus})^{\sim} \stackrel{\text{def}}{=} A^{\oplus}$$

System PRK – Properties

Theorem (Embedding + conservative extension)

$\vdash A$ holds classically if and only if $\vdash A^{\oplus}$ holds in PRK

Strong propositions behave constructively

The classical excluded middle $\vdash (A \vee \neg A)^{\oplus}$ always holds.

The strong excluded middle $\vdash (A \vee \neg A)^{+}$ does not hold in general.

Outline

The calculus λ^{PRK}

We assign explicit witnesses to proofs:

$$
\begin{array}{rcll}
t, s, u, \ldots & ::= & x & \text{variable} \\
& \mid & t \bowtie_P s & \text{absurdity} \\
& \mid & \langle t, s \rangle^{\pm} & \wedge^+ \;/\; \vee^- \text{ introduction} \\
& \mid & \pi_i^{\pm}(t) & \wedge^+ \;/\; \vee^- \text{ elimination} \\
& \mid & \mathsf{in}_i^{\pm}(t) & \vee^+ \;/\; \wedge^- \text{ introduction} \\
& \mid & \mathsf{case}^{\pm}\, t\, [_{x:P}.s][_{y:Q}.u] & \vee^+ \;/\; \wedge^- \text{ elimination} \\
& \mid & \nu^{\pm} t & \neg^+ \;/\; \neg^- \text{ introduction} \\
& \mid & \mu^{\pm} t & \neg^+ \;/\; \neg^- \text{ elimination} \\
& \mid & \bigcirc^{\pm}_{(x:P)}.\, t & \text{classical introduction} \\
& \mid & t \bullet^{\pm} s & \text{classical elimination}
\end{array}
$$

The calculus λ^{PRK}

Type system (excerpt)

$$\frac{\Gamma \vdash t : A^{+} \qquad \Gamma \vdash s : A^{-}}{\Gamma \vdash t \bowtie_{P} s : P}\ \textsc{Abs} \qquad \dots \qquad \frac{\Gamma \vdash t : A^{\ominus} \qquad \Gamma \vdash s : B^{\ominus}}{\Gamma \vdash \langle t, s\rangle^{-} : (A \vee B)^{-}}\ \mathrm{I}{\vee}^{-}$$

$$\frac{\Gamma, x : A^{\ominus} \vdash t : A^{+}}{\Gamma \vdash \bigcirc^{+}_{(x:A^{\ominus})}.\, t : A^{\oplus}}\ \mathrm{I}{\bigcirc}^{+} \qquad \dots \qquad \frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : A^{\ominus}}{\Gamma \vdash t \bullet^{+} s : A^{+}}\ \mathrm{E}{\bigcirc}^{+}$$

The calculus λ^{PRK}

Reduction rules

$$
\begin{array}{rcl}
\pi_i^{\pm}(\langle t_1, t_2\rangle^{\pm}) & \xrightarrow{\beta^+_\wedge \,/\, \beta^-_\vee} & t_i \\
\mathsf{case}^{\pm}(\mathsf{in}_i^{\pm}(t))\,[_x.s_1][_x.s_2] & \xrightarrow{\beta^+_\vee \,/\, \beta^-_\wedge} & s_i[x := t] \\
\mu^{\pm}(\nu^{\pm} t) & \xrightarrow{\beta^+_\neg \,/\, \beta^-_\neg} & t \\
(\bigcirc_x^{\pm}.\, t) \bullet^{\pm} s & \xrightarrow{\beta^+_\circ \,/\, \beta^-_\circ} & t[x := s] \\
\langle t_1, t_2\rangle^{+} \bowtie \mathsf{in}_i^{-}(s) & \xrightarrow{\bowtie_\wedge} & (t_i \bullet^{+} s) \bowtie (s \bullet^{-} t_i) \\
\mathsf{in}_i^{+}(t) \bowtie \langle s_1, s_2\rangle^{-} & \xrightarrow{\bowtie_\vee} & (t \bullet^{+} s_i) \bowtie (s_i \bullet^{-} t) \\
(\nu^{+} t) \bowtie (\nu^{-} s) & \xrightarrow{\bowtie_\neg} & (s \bullet^{+} t) \bowtie (t \bullet^{-} s) \\
 & & \\
\bigcirc_x^{\pm}.\,(t \bullet^{\pm} x) & \xrightarrow{\eta_\circ} & t \quad \text{if } x \notin \mathsf{fv}(t)
\end{array}
$$

The calculus λ^{PRK}

Theorem (Subject Reduction)

If $\Gamma \vdash t : P$ and $t \to s$ then $\Gamma \vdash s : P$.

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Theorem (Duality)

1. $\Gamma \vdash t : P$ if and only if $\Gamma^\perp \vdash t^\perp : P^\perp$
2. $t \to s$ if and only if $t^\perp \to s^\perp$

where $-^\perp$ flips all the signs and exchanges dual connectives $(\wedge, \vee)$.

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Theorem (Subject Reduction)

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where $-^{\perp}$ flips all the signs and exchanges dual connectives $(\wedge, \vee)$.

Theorem (Convergence)

λ^{PRK} is confluent and strongly normalizing.

- The main difficulty in the SN proof is how to deal with the mutually recursive types $A^{\oplus} \simeq A^{\ominus} \Rightarrow A^{+}$ and $A^{\ominus} \simeq A^{\oplus} \Rightarrow A^{-}$.
- The SN proof is via a translation to System F with non-strictly positive recursive types, relying on a result by Mendler.

The calculus λ^{PRK}

There have been many computational interpretations of classical logic:

1. Parigot's $\lambda\mu$.
2. Barbanera and Berardi's symmetric λ-calculus.
3. Curien and Herbelin's $\bar{\lambda}\mu\tilde{\mu}$.
4. Krivine's λ_c.
5. ...

λ^{PRK} provides a new computational interpretation for classical logic.

The calculus λ^{PRK}

Example: conjunction

Taking:

$$\begin{array}{rcl}
\langle t, s\rangle & \stackrel{\text{def}}{=} & \bigcirc^{+}_{(_:(A\wedge B)^{\ominus})}.\,\langle t, s\rangle^{+} \\
\pi_i(t) & \stackrel{\text{def}}{=} & \bigcirc^{+}_{(x:A_i{}^{\ominus})}.\,\pi_i^{+}(t \bullet^{+} \bigcirc^{-}_{(_:(A_1\wedge A_2)^{\oplus})}.\,\mathrm{in}_i^{-}(x)) \bullet^{+} x
\end{array}$$

Classical introduction and elimination of conjunction can be derived:

$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : B^{\oplus}}{\Gamma \vdash \langle t, s\rangle : (A \wedge B)^{\oplus}} \qquad \frac{\Gamma \vdash t : (A_1 \wedge A_2)^{\oplus}}{\Gamma \vdash \pi_i(t) : A_i{}^{\oplus}}$$

The standard computation rule for projection can be recovered:

$$\pi_i(\langle t_1, t_2\rangle) \to^{*} t_i$$

The calculus λ^{PRK}

A more interesting example: implication

In classical logic, implication is derivable from negation and disjunction.
This can be extended to the computational level.

Let $(A \to B) \stackrel{\mathrm{def}}{=} (\neg A \vee B)$.
Abstraction and application can be defined with their expected types:

$$
\begin{array}{rcl}
\lambda x.\, t & \stackrel{\mathrm{def}}{=} & \bigcirc^{+}_{(y:(A \Rightarrow B)^{\ominus})}.\, \mathsf{in}^{+}_{2}(t[x := \mathbf{X}_y]) \\
\mathbf{X}_y & \stackrel{\mathrm{def}}{=} & \bigcirc^{+}_{(z:A^{\ominus})}.\, (\mu^{-}(\mathbf{X}'_{y,z} \bullet^{-} \bigcirc^{+}_{(\_:(\neg A)^{\ominus})}.\, \nu^{+} z)) \bullet^{+} z \\
\mathbf{X}'_{y,z} & \stackrel{\mathrm{def}}{=} & \pi^{+}_{1}(y \bullet^{-} \bigcirc^{+}_{(\_:(A \Rightarrow B)^{\ominus})}.\, \mathsf{in}^{+}_{1}(\bigcirc^{+}_{(\_:(\neg A)^{\ominus})}.\, \nu^{+} z)) \\
t \mathbin{@} s & \stackrel{\mathrm{def}}{=} & \mathsf{IC}^{+}_{(x:B^{\ominus})}. \\
& & \quad \mathsf{case}^{+}\ (t \bullet^{+} \bigcirc^{-}_{(\_:(A \to B)^{\oplus})}.\, \langle(\bigcirc^{-}_{(\_:(\neg A)^{\oplus})}.\, \nu^{-} s), x\rangle^{-}) \\
& & \qquad [_{(y:(\neg A)^{\oplus})}.s \bowtie_{B^{+}} \mu^{-}(y \bullet^{+} \bigcirc^{-}_{(\_:(\neg A)^{\oplus})}.\, \nu^{-} x)] \\
& & \qquad [_{(z:B^{\oplus})}.z \bullet^{+} x]
\end{array}
$$

The standard β-reduction rule can be recovered:

$$(\lambda x.\, t) \mathbin{@} s \to^{*} t[x := s]$$

Outline

Kripke Semantics

A Kripke model for PRK is a structure $\mathcal{M} = (\mathcal{W}, \leq, \mathcal{V}^+, \mathcal{V}^-)$.
(Enjoying appropriate technical conditions).

Forcing (excerpt)

$$
\begin{array}{lcl}
\mathcal{M}, w \Vdash \alpha^+ & \iff & \alpha \in \mathcal{V}^+_w \\
\mathcal{M}, w \Vdash \alpha^- & \iff & \alpha \in \mathcal{V}^-_w \\
\vdots & & \\
\mathcal{M}, w \Vdash (A \vee B)^- & \iff & \mathcal{M}, w \Vdash A^\ominus \text{ and } \quad \mathcal{M}, w \Vdash B^\ominus \\
\vdots & & \\
\mathcal{M}, w \Vdash A^\oplus & \iff & \mathcal{M}, w' \nVdash A^- \text{ for all } w' \geq w \\
\vdots & &
\end{array}
$$

Theorem (Soundness and Completeness)

$$\Gamma \vdash P \qquad \text{if and only if} \qquad \Gamma \Vdash P$$

Outline

Further Extensions

Second Order λ^{PRK}

We have extended λ^{PRK} with implication, co-implication, and second-order quantifiers:

$$\text{Pure propositions} \quad A \quad ::= \quad \ldots \mid A \to A \mid A \ltimes A \mid \forall \alpha. A \mid \exists \alpha. A$$

- All of the previous results can be extended to this setting.
- The SN proof requires a completely different strategy, using reducibility candidates.

Intuitionistic λ^{PRK}

We have identified an intuitionistic subset of λ^{PRK}.
The key is, essentialy, to identify $A^{\oplus}$ with A^{+} rather than with $A^{\ominus} \to A^{+}$.

Outline

Contributions

- We studied an extension of the BHK interpretation.

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  PRK is a **conservative extension** of classical logic.

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- This interpretation motivates the logical system PRK.
  PRK is a **conservative extension** of classical logic.

- **Propositions-as-types.**
  PRK corresponds to a confluent and terminating calculus λ^{PRK}.
  It has been extended to second-order logic.
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Future Work

- Relate λ^{PRK} with existing classical calculi.
- Extend λ^{PRK} with dependent types.
- In System F, $\{\exists, \wedge, \vee, \bot, \top, \neg\}$ can be derived from $\{\forall, \rightarrow\}$.
  This is not true in second-order PRK (!)
  Can we identify subsets of "computationally adequate" connectives?