A Constructive Logic with Classical Proofs and Refutations

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Pablo Barenbaum[1,2] Teodoro Freund[1]

[1] Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Argentina

[2] Universidad Nacional de Quilmes
Argentina

Outline

The BHK Interpretation ($\{\wedge, \vee, \neg\}$ fragment)

$$
\begin{aligned}
(A \wedge B)^+ &\simeq A^+ \times B^+ \\
(A \vee B)^+ &\simeq A^+ \uplus B^+ \\
(\neg A)^+ &\simeq A^+ \Rightarrow \varnothing
\end{aligned}
$$

$A^+ =$ "proofs of A"

Nelson's Strong Negation

(Nelson, 1949)

$$
\begin{aligned}
(A \wedge B)^+ &\simeq A^+ \times B^+ & \qquad (A \wedge B)^- &\simeq A^- \uplus B^- \\
(A \vee B)^+ &\simeq A^+ \uplus B^+ & \qquad (A \vee B)^- &\simeq A^- \times B^- \\
(\neg A)^+ &\simeq A^- & \qquad (\neg A)^- &\simeq A^+
\end{aligned}
$$

$A^+ =$ "proofs of A"

$A^- =$ "refutations of A"

Starting point: a BHK interpretation for classical logic

$$
\begin{array}{rcl@{\qquad}rcl}
(A \wedge B)^+ & \simeq & A^\oplus \times B^\oplus & (A \wedge B)^- & \simeq & A^\ominus \uplus B^\ominus \\
(A \vee B)^+ & \simeq & A^\oplus \uplus B^\oplus & (A \vee B)^- & \simeq & A^\ominus \times B^\ominus \\
(\neg A)^+ & \simeq & A^\ominus & (\neg A)^- & \simeq & A^\oplus \\
A^\oplus & \simeq & A^\ominus \Rightarrow A^+ & A^\ominus & \simeq & A^\oplus \Rightarrow A^-
\end{array}
$$

$A^+ =$ "strong proofs of A"

$A^\oplus =$ "classical proofs of A"

$A^- =$ "strong refutations of A"

$A^\ominus =$ "classical refutations of A"

Outline

- Introduction
- The Logical System PRK
- Kripke Semantics
- Propositions as types
- Conclusion

System PRK – Natural Deduction

Pure propositions	A	$::=$	$\alpha \mid A \wedge A \mid A \vee A \mid \neg A$	
Propositions	P	$::=$	A^{+}	strong affirmation
		$\mid$	A^{-}	strong denial
		$\mid$	$A^{\oplus}$	classical affimation
		$\mid$	$A^{\ominus}$	classical denial

Inference rules (excerpt)

$$\frac{\Gamma \vdash A^{+} \qquad \Gamma \vdash A^{-}}{\Gamma \vdash Q}\ \textsc{Abs} \qquad\qquad \frac{\Gamma \vdash A^{\ominus} \qquad \Gamma \vdash B^{\ominus}}{\Gamma \vdash (A \vee B)^{-}}\ \mathrm{I}\vee^{-}$$

$$\frac{\Gamma, A^{\ominus} \vdash A^{+}}{\Gamma \vdash A^{\oplus}}\ \mathrm{IC}^{+} \qquad\qquad \frac{\Gamma \vdash A^{\oplus} \qquad \Gamma \vdash A^{\ominus}}{\Gamma \vdash A^{+}}\ \mathrm{EC}^{+}$$

System PRK – Natural Deduction

Conservative extension

$\vdash A$ holds classically if and only if $\vdash A^{\oplus}$ holds in PRK

Strong propositions behave constructively

The classical excluded middle $\vdash (A \vee \neg A)^{\oplus}$ always holds.

The strong excluded middle $\vdash (A \vee \neg A)^{+}$ does not hold in general.

Outline

Kripke Semantics

A Kripke model for PRK is a structure $\mathcal{M} = (\mathcal{W}, \leq, \mathcal{V}^+, \mathcal{V}^-)$.
(Enjoying appropriate technical conditions).

Forcing (excerpt)

$$\begin{array}{lcl}
\mathcal{M}, w \Vdash \alpha^+ & \iff & \alpha \in \mathcal{V}^+_w \\
\mathcal{M}, w \Vdash \alpha^- & \iff & \alpha \in \mathcal{V}^-_w \\
\mathcal{M}, w \Vdash (A \vee B)^- & \iff & \mathcal{M}, w \Vdash A^\ominus \ \text{ and } \ \mathcal{M}, w \Vdash B^\ominus \\
\mathcal{M}, w \Vdash A^\oplus & \iff & \mathcal{M}, w' \nVdash A^- \text{ for all } w' \geq w
\end{array}$$

Theorem (Soundness and Completeness)

$$\Gamma \vdash P \quad \text{if and only if} \quad \Gamma \Vdash P$$

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The calculus λ^{PRK}

Type system (excerpt)

$$\frac{\Gamma \vdash t : A^{+} \qquad \Gamma \vdash s : A^{-}}{\Gamma \vdash t \bowtie s : Q}\,\textsc{Abs}$$

$$\frac{\Gamma \vdash t : A^{\ominus} \qquad \Gamma \vdash s : B^{\ominus}}{\Gamma \vdash \langle t, s\rangle^{-} : (A \vee B)^{-}}\,\mathrm{I}\vee^{-}$$

$$\frac{\Gamma, x : A^{\ominus} \vdash t : A^{+}}{\Gamma \vdash \mathsf{IC}^{+}_{x}.\, t : A^{\oplus}}\,\mathrm{IC}^{+}$$

$$\frac{\Gamma \vdash t : A^{\oplus} \qquad \Gamma \vdash s : A^{\ominus}}{\Gamma \vdash t \bullet^{+} s : A^{+}}\,\mathrm{EC}^{+}$$

The calculus λ^{PRK}

λ^{PRK} is provided with a notion of reduction with good properties:

Theorem – Subject Reduction

If $\Gamma \vdash t : P$ and $t \to s$ then $\Gamma \vdash s : P$.

Theorem

λ^{PRK} is confluent and strongly normalizing.

`The proof of strong normalization relies on a translation to System F with Mendler-style recursion.`

The embedding of classical logic into PRK has good computational behavior. For example:

$$\pi_i^{\mathcal{C}}(\langle t_1, t_2\rangle^{\mathcal{C}}) \to^* t_i$$

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Contributions

- We studied an extension of the BHK interpretation.

  Key idea: $A^{\oplus} \simeq A^{\ominus} \Rightarrow A^{+}$

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  PRK is a **conservative extension** of classical logic.

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- **Propositions-as-types.**
  PRK corresponds to a confluent and terminating calculus λ^{PRK}.

- λ^{PRK} is thus a **computational interpretation** of classical logic.
  There are many other ones—*e.g.* by Griffin, Parigot, Barbanera–Berardi, Curien–Herbelin, Krivine, ...

Future Work

- Extend PRK to second-order logic and dependent types.