Sharing and Linear Logic with Restricted Access

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Abstract. The two Girard translations provide two different means of obtaining embeddings of Intuitionistic Logic into Linear Logic, corresponding to different lambda-calculus calling mechanisms. The translations, mapping $A \to B$ respectively to $!A \multimap B$ and $!(A \multimap B)$, have been shown to correspond respectively to call-by-name and call-by-value.
In this work, we split the *of-course* modality of linear logic into two modalities, written "!" and "•". Intuitively, the modality "!" specifies a subproof that can be duplicated and erased, but may not necessarily be "accessed", *i.e.* interacted with, while the combined modality "!•" specifies a subproof that can moreover be accessed. The resulting system, called MSCLL, enjoys cut-elimination and is conservative over MELL.
We study how restricting access to subproofs provides ways to control *sharing* in evaluation strategies. For this, we introduce a term-assignment for an intuitionistic fragment of MSCLL, called the $\lambda^{!\bullet}$-calculus, which we show to enjoy subject reduction, confluence, and strong normalization of the simply typed fragment. We propose three sound and complete translations that respectively simulate call-by-name, call-by-value, and a variant of call-by-name that *shares* the evaluation of its arguments (similarly as in call-by-need). The translations are extended to simulate the Bang-calculus, as well as weak reduction strategies.

Keywords: Linear Logic, Lambda Calculus, Sharing, Calling Mechanisms, Call-By-Value, Call-By-Name, Call-By-Need

1 Introduction

The propositions-as-types correspondence links *computation* and *logic*, relating types with propositions, programs with proofs, and program evaluation with proof normalization. The prime example is the simply typed λ-calculus, which corresponds to intuitionistic propositional logic. The correspondence has been extended to many other calculi and logics, including Linear Logic (LL) [16]. Linear Logic proposes a *resource conscious* approach to logic, in that only formulae prefixed with an *exponential modality* can be duplicated and erased. In LL, there are two exponential modalities: *of-course* ("!"), allowing duplication/erasure on the left, and *why-not* ("?"), allowing duplication/erasure on the right. These modalities recover the ability to duplicate and erase formulae in a controlled way, making LL a suitable language to model resource-sensitive phenomena such as concurrency, memory management, and computational complexity.

Girard [16] discusses two possible ways of embedding intuitionistic logic into LL, mapping the intuitionistic implication $A \to B$ respectively to $!A \multimap B$ and to $!A \multimap !B$, where "$\multimap$" stands for the linear implication[3]. Maraist *et al.* [23] observed that these translations can be used to extend the propositions-as-types correspondence to provide a logical foundation for *evaluation mechanisms*.

The most well-known evaluation mechanism for the λ-calculus is perhaps *call-by-name* (CBN), in which arguments to functions are re-evaluated upon each use. The theory of CBN has been thoroughly developed, *e.g.* in Barendregt's book [11]. On the other hand, in the *call-by-value* (CBV) evaluation mechanism [27], arguments to functions are evaluated once and for all; then their value can be recalled upon each use. Call-by-value is less deeply studied in the literature than CBN, but it has been gaining attention since its theory is subtle and corresponds more closely to the evaluation mechanism behind most programming languages.

Embeddings Encode Evaluation Mechanisms. The first Girard translation $(\cdot)^{\mathsf{N}}$ operates on *formulae* by mapping $A \to B$ to $!A^{\mathsf{N}} \multimap B^{\mathsf{N}}$. It operates on *terms* in such a way that a λ-calculus application $t\,s$ is mapped to $t^{\mathsf{N}}\,!s^{\mathsf{N}}$ in a *linear* λ-calculus[4]. Here "!" is a *term constructor*, which corresponds to an instance of the !-*introduction* rule, also called *promotion*. The key point is that the argument s of an application is prefixed with "!". This enables the (arbitrary) term s^{N} to be freely copied or discarded in t^{N}, as dictated by CBN. In this sense, the first translation provides a logical foundation for the CBN evaluation mechanism.

The second Girard translation $(\cdot)^{\mathsf{V}}$ operates on formulae by mapping the intuitionistic implication $A \to B$ to $!A^{\mathsf{V}} \multimap !B^{\mathsf{V}}$. It operates on terms in such a way that a λ-calculus application $t\,s$ is mapped to $\mathtt{der}(t^{\mathsf{V}})\,s^{\mathsf{V}}$. Here, $\mathtt{der}(\cdot)$ stands for an appropriate operation in the target language that corresponds to the !-*elimination* rule, also called *dereliction*. The key point is that the argument is not prefixed with "!", which means that it must be evaluated before being consumed, as dictated by CBV. This translation prefixes *values*, such as $\lambda x.\, r$, with a promotion, resulting in $!(\lambda x.\, r^{\mathsf{V}})$. Consequently, only values will end up being copied or discarded.

Linear Logic with Restricted Access. In this work, we propose a logical system that arises from splitting each exponential modality into two new modalities. We refer to this new system as *Linear Logic with Restricted Access*, and formally as MSCLL, to reflect that we study the fragment with **M**ultiplicative, **S**haring, and a**C**cess connectives. The *exponential* of-course ("!") modality of LL is split into two modalities in MSCLL: a *sharing* modality "!" and an *access* modality "$\bullet$", that "grants" access to a subproof. The *sharing of-course* of MSCLL turns out to be weaker than the *exponential of-course* of LL but, by abuse of notation, we denote it using the same symbol "!". Dually, the exponential why-not ("?") modality of LL is split into a sharing modality "?" and an access

[3] Sometimes the second embedding is defined as mapping $A \to B$ to $!(A \multimap B)$. This is just an apparent difference, which is canceled out by adjusting the translation of sequents accordingly.

[4] Girard's original translations target Linear Logic presented in sequent calculus style. We follow here Maraist *et al.* [23], in which the target language is presented as a linear λ-calculus, in natural deduction style.

modality "$\circ$" in MSCLL. The resulting system MSCLL is a conservative extension of Multiplicative Exponential Linear Logic (MELL), in which the combined modality "$!\bullet$" plays the role of the exponential of-course modality of LL. The operational intuition is that the sharing modality "!" in MSCLL specifies an expression that may be duplicated and erased, but it may not necessarily be used or *accessed*. The combined modality "$!\bullet$" specifies an expression that may, additionally, be accessed, *i.e.* its contents can be made to interact with the surrounding computational context.

Embedding Call-by-Name and Call-by-Value. Following, we revisit Girard's translations, but this time targetting the $\lambda^{!\bullet}$-*calculus*, a linear λ-calculus based on MSCLL. Our CBN translation $(\cdot)^{\mathsf{N}}$ operates on formulae by mapping $A \to B$ to $!\bullet A^{\mathsf{N}} \multimap B^{\mathsf{N}}$. It operates on terms by mapping a λ-calculus application $t\,s$ to $t^{\mathsf{N}}\;!\bullet s^{\mathsf{N}}$, indicating that the argument may be freely copied, discarded, and accessed. This mimics the original translation of CBN to LL. Our CBV translation $(\cdot)^{\mathsf{V}}$ operates on formulae by mapping $A \to B$ to $!\bullet A^{\mathsf{V}} \multimap\; !\bullet B^{\mathsf{V}}$. It operates on terms by leaving the translation of the argument of an application intact, mapping $t\,s$ to $\mathtt{req}(\mathtt{der}(t^{\mathsf{V}}))\;s^{\mathsf{V}}$. As before, $\mathtt{der}(\cdot)$ is an appropriate operator corresponding to !-elimination, while $\mathtt{req}(\cdot)$ stands for $\bullet$-elimination. At the same time, the translation maps values such as $\lambda x.\,t$ (roughly[5]) to $!\bullet\lambda x.\,t^{\mathsf{V}}$. Thus an argument s^{V} cannot be copied, discarded, or accessed at all unless later, after further evaluation, it takes the form $!\bullet\mathtt{v}$, for some value $\mathtt{v}$, in which case $\mathtt{v}$ can be copied. This too mimics the original translation from CBV to LL. The CBN and CBV translations are proved to be sound and complete, in the sense that two λ-terms are interconvertible in the source language if and only if they are mapped to interconvertible terms in the target language.

Call-by-Sharing. The two translations above suggest a "missing link" translation $(\cdot)^{\mathsf{S}}$ that maps $A \to B$ to $!\bullet A^{\mathsf{S}} \multimap \bullet B^{\mathsf{S}}$. This translation cannot be expressed directly in a linear λ-calculus, because the "$\bullet$" modality is used as a stand-alone operator. A λ-calculus application $t\,s$ is now mapped to $t^{\mathsf{S}}\;!s^{\mathsf{S}}$, meaning that the argument can be copied and discarded, but *not accessed* yet. A value such as $\lambda x.\,r$ is now (roughly) mapped to $\bullet\lambda x.\,r^{\mathsf{S}}$. Thus an argument s^{S} cannot be accessed unless, after further evaluation, it takes the form $\bullet\mathtt{v}$, for some value $\mathtt{v}$, in which case $\mathtt{v}$ can be accessed. The fact that arguments cannot be accessed until they become values means that the evaluation mechanism can keep *references* to a single *shared* copy of the argument, until it becomes accessible. This translation suggests an evaluation mechanism that we dub *call-by-sharing* (CBS).

Call-by-sharing bears a strong resemblance to call-by-need (CBNd), an evaluation mechanism introduced by Wadsworth in 1971 [29]. Both in CBNd and in CBS, arguments that are not used may be discarded without being evaluated. A *reference* to a shared argument may be freely copied, but the argument itself can only be copied after it has been evaluated to a value. Nevertheless, there are some subtle differences between CBNd and CBS, and in particular CBS achieves less sharing than CBNd (see the discussion in Section 5). Unfortunately, there does not seem to be a way to embed CBNd into $\lambda^{!\bullet}$.

[5] Intuitionistic variables will be mapped to modal variables.

Bang Calculus. Another approach towards providing a common framework to explain CBV and CBN is the Bang-calculus [15]. It is an untyped lambda calculus that has explicit constructors in the syntax for promotion (!-introduction) and dereliction (!-elimination). It was motivated by the fact that Girard's original CBN and CBV translations of the intuitionistic logic into LL made use of logical exponentials (promotion and dereliction) that were not reflected in the syntax. The aim was thus to introduce an intermediate formalism between lambda calculus and proof nets, a graphical notation for LL proofs [16], that allows explicit use of "boxes" to mark values. Soundness and completeness of these translations with respect to reduction was proved by Guerrieri and Manzonetto [17] for slightly different notion of reduction for the Bang-calculus than that of [15]. The Bang-calculus can in fact be embedded into our $\lambda^{!\bullet}$-calculus and this embedding is both sound and complete.

Contributions. A summary of the contributions are as follows:
1. The introduction of a new logic called MSCLL ("Linear Logic with Restricted Access) that enjoys cut-elimination and is conservative over MELL. It provides a split of each exponential modality into a *sharing modality* and an *access modality*.
2. A term assignment for MSCLL, the $\lambda^{!\bullet}$*-calculus*, which operationally distinguishes between two kinds of expressions. *Sharable expressions* can be discarded, and *references* to shared expressions can be duplicated, but they cannot be accessed, and thus they can remain shared. *Sharable accessible expressions* can moreover be accessed, and they are copied whenever access to them is requested. This distinction allows to formulate the CBS evaluation mechanism.
3. Translations from CBV and CBN to $\lambda^{!\bullet}$ that are sound, complete and preserve normal forms.
4. The presentation of the *call-by-sharing* calculus (CBS), and a translation from CBS to $\lambda^{!\bullet}$ that is sound, complete and preserves normal forms.
5. A weak *evaluation mechanism* for $\lambda^{!\bullet}$ that can simulate weak evaluation strategies in CBN, CBV and CBS, with soundness and completeness results.
6. A translation from Ehrhard's Bang-calculus to $\lambda^{!\bullet}$ which is sound, complete and preserves normal forms.

Structure of the paper. We review some background notions in Section 2. We present MSCLL in Section 3 and a term assignment for this logic, the $\lambda^{!\bullet}$-calculus, in Section 4. Definitions of the CBV, CBN, and CBS calculi and their translations to $\lambda^{!\bullet}$ are presented in Section 5 together with results of soundness, completeness, and preservation of normal forms. Section 7 shows how $\lambda^{!\bullet}$ can also embed the Bang-calculus via a sound and complete translation. Section 6 presents a notion of weak evaluation for $\lambda^{!\bullet}$ which is shown to simulate weak CBV, CBN, and CBS evaluation. Finally, we conclude and discuss future work. Most proofs are omitted from the main body of the paper and can be found in the appendix.

2 Preliminary Notions

In this section we present some background notions and results that we use throughout the paper.

Recall that an *abstract rewriting system* (ARS) is a pair $\mathcal{X} = (X, \to_{\mathcal{X}})$ where X is a set and $\to_{\mathcal{X}} \subseteq X^2$ is a binary relation called *reduction*. We write $\to_{\mathcal{X}}^*$ for the reflexive–transitive closure of $\to_{\mathcal{X}}$, $\to_{\mathcal{X}}^+$ for the transitive closure, $\to_{\mathcal{X}}^=$ for the reflexive closure, $\leftrightarrow_{\mathcal{X}}$ for the symmetric closure, $\leftarrow_{\mathcal{X}}$ or $\to_{\mathcal{X}}^{-1}$ for the inverse relation, and $\to_{\mathcal{X}}^n$ for the composition of $\to_{\mathcal{X}}$ with itself n times. An ARS is confluent (CR) if ${}_{\mathcal{X}}^*\!\leftarrow \to_{\mathcal{X}}^* \subseteq \to_{\mathcal{X}}^* \,{}_{\mathcal{X}}^*\!\leftarrow$ and strongly normalizing (SN) if there are no infinite reductions $x_1 \to_{\mathcal{X}} x_2 \to_{\mathcal{X}} \ldots$.

Abstract Results on Translations Given ARSs $\mathcal{X} = (X, \to_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \to_{\mathcal{Y}})$, a *translation* $T : \mathcal{X} \to \mathcal{Y}$ is a function $T : X \to Y$, also written T, by abuse of notation. A translation is *sound* if $x_1 \to_{\mathcal{X}}^* x_2$ implies $T(x_1) \to_{\mathcal{Y}}^* T(x_2)$ for all $x_1, x_2 \in X$, and *complete* if $T(x_1) \to_{\mathcal{Y}}^* T(x_2)$ implies $x_1 \to_{\mathcal{X}}^* x_2$ for all $x_1, x_2 \in X$. The following are easy results on a translation T:

Proposition 2.1 (Conditions for soundness [proof in Prop. A.1]). *If $x_1 \to_{\mathcal{X}} x_2$ implies $T(x_1) \to_{\mathcal{Y}}^* T(x_2)$ for every $x_1, x_2 \in X$, then T is sound.*

Theorem 2.1 (Conditions for completeness [proof in Thm. A.1]). *Let $Y' \subseteq Y$ and let $T^{-1} : Y' \to X$ be a function. Suppose that T^{-1} is the left-inverse of T,* i.e. *for all $x \in X$ we have that $T(x) \in Y'$ and $T^{-1}(T(x)) = x$. Suppose moreover that T^{-1} simulates reduction,* i.e. *for all $y_1 \in Y'$ and $y_2 \in Y$ such that $y_1 \to_{\mathcal{Y}} y_2$, we have that $y_2 \in Y'$ and $T^{-1}(y_1) \to_{\mathcal{X}}^* T^{-1}(y_2)$. Then T is complete.*

The Linear Substitution Calculus The LSC is a refinement of the λ-calculus with explicit substitutions, introduced by Accattoli and Kesner [1,3] as a variation over a calculus by Milner [25]. The set of LSC *terms* ($\mathcal{T}_{\mathsf{LSC}}$) is defined as follows, where $[x/s]$ is called an *explicit substitution* (ES):

$$t, s, \ldots ::= x \mid \lambda x.\, t \mid t\, s \mid t[x/s]$$

A *context* is a term with a single occurrence of a *hole* "$\Box$". A *substitution context* is a list of ESs. Formally:

$$\begin{array}{ll} \text{General contexts} & \mathsf{C} ::= \Box \mid \lambda x.\,\mathsf{C} \mid \mathsf{C}\, t \mid t\, \mathsf{C} \mid \mathsf{C}[x/t] \mid t[x/\mathsf{C}] \\ \text{Substitution contexts} & \mathsf{L} ::= \Box \mid \mathsf{L}[x/t] \end{array}$$

Free and *bound* occurrences of variables are defined as expected, and $\mathsf{fv}(t)$ denotes the set of free variables of t. Terms are defined up to α-renaming of bound variables.

We write $\mathsf{C}\langle t\rangle$ for the *variable-capturing* substitution of $\Box$ in C by t. We write $\mathsf{C}\langle\!\langle t\rangle\!\rangle$ for the *capture-avoiding* substitution of $\Box$ in C by t. For example, if $\mathsf{C} = \lambda x.\,\Box$ then $\mathsf{C}\langle x\rangle = \lambda x.\, x$, while $\mathsf{C}\langle\!\langle x\rangle\!\rangle = \lambda y.\, x \neq \lambda x.\, x$. In the case of substitution contexts, we usually write $t\mathsf{L}$ rather than $\mathsf{L}\langle t\rangle$. The *domain* of L, written $\mathsf{dom}(\mathsf{L})$, is the set of variables bound by L. The LSC has three rewriting rules, closed by congruence under arbitrary contexts:

$$(\lambda x.\, t)\mathsf{L}\; s \to_{\mathsf{db}} t[x/s]\mathsf{L} \qquad \mathsf{C}\langle\!\langle x\rangle\!\rangle[x/t] \to_{\mathsf{ls}} \mathsf{C}\langle\!\langle t\rangle\!\rangle[x/t] \qquad t[x/s] \to_{\mathsf{gc}} t \text{ (if } x \notin \mathsf{fv}(t))$$

Distance beta (db) performs a β-step but creates an ES rather than doing meta-level substitution, *linear substitution* (ls) replaces a *single* occurrence of x to a term t when

x is bound to t by an ES, and *garbage collection* (gc) removes an unreachable ES. Reduction in LSC is the union $\to_{\mathsf{lsc}} := \to_{\mathsf{db}} \cup \to_{\mathsf{ls}} \cup \to_{\mathsf{gc}}$.

The rewriting rules of LSC are said to operate "at a distance" due to their peculiar use of contexts. This avoids reduction getting stuck in the presence of explicit substitutions. For example, if the left-hand side of db were declared to be $(\lambda x.\, t)\, s$, then an expression such as $(\lambda x.\, t)[y/r]\, s$ would not be a redex. This notation has a strong connection to proof nets [2].

We write $\Gamma \vdash_{\mathrm{lsc}} t : A$ is t has type A under the typing context Γ, with standard simple type assignment rules. Recall from [19, Theorem 6.11] that simply typed terms are SN:

Theorem 2.2. *If $\Gamma \vdash_{\mathrm{lsc}} t : A$, there are no infinite reduction sequences $t \to_{\mathrm{lsc}} t_1 \to_{\mathrm{lsc}} \ldots$.*

3 Linear Logic with Restricted Access

We start by defining a one-sided sequent calculus presentation for a linear logic with *multiplicative* connectives ($\otimes, \parr$, called *tensor* and *par*), *sharing* modalities ($!, ?$, called *of-course* and *why-not*), and *access* modalities ($\bullet$, $\circ$, called *grant* and *demand*), which we dub MSCLL.

Formulae and Sequent Calculus Presentation. We assume given a denumerable set of atomic formulae ($\alpha, \beta, \ldots$), each of which has a corresponding negative version ($\overline{\alpha}, \overline{\beta}, \ldots$). The set of formulae is given by the grammar:

$$A, B, \ldots ::= \alpha \mid \overline{\alpha} \mid A \otimes B \mid A \parr B \mid !A \mid ?A \mid \bullet A \mid \circ A$$

Linear negation is the involutive operator $(\cdot)^{\perp}$ given by:

$$\begin{array}{cccc} \alpha^{\perp} := \overline{\alpha} & \overline{\alpha}^{\perp} := \alpha & (A \otimes B)^{\perp} := A^{\perp} \parr B^{\perp} & (A \parr B)^{\perp} := A^{\perp} \otimes B^{\perp} \\ (!A)^{\perp} := ?A^{\perp} & (?A)^{\perp} := !A^{\perp} & (\bullet A)^{\perp} := \circ A^{\perp} & (\circ A)^{\perp} := \bullet A^{\perp} \end{array}$$

Sequents are of the form $\vdash \Gamma$, where Γ is a finite *multiset* of formulae (note that we do not include an explicit exchange rule). If $\Gamma = A_1, \ldots, A_n$ and $\mathfrak{m}$ is one of the modalities, we write $\mathfrak{m}\Gamma$ to stand for $\mathfrak{m}A_1, \ldots, \mathfrak{m}A_n$, so for instance $\circ\Gamma = \circ A_1, \ldots, \circ A_n$. Derivable sequents are given inductively by the following rules:

$$\frac{}{\vdash A, A^{\perp}}\,\mathtt{ax} \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,\mathtt{cut} \quad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,\mathtt{!p}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,\mathtt{?w} \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,\mathtt{?c} \quad \frac{\vdash \Gamma, \circ A}{\vdash \Gamma, ?\circ A}\,\mathtt{?\circ d} \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, \bullet A}\,\bullet \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, \circ A}\,\circ$$

Most rules are standard rules from multiplicative-exponential linear logic (MELL), including the standard weakening ($\mathtt{?w}$), contraction ($\mathtt{?c}$), and promotion ($\mathtt{!p}$) rules. The atypical rule is dereliction ($\mathtt{?\circ d}$), which requires the conclusion to be $\Gamma, ?\circ A$ instead of the usual $\Gamma, ?A$. The $\bullet$ and $\circ$ rules are (trivial) introduction rules for $\bullet$ and $\circ$.

As discussed in the introduction, the intuition is that a proof of $!A$ lies inside a box which may be duplicated or erased, but it may not necessarily be possible to *access* the contents of the box, which means that having a proof does not necessarily enable one to interact with its contents. A proof of $!\bullet A$ lies inside a box which may both duplicated, erased, and accessed. Informally speaking, the combined modalities $!\bullet A$ and $?\circ A$ in MSCLL play the role of the usual $!A$ and $?A$ modalities in MELL.

Remark 3.1. If we define linear equivalence of formulae $A \multimapboth B$ as usual in linear logic, it is immediate to show that $\vdash A \multimapboth \circ A$ holds, but in general $\vdash ?A \multimapboth ?\circ A$ does not hold. Hence linear equivalence is not a congruence with respect to the sharing modalities.

Basic Properties. Consider the mapping $(\cdot)^\bullet$ from formulae in MELL to formulae in MSCLL which replaces each occurrence of "!" with "$!\bullet$", each occurrence of "?" with "$?\circ$", and leaves the other connectives unaltered. The following theorem is easy to prove by induction on the derivation of the sequents:

Theorem 3.1 (Conservativity). *$\vdash \Gamma$ holds in* MELL *if and only if $\vdash \Gamma^\bullet$ holds in* MSCLL

By the usual techniques, one can show that MSCLL enjoys cut-elimination:

Theorem 3.2 (Cut elimination [proof in Thm. B.1]). *If $\vdash \Gamma$ is provable in* MSCLL*, then there is a derivation of $\vdash \Gamma$ without instances of the* `cut` *rule.*

4 A Sharing Linear λ-Calculus

In this section, we present a *sharing* linear λ-calculus based on MSCLL, called the $\lambda^{!\bullet}$-calculus. The relationship between the $\lambda^{!\bullet}$-calculus and MSCLL is akin to that between linear λ-calculi and MELL. In particular, typing rules for the $\lambda^{!\bullet}$-calculus are presented in natural deduction (rather than sequent calculus) style, and, furthermore, the $\lambda^{!\bullet}$-calculus is intuitionistic (rather than classical).

Syntax and Typing System We assume given denumerable sets of *linear variables* $(a, b, \ldots)$ and *unrestricted variables* $(u, v, \ldots)$. The set of *types* $(A, B, \ldots)$ and the set $\mathcal{T}_\bullet$ of $\lambda^{!\bullet}$*-terms* $(t, s, \ldots)$, or just *terms*, are given by:

$$A ::= \alpha \mid A \multimap B \mid \bullet A \mid !A \qquad t ::= a \mid u \mid \lambda a.\, t \mid t\, s \mid \bullet t \mid \mathtt{req}(t) \mid !t \mid t[u/s]$$

A term may be a *linear* or an *unrestricted variable*, an *abstraction* $\lambda a.\, t$ (binding a linear variable), an *application* $t\, s$, an access *grant* $\bullet t$, an access *request* $\mathtt{req}(t)$, a *promotion* $!t$ or a *substitution* $t[u/s]$ (binding an unrestricted variable).

Free and bound occurrences of variables are defined as expected. We write $\mathsf{fv}(t)$ for the set of free variables of t. Terms are defined up to α-renaming of bound variables. By convention, we assume that $!t[x/s]$ stands for $!(t[x/s])$. Similarly, $\bullet t[x/s]$, and $\lambda a.\, t[x/s]$ stand, respectively, for $\bullet(t[x/s])$, and $\lambda a.\, (t[x/s])$.

Unrestricted typing environments $(\Delta, \Delta', \ldots)$ are partial functions mapping unrestricted variables to types, written $u_1 : A_1, \ldots, u_n : A_n$. *Linear typing environments* $(\Gamma, \Gamma', \ldots)$ map linear variables to types, written $a_1 : A_1, \ldots, a_n : A_n$. We assume that typing environments have finite domain.

Typing judgments are of the form $\Delta; \Gamma \vdash t : A$. Derivable judgments are defined inductively by the following rules. The types of unrestricted variables in Δ may be thought of as being implicitly prefixed by "!•", as attested by the rule sub.

$$\frac{}{\Delta; a : A \vdash a : A}\ \texttt{lvar} \qquad \frac{}{\Delta, u : A; \cdot \vdash u : \bullet A}\ \texttt{uvar} \qquad \frac{\Delta; \Gamma_1 \vdash t : A \multimap B \quad \Delta; \Gamma_2 \vdash s : A}{\Delta; \Gamma_1, \Gamma_2 \vdash t\, s : B}\ \texttt{app}$$

$$\frac{\Delta; \Gamma, a : A \vdash t : B}{\Delta; \Gamma \vdash \lambda a.\, t : A \multimap B}\ \texttt{abs} \qquad \frac{\Delta; \Gamma \vdash t : A}{\Delta; \Gamma \vdash \bullet t : \bullet A}\ \texttt{grant} \qquad \frac{\Delta; \Gamma \vdash t : \bullet A}{\Delta; \Gamma \vdash \texttt{req}(t) : A}\ \texttt{request}$$

$$\frac{\Delta; \cdot \vdash t : A}{\Delta; \cdot \vdash !t : !A}\ \texttt{prom} \qquad \frac{\Delta, u : A; \Gamma_1 \vdash t : B \quad \Delta; \Gamma_2 \vdash s : !\bullet A}{\Delta; \Gamma_1, \Gamma_2 \vdash t[u/s] : B}\ \texttt{sub}$$

Example 4.1. $\lambda a.\, (!\bullet !u)[u/a]$ has type $!\bullet A \multimap !\bullet !\bullet A$, under the empty contexts.

Logical Soundness. Types of $\lambda^{!\bullet}$ encode *formulae* of MSCLL, while terms encode *proofs*. Indeed, consider the translation $(\cdot)^\star$ on types below, and the following result:

$$\alpha^\star := \alpha \qquad (A \multimap B)^\star := A^{\star\perp} \parr B^\star \qquad (\bullet A)^\star := \bullet A^\star \qquad (!A)^\star := !A^\star$$

Proposition 4.1 (Logical Soundness of $\lambda^{!\bullet}$ [PROOF IN PROP. C.1]). *If $\Delta; \Gamma \vdash t : A$ holds in $\lambda^{!\bullet}$, then $\vdash ?\circ(\Delta^{\star\perp}), \Gamma^{\star\perp}, A^\star$ holds in* MSCLL.

Note that, by design, $\lambda^{!\bullet}$ is not intended to be *complete* with respect to MSCLL. For example, the type $(!\alpha \multimap !\alpha \multimap \beta) \multimap !\alpha \multimap \beta$ is not inhabited in $\lambda^{!\bullet}$, whereas the corresponding formula $(!\alpha \otimes !\alpha \otimes \overline{\beta}) \parr ?\overline{\alpha} \parr \beta$ is provable in MSCLL.

Reduction Semantics Let us write $\mathsf{Ctxs}_\bullet$ for the the set of $\lambda^{!\bullet}$-*contexts* $(\mathsf{C}, \mathsf{C}', \ldots)$, or just *contexts*, which are $\lambda^{!\bullet}$-terms with a single occurrence of a *hole* "$\Box$", and $\mathsf{SCtxs}_\bullet$ for the the set of *substitution contexts* $(\mathsf{L}, \mathsf{L}', \ldots)$, which are lists of ESs:

$$\mathsf{C} ::= \Box \mid \lambda a.\, \mathsf{C} \mid \mathsf{C}\, t \mid t\, \mathsf{C} \mid \bullet\mathsf{C} \mid \texttt{req}(\mathsf{C}) \mid !\mathsf{C} \mid \mathsf{C}[u/t] \mid t[u/\mathsf{C}] \qquad \mathsf{L} ::= \Box \mid \mathsf{L}[u/t]$$

We write $t\{a := s\}$ for the capture-avoiding substitution of the free occurrences of a in t by s. The domain of a substitution context $(\mathsf{dom}(\mathsf{L}))$, and the plugging of a term into a context, both with capture ($\mathsf{C}\langle t\rangle$ and $t\mathsf{L}$) and avoiding capture ($\mathsf{C}\langle\!\langle t\rangle\!\rangle$), are defined similarly as for the LSC (see Section 2).

There are four **rewriting rules**, closed by compatibility under arbitrary contexts:

$$\begin{array}{rl} (\lambda a.\, t)\mathsf{L}\, s \to_{\bullet\mathsf{db}} & t\{a := s\}\mathsf{L} \\ \texttt{req}((\bullet t)\mathsf{L}) \to_{\bullet\mathsf{req}} & t\mathsf{L} \\ \mathsf{C}\langle\!\langle u\rangle\!\rangle[u/(!(\bullet t)\mathsf{L}_1)\mathsf{L}_2] \to_{\bullet\mathsf{ls}} & \mathsf{C}\langle\!\langle(\bullet t)\mathsf{L}_1\rangle\!\rangle[u/!(\bullet t)\mathsf{L}_1]\mathsf{L}_2 \\ t[u/(!s)\mathsf{L}] \to_{\bullet\mathsf{gc}} & t\mathsf{L} \quad (\text{if } u \notin \mathsf{fv}(t)) \end{array}$$

Reduction in $\lambda^{!\bullet}$ is defined as the union $\to_\bullet := \to_{\bullet\mathsf{db}} \cup \to_{\bullet\mathsf{req}} \cup \to_{\bullet\mathsf{ls}} \cup \to_{\bullet\mathsf{gc}}$.

The •db rule is a distant β-rule. The calculus does not assume that terms are typable; but, in typable terms, there is exactly one occurrence of a in the body of $\lambda a.\, t$ by linearity. The •req rule requests access to a term. The •ls rule substitutes a single occurrence of u by $(\bullet t)\mathsf{L}_1$, provided that u is bound to a term of the form $(!(\bullet t)\mathsf{L}_1)\mathsf{L}_2$. Note that the subterm $(\bullet t)\mathtt{L}_1$ is *copied* by $\bullet\mathsf{ls}$, while $\mathtt{L}_2$ is moved outside so that its bindings remain shared. The $\bullet\mathsf{gc}$ rule may erase an unused substitution if u is bound to a term of the form $(!s)\mathtt{L}$. Allowing to erase *any* unused substitution, *e.g.* with the usual gc rule of LSC, would break subject reduction, as it could lead to weakening (erasure) of a linear variable.

The $\bullet\mathsf{gc}$ rule requires that it first be evaluated to a *sharable* term, of the form $!s'$, or more in general $(!s')\mathtt{L}$. A sharable term may not, a priori, be accessed. For example, in $(\mathtt{req}(u)\,\mathtt{req}(u))[u/!v]$ the two occurrences of u cannot be substituted by v, because the $\bullet\mathsf{ls}$ rule requires the argument to be a *sharable accessible* term, of the form $!\bullet s'$, or more in general $(!(\bullet s')\mathtt{L}_1)\mathtt{L}_2$.

A first routine result is:

Proposition 4.2 (Subject Reduction). *If $\Delta;\Gamma \vdash t : A$ and $t \to_{\bullet} s$, then $\Delta;\Gamma \vdash s : A$.*

Confluence. Confluence is tricky since standard techniques are not immediately applicable. A variant of Tait–Martin-Löf's method [11, § 3.2] would require to define parallel reduction, which is not immediate due to the fact that rewrite rules operate at a distance. The interpretation method, used for confluence of the structural lambda calculus [4], a close relative of LSC, does not apply since *full composition*, *i.e.* $t[x/s] \to^* t\{x := s\}$, does not hold for $\lambda^{!\bullet}$. An axiomatic rewriting approach based on residuals and orthogonality fails for $\lambda^{!\bullet}$ too. A simple example is the following, where we use labels α and β to mark redexes:

$$t[v/(!\bullet s)\mathtt{L}] \leftarrow^{\alpha}_{\bullet\mathsf{gc}} t[u^{\alpha}/(!v^{\beta})[v/(!\bullet s)\mathtt{L}]] \to^{\beta}_{\bullet\mathsf{ls}} t[u^{\alpha}/(!\bullet s)[v/(!\bullet s)]\mathtt{L}] \to^{\alpha}_{\bullet\mathsf{gc}} t[v/!\bullet s]\mathtt{L}$$

where, however, $t[v/(!\bullet s)\mathtt{L}] \neq t[v/!\bullet s]\mathtt{L}$. Nevertheless, confluence holds for $\lambda^{!\bullet}$ modulo the congruence generated by $t[u/s[v/r]] \equiv t[u/s][v/r]$, provided that $v \notin \mathsf{fv}(t)$. To prove this, we develop a theory of residuals for $\lambda^{!\bullet}$ modulo $\equiv$. We then resort to the axiomatic rewriting framework due to Melliès [24], verifying that $\lambda^{!\bullet}$ modulo $\equiv$ can be modeled as an *orthogonal axiomatic rewriting system*, which entails confluence (see [24, Theorem 2.4]):

Proposition 4.3 (Confluence [Proof in Sec. C.2]). *$\lambda^{!\bullet}$ modulo $\equiv$ is confluent.*

Strong Normalization. Typable terms in $\lambda^{!\bullet}$ are strongly normalizing. To prove this, we reduce SN of typed $\lambda^{!\bullet}$ to SN of simply typed LSC.

Let us write $\to_{\bullet\mathsf{i}}$ for $\to_{\bullet} \setminus \to_{\bullet\mathsf{gc}}$ and $\to_{\mathsf{lsci}}$ for $\to_{\mathsf{lsc}} \setminus \to_{\bullet\mathsf{gc}}$. Garbage collection $\to_{\bullet\mathsf{gc}}$ can be *postponed*, so to show there are no infinite reduction sequences $t \to_{\bullet} t_1 \to_{\bullet} t_2 \ldots$ we may assume without loss of generality that the sequence consists of $\to_{\bullet\mathsf{i}}$ steps.

We start by defining a translation $[\![\cdot]\!]$ from $\lambda^{!\bullet}$ to LSC. Let $\mathbf{1}$ be any inhabited type in simply typed LSC, and $*$ a closed inhabitant of $\mathbf{1}$ in normal form. Types and terms are translated as follows (where z is assumed to be fresh in the "$\bullet$" case):

$$[\![\alpha]\!] := \alpha \qquad [\![A \multimap B]\!] := [\![A]\!] \to [\![B]\!] \qquad [\![\bullet A]\!] := \mathbf{1} \to [\![A]\!] \qquad [\![!A]\!] := [\![A]\!]$$

$$\begin{array}{llll} [\![a]\!] := a & [\![u]\!] := u & [\![\bullet t]\!] := \lambda z.\,[\![t]\!] & [\![\mathtt{req}(t)]\!] := [\![t]\!]\,* \\ [\![\lambda a.\,t]\!] := \lambda a.\,[\![t]\!] & [\![t\,s]\!] := [\![t]\!]\,[\![s]\!] & [\![!t]\!] := [\![t]\!] & [\![t[x/s]]\!] := [\![t]\!][x/[\![s]\!]] \end{array}$$

An unrestricted variable $u : A$ in the environment is translated as $u : \mathbf{1} \to [\![A]\!]$, while a linear variable $a : A$ is translated as $a : [\![A]\!]$. It is easy to show that the translation preserves typing, in the sense that $\Delta; \Gamma \vdash t : A$ implies $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : [\![A]\!]$.

To conclude, it would suffice to show that LSC simulates $\lambda^{!\bullet}$ reduction; more precisely that $t \to_\bullet s$ implies $[\![t]\!] \to^+_{\mathrm{lsc}} [\![s]\!]$. Unfortunately, this is not the case; for instance, in the following example, the $\to_{\bullet\mathsf{ls}}$ rule of $\lambda^{!\bullet}$ *extrudes* the $[y/z]$ outside to share the binding, while the $\to_{\mathsf{ls}}$ rule of LSC makes a *copy* of $[y/z]$:

$$x[x/(!\bullet y)[y/z]] \to_{\bullet\mathsf{ls}} (\bullet y)[x/!\bullet y][y/z] \qquad x[x/t[y/z]] \to_{\mathsf{ls}} (t[y/z])[x/t[y/z]]$$

To address this, we define a binary relation $\Rrightarrow$ on LSC terms called *fusion*, that allows to "extrude and fuse" ESs, given by the reflexive, transitive, and contextual closure of the following rules, avoiding capture:

$$t[x/s] \Rrightarrow t \;\; (\text{if } x \notin \mathsf{fv}(t)) \qquad t[x/s][y/s] \Rrightarrow t\{x := y\}[y/s] \qquad \mathsf{C}\langle t[x/s]\rangle \Rrightarrow \mathsf{C}\langle t\rangle[x/s]$$

The two key technical properties are, first, that the $\to_{\mathrm{lsci}}$ simulates $\to_{\bullet\mathsf{i}}$ up to fusion, more precisely that $t \to_{\bullet\mathsf{i}} s$ implies $[\![t]\!] \to^+_{\mathrm{lsci}}\Rrightarrow [\![s]\!]$. Second, that fusion can be postponed, *i.e.* that $\Rrightarrow\to_{\bullet\mathsf{i}} \subseteq \to^+_{\bullet\mathsf{i}}\Rrightarrow$. Thus an infinite reduction sequence $t \to_{\bullet\mathsf{i}} t_1 \to_{\bullet\mathsf{i}} t_2 \ldots$ can be mapped to $[\![t]\!] \to^+_{\mathrm{lsci}}\Rrightarrow [\![t_1]\!] \to^+_{\mathrm{lsci}}\Rrightarrow [\![t_2]\!] \ldots$ and by postponing fusion we obtain an infinite reduction in LSC. If t is typable in $\lambda^{!\bullet}$ then $[\![t]\!]$ is typable in LSC, contradicting Thm. 2.2. To sum up:

Theorem 4.1 (Termination [PROOF IN THM. D.3]). *If t is typable in $\lambda^{!\bullet}$ then t is $\to_\bullet$-SN.*

To conclude this section, we remark that $\to_\bullet$-normal forms can be characterized inductively. The full characterization can be found in the appendix (Section D.4).

5 Embedding CBN, CBV and CBS

In this section, we recall known CBN and CBV calculi, and we introduce a sharing variant of call-by-name we dub *call-by-sharing* (CBS). Second, we define translations that provide embeddings into $\lambda^{!\bullet}$, and study their properties.

Our first step is to give precise definitions of each of the calculi we will work with. These calculi operate on LSC terms, and some of them also use the notions of *strict values* ($\mathtt{v}, \mathtt{w}, \ldots$) and *lax values* ($\mathtt{v}^+, \mathtt{w}^+, \ldots$), defined as follows:

$$\text{Strict values} \quad \mathtt{v} ::= \lambda x.\, t \qquad \text{Lax values} \quad \mathtt{v}^+ ::= x \mid \lambda x.\, t$$

We define the following rewriting rules on LSC terms, closed by compatibility under arbitrary contexts:

$$\begin{array}{rlrl}
(\lambda x.\, t)\mathsf{L}\; s & \to_{\mathsf{db}} t[x/s]\mathsf{L} & \mathsf{C}\langle\!\langle x\rangle\!\rangle[x/t] & \to_{\mathsf{ls}} \mathsf{C}\langle\!\langle t\rangle\!\rangle[x/t] \\
\mathsf{C}\langle\!\langle x\rangle\!\rangle[x/\mathtt{v}\mathsf{L}] & \to_{\mathsf{lsv}} \mathsf{C}\langle\!\langle \mathtt{v}\rangle\!\rangle[x/\mathtt{v}]\mathsf{L} & \mathsf{C}\langle\!\langle x\rangle\!\rangle[x/\mathtt{v}\mathsf{L}] & \to_{\mathsf{lsw}} \mathsf{C}\langle\!\langle \mathtt{v}\mathsf{L}\rangle\!\rangle[x/\mathtt{v}\mathsf{L}] \\
t[x/s] & \to_{\mathsf{gc}} t \quad (\text{if } x \notin \mathsf{fv}(t)) & t[x/\mathtt{v}^+\mathsf{L}] & \to_{\mathsf{gcv+}} t\mathsf{L} \quad (\text{if } x \notin \mathsf{fv}(t))
\end{array}$$

Definition 5.1 (Notions of reduction). *The relations corresponding to* CBN *($\to_{\mathsf{N}}$),* CBV *($\to_{\mathsf{V}}$), and* CBS *($\to_{\mathsf{S}}$) reduction are defined by:*

$$\to_{\mathsf{N}} := \to_{\mathsf{db}} \cup \to_{\mathsf{ls}} \cup \to_{\mathsf{gc}} \qquad \to_{\mathsf{V}} := \to_{\mathsf{db}} \cup \to_{\mathsf{lsv}} \cup \to_{\mathsf{gcv+}} \qquad \to_{\mathsf{S}} := \to_{\mathsf{db}} \cup \to_{\mathsf{lsw}} \cup \to_{\mathsf{gc}}$$

The call-by-name (CBN) calculus $\to_{\mathsf{N}}$ corresponds to usual reduction in LSC [3].

The call-by-value (CBV) calculus $\to_{\mathsf{V}}$ is a variant of Accattoli and Paolini's value-substitution calculus [5] (VSC), with two differences. First, the rule $\to_{\mathsf{lsv}}$ is *linear* in that it substitutes one occurrence of a variable x at a time, while the corresponding rule of VSC substitutes all occurrences of x at once. This difference allows us to present the calculi in a uniform way. Second, $\to_{\mathsf{lsv}}$ allows substituting variables for *strict values* (abstractions), while VSC allows substituting *lax values* (both abstractions and variables). This is necessary to be able to define a *complete* embedding (see also Rem. 5.2).

The call-by-sharing (CBS) calculus $\to_{\mathsf{S}}$ is a sharing variant of CBN, in which the argument may be discarded without being evaluated (using gc). At the same time, the evaluation of the argument is *shared*, in the sense that lsw only allows copying arguments when they have been evaluated to the form vL. The CBS calculus bears a strong resemblance to the call-by-need λ-calculus of Ariola *et al.* [7], which can be obtained by changing $\to_{\mathsf{lsw}}$ to $\to_{\mathsf{lsv}}$, *i.e.* call-by-need is $\to_{\mathsf{Nd}} := \to_{\mathsf{db}} \cup \to_{\mathsf{lsv}} \cup \to_{\mathsf{gc}}$ (see for instance [9]). Note that CBS achieves *less* sharing that CBNd, because the $\to_{\mathsf{lsw}}$ rule makes two copies of L, whereas $\to_{\mathsf{lsv}}$ keeps a single shared copy of L. Unfortunately, it does not seem possible to give a sound embedding of CBNd into $\lambda^{!\bullet}$.

Remark 5.1. The reduction relations above are *calculi*, *i.e.* orientations of equational theories, not *evaluation* mechanisms. We shall turn our attention to evaluation in Section 6.

Next, we describe the translations $(\cdot)^{\mathsf{N}}$, $(\cdot)^{\mathsf{V}}$, and $(\cdot)^{\mathsf{S}}$. Each translation maps a simple type into a $\lambda^{!\bullet}$-type, an LSC term into a $\lambda^{!\bullet}$-term, and an LSC typing judgment into an $\lambda^{!\bullet}$ typing judgment.

Embedding Call-by-Name The CBN translation $(\cdot)^{\mathsf{N}}$ is defined on types and terms by:

$$\alpha^{\mathsf{N}} := \alpha \qquad (A \to B)^{\mathsf{N}} := {!\bullet}A^{\mathsf{N}} \multimap B^{\mathsf{N}}$$

$$x^{\mathsf{N}} := \mathtt{req}(x) \qquad (\lambda x.\, t)^{\mathsf{N}} := \lambda a.\, t^{\mathsf{N}}[x/a] \qquad (t\, s)^{\mathsf{N}} := t^{\mathsf{N}}\, {!\bullet}s^{\mathsf{N}} \qquad t[x/s]^{\mathsf{N}} := t^{\mathsf{N}}[x/{!\bullet}s^{\mathsf{N}}]$$

In the abstraction case, a is assumed to be fresh, *i.e.* $a \notin \mathsf{fv}(t^{\mathsf{N}})$. The translation is extended to typing environments: $(x_1 : A_1, \ldots, x_n : A_n)^{\mathsf{N}} := x_1 : A_1^{\mathsf{N}}, \ldots, x_n : A_n^{\mathsf{N}}$, and judgments: $(\Gamma \vdash t : A)^{\mathsf{N}} := \Gamma^{\mathsf{N}}; \cdot \vdash t^{\mathsf{N}} : A^{\mathsf{N}}$.

Proposition 5.1 (CBN **typing** [PROOF IN PROP. E.1]). *If $\Gamma \vdash t : A$ then $\Gamma^{\mathsf{N}}; \cdot \vdash t^{\mathsf{N}} : A^{\mathsf{N}}$.*

Lemma 5.1 (CBN **simulation**). *If $t \to_{\mathsf{N}} s$ then $t^{\mathsf{N}} \to_{\bullet}^{*} s^{\mathsf{N}}$. Furthermore, the reduction uses either at least one, and at most two $\to_{\bullet}$ steps.*

Proof. By induction on the derivation of $t \to_{\mathsf{N}} s$. The interesting cases are when there is a db, ls, or gc step at the root. If $(\lambda x.\, t)\mathsf{L}\, s \to_{\mathsf{db}} t[x/s]\mathsf{L}$, then:

$$((\lambda x.\, t)\mathsf{L}\, s)^{\mathsf{N}} = (\lambda a.\, t^{\mathsf{N}}[x/a])\mathsf{L}^{\mathsf{N}}\, {!\bullet}s^{\mathsf{N}} \to_{\bullet\mathsf{db}} t^{\mathsf{N}}[x/{!\bullet}s^{\mathsf{N}}]\mathsf{L}^{\mathsf{N}} = (t[x/s]\mathsf{L})^{\mathsf{N}}$$

If $\mathsf{C}\langle\!\langle x\rangle\!\rangle[x/t] \to_{\mathsf{ls}} \mathsf{C}\langle\!\langle t\rangle\!\rangle[x/t]$, then:

$$\begin{aligned}\mathsf{C}\langle\!\langle x\rangle\!\rangle[x/t]^{\mathsf{N}} = \mathsf{C}^{\mathsf{N}}\langle\!\langle \mathtt{req}(x)\rangle\!\rangle[x/{!\bullet}t^{\mathsf{N}}] &\to_{\bullet\mathsf{ls}} \mathsf{C}^{\mathsf{N}}\langle\!\langle \mathtt{req}(\bullet t^{\mathsf{N}})\rangle\!\rangle[x/{!\bullet}t^{\mathsf{N}}] \\ &\to_{\bullet\mathsf{req}} \mathsf{C}^{\mathsf{N}}\langle\!\langle t^{\mathsf{N}}\rangle\!\rangle[x/{!\bullet}t^{\mathsf{N}}] = \mathsf{C}\langle\!\langle t\rangle\!\rangle[x/t]^{\mathsf{N}}\end{aligned}$$

If $t[x/s] \to_{\mathsf{gc}} t$ with $x \notin \mathsf{fv}(t)$, then $t[x/s]^{\mathsf{N}} = t^{\mathsf{N}}[x/!\bullet s^{\mathsf{N}}] \to_{\bullet\mathsf{gc}} t^{\mathsf{N}}$, where that $x \notin \mathsf{fv}(t^{\mathsf{N}})$ because $\mathsf{fv}(t^{\mathsf{N}}) = \mathsf{fv}(t)$. ∎

For completeness, we define an **inverse** CBN **translation**. We define a subset $\mathcal{T}_{\bullet}^{\mathsf{N}} \subseteq \mathcal{T}_{\bullet}$, containing the closure by $\to_{\bullet}$-reduction of the image of $(\cdot)^{\mathsf{N}}$:

$$\underline{t}, \underline{s}, \ldots ::= \mathtt{req}(u) \mid \mathtt{req}(\bullet\underline{t}) \mid \lambda a.\, \underline{t}[u/a] \mid \underline{t}\;!\bullet\underline{s} \mid \underline{t}[u/!\bullet\underline{s}]$$

where, in the production $\underline{t} ::= \lambda a.\, \underline{t}[u/a]$ we assume that a is fresh, that is, $a \notin \mathsf{fv}(\underline{t})$. The *inverse* CBN *translation* is a function $(\cdot)^{-\mathsf{N}} : \mathcal{T}_{\bullet}^{\mathsf{N}} \to \mathcal{T}_{\mathsf{LSC}}$ defined as follows, by induction on the derivation of a term with the grammar above[6].

$$\begin{array}{lll} \mathtt{req}(x)^{-\mathsf{N}} := x & \mathtt{req}(\bullet\underline{t})^{-\mathsf{N}} := \underline{t}^{-\mathsf{N}} & (\lambda a.\, \underline{t}[x/a])^{-\mathsf{N}} := \lambda x.\, \underline{t}^{-\mathsf{N}} \\ (\underline{t}\;!\bullet\underline{s})^{-\mathsf{N}} := \underline{t}^{-\mathsf{N}}\, \underline{s}^{-\mathsf{N}} & \underline{t}[x/!\bullet\underline{s}]^{-\mathsf{N}} := \underline{t}^{-\mathsf{N}}[x/\underline{s}^{-\mathsf{N}}] & \end{array}$$

It is easy to check that $(\cdot)^{-\mathsf{N}}$ is the left-inverse of $(\cdot)^{\mathsf{N}}$, *i.e.* if $t \in \mathcal{T}_{\mathsf{LSC}}$ then $t^{\mathsf{N}} \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and $(t^{\mathsf{N}})^{-\mathsf{N}} = t$. Moreover:

Lemma 5.2 (Inverse CBN simulation [PROOF IN LEMMA E.2]**).** *Let $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and $s \in \mathcal{T}_{\bullet}$ such that $\underline{t} \to_{\bullet} s$. Then $s \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and $\underline{t}^{-\mathsf{N}} \to_{\mathsf{N}}^{=} s^{-\mathsf{N}}$.*

Using the abstract soundness and completeness results (Prop. 2.1, Thm. 2.1) together with the lemmas above, we obtain:

Theorem 5.1 (Sound and complete CBN embedding). *Given terms $t, s \in \mathcal{T}_{\mathsf{LSC}}$, $t \to_{\mathsf{N}}^{*} s$ if and only if $t^{\mathsf{N}} \to_{\bullet}^{*} s^{\mathsf{N}}$. Moreover, t is in $\to_{\mathsf{N}}$-normal form iff t^{N} is in $\to_{\bullet}$-normal form.*

Embedding Call-by-Value The CBV translation $(\cdot)^{\mathsf{V}}$ is defined on types and terms by:

$$\alpha^{\mathsf{V}} := \alpha \qquad (A \to B)^{\mathsf{V}} := !\bullet A^{\mathsf{V}} \multimap !\bullet B^{\mathsf{V}}$$

$$x^{\mathsf{V}} := !x \qquad (\lambda x.\, t)^{\mathsf{V}} := !\bullet\lambda a.\, t^{\mathsf{V}}[x/a] \qquad (t\, s)^{\mathsf{V}} := \mathtt{req}(u)[u/t^{\mathsf{V}}]\, s^{\mathsf{V}} \qquad t[x/s]^{\mathsf{V}} := t^{\mathsf{V}}[x/s^{\mathsf{V}}]$$

where, as for CBN, a is assumed to be fresh in the abstraction case. The translation is extended to typing environments: $(x_1 : A_1, \ldots, x_n : A_n)^{\mathsf{V}} := x_1 : A_1^{\mathsf{V}}, \ldots, x_n : A_n^{\mathsf{V}}$, and judgments: $(\Gamma \vdash t : A)^{\mathsf{V}} := \Gamma^{\mathsf{V}}; \cdot \vdash t^{\mathsf{V}} : !\bullet A^{\mathsf{V}}$.

Proposition 5.2 (CBV **typing** [PROOF IN PROP. E.2]**).** *If $\Gamma \vdash t : A$ then $\Gamma^{\mathsf{V}}; \cdot \vdash t^{\mathsf{V}} : !\bullet A^{\mathsf{V}}$.*

Lemma 5.3 (CBV **simulation** [PROOF IN LEMMA E.3]**).** *If $t \to_{\mathsf{V}} s$ then $t^{\mathsf{V}} \to_{\bullet}^{*} s^{\mathsf{V}}$. Furthermore, the reduction uses at least one, and at most four $\to_{\bullet}$ steps.*

We turn our attention to **completeness** for $(\cdot)^{\mathsf{V}}$. A first comment is that the CBV translation only turns out to be complete up to garbage collection. More precisely, soundness with respect to reduction holds, in the sense that $t \to_{\mathsf{V}}^{*} s$ implies $t^{\mathsf{V}} \to_{\bullet}^{*} s^{\mathsf{V}}$, but completeness only holds in the following weak form: $t^{\mathsf{V}} \to_{\bullet}^{*} s^{\mathsf{V}}$ implies $t \rhd_{\mathsf{V}}^{*} s$, where $\rhd_{\mathsf{V}} := (\to_{\mathsf{V}} \cup \to_{\mathsf{gcv+}}^{-1})$. Resorting to confluence, it is possible recover "plain" soundness and completeness of the translation, *i.e.* with respect to the equational theory and not to reduction; more precisely $t \leftrightarrow_{\mathsf{V}}^{*} s$ if and only if $t^{\mathsf{V}} \leftrightarrow_{\bullet}^{*} s^{\mathsf{V}}$. Besides:

[6] Observe that the derivation is unique since, as can be easily seen, the grammar is unambiguous.

Remark 5.2. The study of completeness motivates the fact that in CBV the ls rule can substitute only *strict* values (abstractions) while the $\mathsf{gcv+}$ rule can erase *lax* values (abstractions and variables). To allow substituting variables, the translation of a variable x should be a term of the form $!\bullet t$. A preliminary version of this work used a CBV translation $(\cdot)^{\mathsf{V+}}$ similar to $(\cdot)^{\mathsf{V}}$ but with $x^{\mathsf{V+}} := !\bullet\mathtt{req}(x)$. However, $(\cdot)^{\mathsf{V+}}$ is not complete. In fact, it can be checked $x[x/t]\,s \leftrightarrow^*_{\mathsf{V}} t\,s$ does not hold in general (because t may not be convertible to a value), while it can be seen that $(x[x/t]\,s)^{\mathsf{V+}} \leftrightarrow^*_{\bullet} (t\,s)^{\mathsf{V+}}$ always holds. On the other hand, if the $\mathsf{gcv+}$ rule were not allowed to erase variables, this would again lead to incompleteness, as $x[y/z] \leftrightarrow^*_{\mathsf{V}} x$ would not hold, but $(x[y/z])^{\mathsf{V}} \leftrightarrow^*_{\bullet} x^{\mathsf{V}}$ would hold, since $(x[y/z])^{\mathsf{V}} = !x[y/!z] \to_{\bullet\mathsf{gc}} !x = x^{\mathsf{V}}$.

Next, we define an **inverse** CBV **translation**. First, we define a subset $\mathcal{T}^{\mathsf{V}}_{\bullet} \subseteq \mathcal{T}_{\bullet}$, containing the closure by $\to_{\bullet}$-reduction of the image of $(\cdot)^{\mathsf{V}}$, as well as a subset $\mathsf{SCtxs}^{\mathsf{V}}_{\bullet} \subseteq \mathsf{SCtxs}_{\bullet}$:

$$\begin{aligned} \underline{t}, \underline{s}, \ldots &::= !x \mid !\bullet\lambda a.\,\underline{t}[x/a] \mid \mathtt{req}(u)[u/\underline{t}] \mid \mathtt{req}(\bullet\lambda a.\,\underline{t}[x/a]) \mid \lambda a.\,\underline{t}[x/a] \mid \underline{t}\,\underline{s} \mid \underline{t}[u/\underline{s}] \\ \underline{\mathsf{L}} &::= \Box \mid \underline{\mathsf{L}}[u/\underline{t}] \end{aligned}$$

where in the occurrences of $\lambda a.\,\underline{t}[x/a]$ we assume that a is fresh. The *inverse* CBV *translation* is a function $(\cdot)^{-\mathsf{V}} : \mathcal{T}^{\mathsf{V}}_{\bullet} \to \mathcal{T}_{\mathsf{LSC}}$ defined as follows, by induction on the derivation of a term with the (unambiguous) grammar above:

$$\begin{aligned} (!x)^{-\mathsf{V}} &:= x & (!\bullet\lambda a.\,\underline{t}[x/a])^{-\mathsf{V}} &:= \lambda x.\,\underline{t}^{-\mathsf{V}} & \underline{t}\,\underline{s}^{-\mathsf{V}} &:= \underline{t}^{-\mathsf{V}}\,\underline{s}^{-\mathsf{V}} \\ (\mathtt{req}(\bullet\lambda a.\,\underline{t}[x/a]))^{-\mathsf{V}} &:= \lambda x.\,\underline{t}^{-\mathsf{V}} & (\lambda a.\,\underline{t}[x/a])^{-\mathsf{V}} &:= \lambda x.\,\underline{t}^{-\mathsf{V}} \\ (\mathtt{req}(u)[u/\underline{t}])^{-\mathsf{V}} &:= \underline{t}^{-\mathsf{V}} & \underline{t}[u/\underline{s}]^{-\mathsf{V}} &:= \underline{t}^{-\mathsf{V}}[u/\underline{s}^{-\mathsf{V}}] \end{aligned}$$

It is easy to check that $(\cdot)^{-\mathsf{V}}$ is the left-inverse of $(\cdot)^{\mathsf{V}}$.

Lemma 5.4 (Inverse CBV **simulation, up to** $\mathsf{gcv+}$ [PROOF IN LEMMA E.5]**).** *Let $\underline{t} \in \mathcal{T}^{\mathsf{V}}_{\bullet}$ and $s \in \mathcal{T}_{\bullet}$ such that $\underline{t} \to_{\bullet} s$. Then $s \in \mathcal{T}^{\mathsf{V}}_{\bullet}$ and $\underline{t}^{-\mathsf{V}} \rhd^*_{\mathsf{V}} s^{-\mathsf{V}}$, where $\rhd_{\mathsf{V}} := (\to_{\mathsf{V}} \cup \to^{-1}_{\mathsf{gcv+}})$.*

Theorem 5.2 (Sound and complete CBV **embedding).** *Given terms $t, s \in \mathcal{T}_{\mathsf{LSC}}$:*

1. *$t \to^*_{\mathsf{V}} s$ implies $t^{\mathsf{V}} \to^*_{\bullet} s^{\mathsf{V}}$*
2. *$t^{\mathsf{V}} \to^*_{\bullet} s^{\mathsf{V}}$ implies $t \rhd^*_{\mathsf{V}} s$ where $\rhd_{\mathsf{V}} := (\to_{\mathsf{V}} \cup \to^{-1}_{\mathsf{gcv+}})$.*
3. *$t \leftrightarrow^*_{\mathsf{V}} s$ if and only if $t^{\mathsf{V}} \leftrightarrow^*_{\bullet} s^{\mathsf{V}}$*
4. *t is in $\to_{\mathsf{V}}$-normal form iff t^{V} is in $\to_{\bullet}$-normal form.*

Remark 5.3. Arrial [8] suggests an additional CBV translation mapping $A \to B$ to $!(A \multimap !B)$. In our setting, this means mapping $A \to B$ to $!\bullet(A \multimap !\bullet B)$. This translation is also sound and complete but still requires $\rhd_{\mathsf{V}}$ to obtain completeness.

Embedding Call-by-Sharing The CBS translation $(\cdot)^{\mathsf{S}}$ is defined on types and terms by:

$$\alpha^{\mathsf{S}} := \alpha \qquad (A \to B)^{\mathsf{S}} := !\bullet A^{\mathsf{S}} \multimap \bullet B^{\mathsf{S}}$$

$$x^{\mathsf{S}} := x \qquad (\lambda x.\,t)^{\mathsf{S}} := \bullet\lambda a.\,t^{\mathsf{S}}[x/a] \qquad (t\,s)^{\mathsf{S}} := \mathtt{req}(t^{\mathsf{S}})\,!s^{\mathsf{S}} \qquad t[x/s]^{\mathsf{S}} := t^{\mathsf{S}}[x/!s^{\mathsf{S}}]$$

where, as before, a is assumed to be fresh in the abstraction case. The translation is extended to typing environments: $(x_1 : A_1, \ldots, x_n : A_n)^{\mathsf{S}} := x_1 : A^{\mathsf{S}}_1, \ldots, x_n : A^{\mathsf{S}}_n$, and judgments: $(\Gamma \vdash t : A)^{\mathsf{S}} := \Gamma^{\mathsf{S}}; \cdot \vdash t^{\mathsf{S}} : \bullet A^{\mathsf{S}}$.

Proposition 5.3 (CBS **typing** [PROOF IN PROP. E.3]). *If $\Gamma \vdash t : A$ then $\Gamma^{\mathsf{S}}; \cdot \vdash t^{\mathsf{S}} : \bullet A^{\mathsf{S}}$.*

Lemma 5.5 (CBS **simulation**). *If $t \to_{\mathsf{S}} s$ then $t^{\mathsf{S}} \to_{\bullet}^{*} s^{\mathsf{S}}$.*

Proof. By induction on the derivation of $t \to_{\mathsf{S}} s$. The interesting cases are when there is a db, lsv, or gc step at the root.
If $(\lambda x.\,t)\mathtt{L}\ s \to_{\mathsf{db}} t[x/s]\mathtt{L}$, then:

$$\begin{aligned}((\lambda x.\,t)\mathtt{L}\ s)^{\mathsf{S}} = \mathtt{req}((\bullet\lambda a.\,t^{\mathsf{S}}[x/a])\mathtt{L}^{\mathsf{S}})\ !s^{\mathsf{S}} &\to_{\bullet\mathsf{req}} (\lambda a.\,t^{\mathsf{S}}[x/a])\mathtt{L}^{\mathsf{S}}\ !s^{\mathsf{S}} \\ &\to_{\bullet\mathsf{db}} t^{\mathsf{S}}[x/!s^{\mathsf{S}}]\mathtt{L}^{\mathsf{S}} = (t[x/s]\mathtt{L})^{\mathsf{S}}\end{aligned}$$

If $\mathtt{C}\langle\!\langle x\rangle\!\rangle[x/\mathtt{vL}] \to_{\mathsf{lsw}} \mathtt{C}\langle\!\langle \mathtt{vL}\rangle\!\rangle[x/\mathtt{vL}]$, note that $\mathtt{v} = \lambda y.\,t$ so $\mathtt{v}^{\mathsf{S}} = \bullet\lambda a.\,t^{\mathsf{S}}[y/a]$. Then:

$$\begin{aligned}&(\mathtt{C}\langle\!\langle x\rangle\!\rangle[x/\mathtt{vL}])^{\mathsf{S}} = \mathtt{C}^{\mathsf{S}}\langle\!\langle x\rangle\!\rangle[x/!(\bullet(\lambda a.\,t^{\mathsf{S}}[y/a])\mathtt{L}^{\mathsf{S}})] \\ &\to_{\bullet\mathsf{ls}} \mathtt{C}^{\mathsf{S}}\langle\!\langle(\bullet(\lambda a.\,t^{\mathsf{S}}[y/a]))\mathtt{L}^{\mathsf{S}}\rangle\!\rangle[x/!(\bullet(\lambda a.\,t^{\mathsf{S}}[y/a]))\mathtt{L}^{\mathsf{S}}] = (\mathtt{C}\langle\!\langle \mathtt{vL}\rangle\!\rangle[x/\mathtt{vL}])^{\mathsf{S}}\end{aligned}$$

If $t[x/s] \to_{\mathsf{gc}} t$, where $x \notin \mathsf{fv}(t)$, then $(t[x/s])^{\mathsf{S}} = t^{\mathsf{S}}[x/!s^{\mathsf{S}}] \to_{\bullet\mathsf{gc}} t^{\mathsf{S}}$, where $x \notin \mathsf{fv}(t^{\mathsf{S}})$ because $\mathsf{fv}(t^{\mathsf{S}}) = \mathsf{fv}(t)$. ∎

For completeness, we define an **inverse** CBS **translation**. First, we define a subset $\mathcal{T}_{\bullet}^{\mathsf{S}} \subseteq \mathcal{T}_{\bullet}$, containing the closure by $\to_{\bullet}$-reduction of the image of $(\cdot)^{\mathsf{S}}$, as well as a subset $\mathsf{SCtxs}_{\bullet}^{\mathsf{S}} \subseteq \mathsf{SCtxs}_{\bullet}$, as follows:

$$\begin{aligned}\underline{t}, \underline{s}, \ldots &::= x \mid \bullet\lambda a.\,\underline{t}[x/a] \mid \mathtt{req}(\underline{t}) \mid \underline{t}[u/!\underline{s}] \mid \lambda a.\,\underline{t}[x/a] \mid \underline{t}\ !\underline{s} \\ \underline{\mathtt{L}} &::= \Box \mid \underline{\mathtt{L}}[u/!\underline{t}]\end{aligned}$$

where, in the productions involving a subterm of the form $\lambda a.\,\underline{t}[x/a]$, we assume that a is fresh, that is, $a \notin \mathsf{fv}(\underline{t})$.

The *inverse* CBS *translation* is a function $(\cdot)^{-\mathsf{S}} : \mathcal{T}_{\bullet}^{\mathsf{S}} \to \mathcal{T}_{\mathsf{LSC}}$ defined as follows, by induction on the derivation of a term with the (unambiguous) grammar above:

$$\begin{array}{rclrclrcl} x^{-\mathsf{S}} &:=& x & (\bullet\lambda a.\,\underline{t}[x/a])^{-\mathsf{S}} &:=& \lambda x.\,\underline{t}^{-\mathsf{S}} & \mathtt{req}(\underline{t})^{-\mathsf{S}} &:=& \underline{t}^{-\mathsf{S}} \\ (\lambda a.\,\underline{t}[x/a])^{-\mathsf{S}} &:=& \lambda x.\,\underline{t}^{-\mathsf{S}} & \underline{t}[u/!\underline{s}]^{-\mathsf{S}} &:=& \underline{t}^{-\mathsf{S}}[u/\underline{s}^{-\mathsf{S}}] & \underline{t}\ !\underline{s}^{-\mathsf{S}} &:=& \underline{t}^{-\mathsf{S}}\ \underline{s}^{-\mathsf{S}}\end{array}$$

It is easy to check that $(\cdot)^{-\mathsf{S}}$ is the left-inverse of $(\cdot)^{\mathsf{S}}$.

Lemma 5.6 (**Inverse** CBS **simulation** [PROOF IN LEMMA E.7]). *Let $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $s \in \mathcal{T}_{\bullet}$ such that $\underline{t} \to_{\bullet} s$. Then $s \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\underline{t}^{-\mathsf{S}} \to_{\mathsf{Nd}}^{=} s^{-\mathsf{S}}$.*

Theorem 5.3 (**Sound and complete** CBS **embedding**). *Let $t, s \in \mathcal{T}_{\mathsf{LSC}}$. Then $t \to_{\mathsf{S}}^{*} s$ if and only if $t^{\mathsf{S}} \to_{\bullet}^{*} s^{\mathsf{S}}$. Moreover, t is in $\to_{\mathsf{S}}$-normal form iff t^{S} is in $\to_{\bullet}$-normal form.*

As previously mentioned, it does not seem possible to embed Wadsworth's call-by-need (*i.e.* CBNd) in $\lambda^{!\bullet}$. One could imagine a variant of $\lambda^{!\bullet}$ that includes the following $\bullet\mathsf{ls}'$ rule rather than $\bullet\mathsf{ls}$:

$$\mathtt{C}\langle\!\langle u\rangle\!\rangle[u/(!(\bullet t)\mathtt{L}_1)\mathtt{L}_2] \mapsto_{\bullet\mathsf{ls}'} \mathtt{C}\langle\!\langle \bullet t\rangle\!\rangle[u/!(\bullet t)]\mathtt{L}_1\mathtt{L}_2$$

The resulting calculus allows embeddings from CBN, CBV, and CBNd. However, it is not well-behaved, as confluence fails. Let $\Omega := (\lambda a.\,a\,a)\,(\lambda a.\,a\,a)$. For example, $x[y/!u][u/!(\bullet v)[v/\Omega]] \to_{\bullet\mathsf{gc}} x[u/!(\bullet v)[v/\Omega]] \to_{\bullet\mathsf{gc}} x$. But also $x[y/!u][u/!(\bullet v)[v/\Omega]] \to_{\bullet\mathsf{ls}} x[y/!\bullet v][u/!(\bullet v)][v/\Omega] \to_{\bullet\mathsf{gc}} x[u/!(\bullet v)][v/\Omega] \to_{\bullet\mathsf{gc}} x[v/\Omega]$.

6 Simulating Weak Evaluation Strategies

In Section 5, we have shown that CBN, CBV, and CBS calculi can be embedded in the $\lambda^{!\bullet}$-calculus. Reduction in these calculi is intended to capture *equivalence*, rather than *evaluation*, of programs. That is, these calculi are orientations of CBN, CBV, and CBS equational theories rather than evaluation mechanisms.

Reduction in the calculi of Section 5 is closed by arbitrary contexts. *E.g.* in the CBV *calculus*, a step $(\lambda x.\, y)\, \lambda z.\, t \to_{\mathsf{V}} (\lambda x.\, y)\, \lambda z.\, t'$ is allowed if $t \to_{\mathsf{V}} t'$, while typically call-by-value *evaluation* would proceed to contract the outermost redex.

In this section, we first define *weak evaluation* relations $\rightsquigarrow^{\mathsf{N}}$, $\rightsquigarrow^{\mathsf{V}}$, and $\rightsquigarrow^{\mathsf{S}}$ for CBN, CBV, and CBS respectively. Recall that evaluation is called *weak* if it does not proceed inside the bodies of λ-abstractions. Second, we define an evaluation relation $\rightsquigarrow$ for $\lambda^{!\bullet}$, which is also "weak" in that it does not reduce inside λ-abstractions, boxes ($\bullet$), nor promotions (!). Finally, we show that evaluation according to $\rightsquigarrow^{\mathsf{N}}$, $\rightsquigarrow^{\mathsf{V}}$, and $\rightsquigarrow^{\mathsf{S}}$ can be simulated by $\rightsquigarrow$ via the translations already introduced in Section 5.

Weak CBN Evaluation The one-step weak CBN evaluation judgment is of the form $t \rightsquigarrow^{\mathsf{N}}_{\rho} t'$, where $t, t' \in \mathcal{T}_{\mathsf{LSC}}$ and the set of CBN-*rulenames* $(\rho, \rho', \ldots)$ is given by $\rho ::= \mathsf{db} \mid \varsigma(x,t) \mid \mathsf{ls} \mid \mathsf{gc}$. Weak CBN evaluation is the union $\rightsquigarrow^{\mathsf{N}} := \rightsquigarrow^{\mathsf{N}}_{\mathsf{db}} \cup \rightsquigarrow^{\mathsf{N}}_{\mathsf{ls}} \cup \rightsquigarrow^{\mathsf{N}}_{\mathsf{gc}}$, excluding auxiliary $\varsigma(x,t)$ steps. It is defined by the following inductive rules:

$$\frac{}{(\lambda x.\, t)\mathtt{L}\, s \rightsquigarrow^{\mathsf{N}}_{\mathsf{db}} t[x/s]\mathtt{L}}\ \mathtt{E}^{\mathsf{N}}\texttt{-db} \qquad \frac{}{x \rightsquigarrow^{\mathsf{N}}_{\varsigma(x,t)} t}\ \mathtt{E}^{\mathsf{N}}\texttt{-}\varsigma \qquad \frac{t \rightsquigarrow^{\mathsf{N}}_{\varsigma(x,s)} t'}{t[x/s] \rightsquigarrow^{\mathsf{N}}_{\mathsf{ls}} t'[x/s]}\ \mathtt{E}^{\mathsf{N}}\texttt{-ls}$$

$$\frac{x \notin \mathsf{fv}(t)}{t[x/s] \rightsquigarrow^{\mathsf{N}}_{\mathsf{gc}} t}\ \mathtt{E}^{\mathsf{N}}\texttt{-gc} \qquad \frac{t \rightsquigarrow^{\mathsf{N}}_{\rho} t'}{t\, s \rightsquigarrow^{\mathsf{N}}_{\rho} t'\, s}\ \mathtt{E}^{\mathsf{N}}\texttt{-app} \qquad \frac{t \rightsquigarrow^{\mathsf{N}}_{\rho} t' \quad x \notin \mathsf{fv}(\rho)}{t[x/s] \rightsquigarrow^{\mathsf{N}}_{\rho} t'[x/s]}\ \mathtt{E}^{\mathsf{N}}\texttt{-subL}$$

The $\mathtt{E}^{\mathsf{N}}$-`db`, $\mathtt{E}^{\mathsf{N}}$-`ls`, and $\mathtt{E}^{\mathsf{N}}$-`gc` rules derive *root reduction* steps. The $\mathtt{E}^{\mathsf{N}}$-`app` and $\mathtt{E}^{\mathsf{N}}$-`subL` rules correspond to congruence closure below weak head evaluation contexts. The side condition in the $\mathtt{E}^{\mathsf{N}}$-`subL` rule is to avoid unwanted variable capture. The somewhat atypical $\mathtt{E}^{\mathsf{N}}$-ς rule derives steps of the form $t \rightsquigarrow^{\mathsf{N}}_{\varsigma(x,s)} t'$, which substitute a single free occurrence of x (in evaluation position) by s. This rule works in synchrony with $\mathtt{E}^{\mathsf{N}}$-`ls` to allow ls steps: for example, $x\, x\, y \rightsquigarrow^{\mathsf{N}}_{\varsigma(x,t)} t\, x\, y$ and $(x\, x\, y)[x/t] \rightsquigarrow^{\mathsf{N}}_{\mathsf{ls}} (t\, x\, y)[x/t]$. This is inspired the formulation of strong call-by-need of Balabonski *et al.* [10].

It is straightforward to show that $\rightsquigarrow^{\mathsf{N}} \subseteq \to_{\mathsf{N}}$. Note also that $\rightsquigarrow^{\mathsf{N}}$ is non-deterministic, although confluent. The source of non-determinism is that gc steps can be performed in any order. For example $((\lambda x.\, x)\, y)[z/s]$ reduces both with a db and with a gc step.

Weak CBV Evaluation The one-step weak CBV evaluation judgment is of the form $t \rightsquigarrow^{\mathsf{V}}_{\rho} t'$, where $t, t' \in \mathcal{T}_{\mathsf{LSC}}$ and the set of CBV-*rulenames* $(\rho, \rho', \ldots)$ is given by $\rho ::= \mathsf{db} \mid \varsigma(x, \mathsf{v}) \mid \mathsf{lsv} \mid \mathsf{gcv+}$. Weak CBV evaluation is $\rightsquigarrow^{\mathsf{V}} := \rightsquigarrow^{\mathsf{V}}_{\mathsf{db}} \cup \rightsquigarrow^{\mathsf{V}}_{\mathsf{lsv}} \cup \rightsquigarrow^{\mathsf{V}}_{\mathsf{gcv+}}$, excluding auxiliary $\varsigma(x, \mathsf{v})$ steps. It is defined by the following inductive rules:

$$\frac{}{(\lambda x.\, t)\mathtt{L}\, s \rightsquigarrow^{\mathsf{V}}_{\mathsf{db}} t[x/s]\mathtt{L}}\ \mathtt{E}^{\mathsf{V}}\texttt{-db} \qquad \frac{}{x \rightsquigarrow^{\mathsf{V}}_{\varsigma(x,\mathsf{v})} \mathsf{v}}\ \mathtt{E}^{\mathsf{V}}\texttt{-}\varsigma$$

$$\frac{x \notin \mathsf{fv}(t)}{t[x/\mathtt{v}^+\mathtt{L}] \rightsquigarrow^{\mathsf{V}}_{\mathsf{gcv+}} t\mathtt{L}}\ \mathtt{E}^{\mathsf{V}}\text{-}\mathtt{gcv+} \quad \frac{t \rightsquigarrow^{\mathsf{V}}_{\varsigma(x,\mathtt{v})} t'}{t[x/\mathtt{vL}] \rightsquigarrow^{\mathsf{V}}_{\mathsf{lsv}} t'[x/\mathtt{v}]\mathtt{L}}\ \mathtt{E}^{\mathsf{V}}\text{-}\mathtt{lsv} \quad \frac{t \rightsquigarrow^{\mathsf{V}}_{\rho} t'}{t\,s \rightsquigarrow^{\mathsf{V}}_{\rho} t'\,s}\ \mathtt{E}^{\mathsf{V}}\text{-}\mathtt{app}$$

$$\frac{t \rightsquigarrow^{\mathsf{V}}_{\rho} t' \quad x \notin \mathsf{fv}(\rho)}{t[x/s] \rightsquigarrow^{\mathsf{V}}_{\rho} t'[x/s]}\ \mathtt{E}^{\mathsf{V}}\text{-}\mathtt{subL} \quad \frac{s \rightsquigarrow^{\mathsf{V}}_{\rho} s'}{t[x/s] \rightsquigarrow^{\mathsf{V}}_{\rho} t[x/s']}\ \mathtt{E}^{\mathsf{V}}\text{-}\mathtt{subR}$$

Similar remarks as for CBN apply, in particular $\rightsquigarrow^{\mathsf{V}} \subseteq \rightarrow_{\mathsf{V}}$. Rules $\mathtt{E}^{\mathsf{V}}$-db, $\mathtt{E}^{\mathsf{V}}$-lsv, and $\mathtt{E}^{\mathsf{V}}$-gcv+ are root reduction rules, while $\mathtt{E}^{\mathsf{V}}$-app, $\mathtt{E}^{\mathsf{V}}$-subL, and $\mathtt{E}^{\mathsf{V}}$-subR correspond to congruence closure rules. The $\mathtt{E}^{\mathsf{V}}$-ς rule plays a similar role as the analogue rule in CBN, but only allows substituting variables for *strict values*. In this notion of CBV evaluation, arguments of applications are not evaluated. The restriction that the argument is a value is not imposed to contract a β-like redex, but rather to perform the substitution. These ideas can already be found in the λ_{CBV}-calculus of [18]. Note that in CBV evaluation ($\rightsquigarrow^{\mathsf{V}}$) there is a second source of non-determinism, namely that $\mathtt{E}^{\mathsf{V}}$-subL and $\mathtt{E}^{\mathsf{V}}$-subR overlap, so the body and the argument of an ES can be evaluated concurrently.

Weak CBS Evaluation The one-step weak CBS evaluation judgment is of the form $t \rightsquigarrow^{\mathsf{S}}_{\rho} t'$, where $t, t' \in \mathcal{T}_{\mathsf{LSC}}$ and the set of CBS-*rulenames* $(\rho, \rho', \ldots)$ is given by $\rho ::= \mathsf{db} \mid \varsigma(x, \mathtt{vL}) \mid \iota(x) \mid \mathsf{lsw} \mid \mathsf{gc}$. Weak CBS evaluation is the union $\rightsquigarrow^{\mathsf{S}} := \rightsquigarrow^{\mathsf{S}}_{\mathsf{db}} \cup \rightsquigarrow^{\mathsf{S}}_{\mathsf{lsw}} \cup \rightsquigarrow^{\mathsf{S}}_{\mathsf{gc}}$, excluding auxiliary $\varsigma(x, \mathtt{vL})$ and $\iota(x)$ steps. It is defined by the following inductive rules:

$$\frac{}{(\lambda x.\, t)\mathtt{L}\, s \rightsquigarrow^{\mathsf{S}}_{\mathsf{db}} t[x/s]\mathtt{L}}\ \mathtt{E}^{\mathsf{S}}\text{-}\mathtt{db} \quad \frac{}{x \rightsquigarrow^{\mathsf{S}}_{\varsigma(x,\mathtt{vL})} \mathtt{vL}}\ \mathtt{E}^{\mathsf{S}}\text{-}\varsigma \quad \frac{}{t[u/x] \rightsquigarrow^{\mathsf{S}}_{\varsigma(x,\mathtt{vL})} t[u/\mathtt{vL}]}\ \mathtt{E}^{\mathsf{S}}\text{-}\varsigma_2$$

$$\frac{}{x \rightsquigarrow^{\mathsf{S}}_{\iota(x)} x}\ \mathtt{E}^{\mathsf{S}}\text{-}\iota \quad \frac{t \rightsquigarrow^{\mathsf{S}}_{\varsigma(x,\mathtt{vL})} t'}{t[x/\mathtt{vL}] \rightsquigarrow^{\mathsf{S}}_{\mathsf{lsw}} t'[x/\mathtt{vL}]}\ \mathtt{E}^{\mathsf{S}}\text{-}\mathtt{lsw} \quad \frac{x \notin \mathsf{fv}(t)}{t[x/s] \rightsquigarrow^{\mathsf{S}}_{\mathsf{gc}} t}\ \mathtt{E}^{\mathsf{S}}\text{-}\mathtt{gc}$$

$$\frac{t \rightsquigarrow^{\mathsf{S}}_{\rho} t' \quad x \notin \mathsf{fv}(\rho)}{t[x/s] \rightsquigarrow^{\mathsf{S}}_{\rho} t'[x/s]}\ \mathtt{E}^{\mathsf{S}}\text{-}\mathtt{subL} \quad \frac{t \rightsquigarrow^{\mathsf{S}}_{\rho} t'}{t\,s \rightsquigarrow^{\mathsf{S}}_{\rho} t'\,s}\ \mathtt{E}^{\mathsf{S}}\text{-}\mathtt{app} \quad \frac{t \rightsquigarrow^{\mathsf{S}}_{\iota(x)} t \quad s \rightsquigarrow^{\mathsf{S}}_{\rho} s'}{t[x/s] \rightsquigarrow^{\mathsf{S}}_{\rho} t[x/s']}\ \mathtt{E}^{\mathsf{S}}\text{-}\mathtt{subR}$$

Similar remarks as for CBN apply, in particular $\rightsquigarrow^{\mathsf{S}} \subseteq \rightarrow_{\mathsf{S}}$. Rules $\mathtt{E}^{\mathsf{S}}$-db, $\mathtt{E}^{\mathsf{S}}$-lsw, and $\mathtt{E}^{\mathsf{S}}$-gc are root reduction rules, while $\mathtt{E}^{\mathsf{S}}$-app, $\mathtt{E}^{\mathsf{S}}$-subL, and $\mathtt{E}^{\mathsf{S}}$-subR are congruence closure rules. The rule $\mathtt{E}^{\mathsf{S}}$-ς plays a similar role as the analogue rules in CBN and CBV, but only allows substituting variables for terms of the form vL (known as *answers* in the literature). The rule $\mathtt{E}^{\mathsf{S}}$-ς_2 is a variant of $\mathtt{E}^{\mathsf{S}}$-ς that acts on the argument of an ES; this rule is not strictly necessary for evaluation, but it is crucial for the embedding into $\lambda^{!\bullet}$ to be complete. The $\mathtt{E}^{\mathsf{S}}$-ι rule is used in synchrony with the congruence rules to derive steps that are always of the form $t \rightsquigarrow^{\mathsf{S}}_{\iota(x)} t$ indicating that x occurs in t in an evaluation position. This is used to check whether x it a *needed* variable. For example, $x\,y \rightsquigarrow^{\mathsf{S}}_{\iota(x)} x\,y$ and $z\,z \rightsquigarrow^{\mathsf{S}}_{\varsigma(z,\mathtt{v})} \mathtt{v}\,z$, so the fact that x is needed on the left triggers the evaluation of the argument: $(x\,y)[x/z\,z] \rightsquigarrow^{\mathsf{S}}_{\varsigma(z,\mathtt{v})} (x\,y)[x/\mathtt{v}\,z]$. From this, one obtains the substitution step $(x\,y)[x/z\,z][z/\mathtt{v}] \rightsquigarrow^{\mathsf{S}}_{\mathsf{lsw}} (x\,y)[x/\mathtt{v}\,z][z/\mathtt{v}]$.

Weak $\lambda^{!\bullet}$-Calculus Evaluation The one-step weak $\lambda^{!\bullet}$ evaluation judgment is of the form $t \rightsquigarrow_\rho t'$, where $t, t' \in \mathcal{T}_\bullet$, and the set of *rulenames* $(\rho, \rho', \ldots)$ is given by $\rho ::= \bullet\mathsf{db} \mid \varsigma(u, (\bullet t)\mathrm{L}) \mid \iota(u) \mid \bullet\mathsf{ls} \mid \bullet\mathsf{gc} \mid \bullet\mathsf{req}$. Weak $\lambda^{!\bullet}$ evaluation is the union $\rightsquigarrow \; := \; \rightsquigarrow_{\bullet\mathsf{db}} \cup \rightsquigarrow_{\bullet\mathsf{ls}} \cup \rightsquigarrow_{\bullet\mathsf{gc}}$, excluding auxiliary $\varsigma(x, \mathrm{vL})$ and $\iota(x)$ steps. It is defined by the following inductive rules:

$$\frac{}{(\lambda a.\, t)\mathrm{L}\; s \rightsquigarrow_{\bullet\mathsf{db}} t\{a := s\}\mathrm{L}}\;\texttt{E}^\bullet\texttt{-db} \qquad \frac{}{u \rightsquigarrow_{\varsigma(u,(\bullet t)\mathrm{L})} (\bullet t)\mathrm{L}}\;\texttt{E}^\bullet\texttt{-}\varsigma \qquad \frac{}{!u \rightsquigarrow_{\varsigma(u,(\bullet t)\mathrm{L})} !(\bullet t)\mathrm{L}}\;\texttt{E}^\bullet\texttt{-}!\varsigma$$

$$\frac{}{u \rightsquigarrow_{\iota(u)} u}\;\texttt{E}^\bullet\texttt{-}\iota \qquad \frac{t \rightsquigarrow_{\varsigma(u,(\bullet s)\mathrm{L}_1)} t'}{t[u/(!(\bullet s)\mathrm{L}_1)\mathrm{L}_2] \rightsquigarrow_{\bullet\mathsf{ls}} t'[u/!(\bullet s)\mathrm{L}_1]\mathrm{L}_2}\;\texttt{E}^\bullet\texttt{-ls} \qquad \frac{t \rightsquigarrow_\rho t'}{t\, s \rightsquigarrow_\rho t'\, s}\;\texttt{E}^\bullet\texttt{-app}$$

$$\frac{u \notin \mathsf{fv}(t)}{t[u/(!s)\mathrm{L}] \rightsquigarrow_{\bullet\mathsf{gc}} t\mathrm{L}}\;\texttt{E}^\bullet\texttt{-gc} \qquad \frac{}{\texttt{req}((\bullet t)\mathrm{L}) \rightsquigarrow_{\bullet\mathsf{req}} t\mathrm{L}}\;\texttt{E}^\bullet\texttt{-req}\bullet \qquad \frac{t \rightsquigarrow_\rho t'}{\texttt{req}(t) \rightsquigarrow_\rho \texttt{req}(t')}\;\texttt{E}^\bullet\texttt{-req}$$

$$\frac{t \rightsquigarrow_\rho t' \quad u \notin \mathsf{fv}(\rho)}{t[u/s] \rightsquigarrow_\rho t'[u/s]}\;\texttt{E}^\bullet\texttt{-esL} \qquad \frac{s \rightsquigarrow_\rho s'}{t[u/s] \rightsquigarrow_\rho t[u/s']}\;\texttt{E}^\bullet\texttt{-esR} \qquad \frac{t \rightsquigarrow_{\iota(u)} t \quad s \rightsquigarrow_\rho s'}{t[u/!s] \rightsquigarrow_\rho t[u/!s']}\;\texttt{E}^\bullet\texttt{-es!}$$

Weak $\lambda^{!\bullet}$ evaluation is a sub-ARS of $\lambda^{!\bullet}$, in the sense that $\rightsquigarrow \subseteq \to_\bullet$. Rules $\texttt{E}^\bullet\texttt{-db}$, $\texttt{E}^\bullet\texttt{-ls}$, $\texttt{E}^\bullet\texttt{-gc}$, and $\texttt{E}^\bullet\texttt{-req}\bullet$ are root reduction rules, while $\texttt{E}^\bullet\texttt{-app}$, $\texttt{E}^\bullet\texttt{-req}$, $\texttt{E}^\bullet\texttt{-esL}$, and $\texttt{E}^\bullet\texttt{-esR}$ are congruence rules. Rule $\texttt{E}^\bullet\texttt{-}\varsigma$ plays a similar role as the analogue rules in CBN, CBV, and CBNd, but only allows substituting a variable by a term of the form $(\bullet t)\mathrm{L}$. Rule $\texttt{E}^\bullet\texttt{-}\iota$ plays a similar role as the analogue rule in CBNd, used to check whether a variable is in evaluation position. Note that there are no congruence rules below λ-abstraction, box $(\bullet)$, nor promotion $(!)$. Evaluation can proceed below promotion in two particular cases. First, the $\texttt{E}^\bullet\texttt{-}!\varsigma$ rule allows to perform substitution immediately below a promotion; for instance $(!u)[u/!\bullet v] \rightsquigarrow_{\bullet\mathsf{ls}} (!\bullet v)[u/!\bullet v]$. Second, the $\texttt{E}^\bullet\texttt{-es!}$ rule allows evaluation below a promotion when a term is the argument of a "needed" substitution; for instance $(uv)[u/!((\lambda a.\, a)(\bullet w))] \rightsquigarrow_{\bullet\mathsf{db}} (uv)[u/!\bullet w] \rightsquigarrow_{\bullet\mathsf{ls}} ((\bullet w)v)[u/!\bullet w]$.

The $\lambda^{!\bullet}$-calculus is designed with the goal in mind of providing a *unifying framework* for call-by-name, call-by-value, and call-by-sharing. As a matter of fact, weak $\lambda^{!\bullet}$ evaluation simulates weak CBN, CBV, and CBS evaluation via the $(\cdot)^{\mathsf{N}}$, $(\cdot)^{\mathsf{V}}$, and $(\cdot)^{\mathsf{S}}$ translations introduced in Section 5:

Theorem 6.1 (Simulation and inverse simulation of evaluation [Proofs in Sec.F]).

	Soundness	Completeness
CBN	*If* $t \rightsquigarrow^{\mathsf{N}} s$ *then* $t^{\mathsf{N}} \rightsquigarrow^* s^{\mathsf{N}}$.	*If* $t^{\mathsf{N}} \rightsquigarrow s$ *then* $s \in \mathcal{T}_\bullet^{\mathsf{N}}$ *and* $t\, (\rightsquigarrow^{\mathsf{N}})^= s^{-\mathsf{N}}$.
CBV	*If* $t \rightsquigarrow^{\mathsf{V}} s$ *then* $t^{\mathsf{V}} \rightsquigarrow^* s^{\mathsf{V}}$.	*If* $t^{\mathsf{V}} \rightsquigarrow s$ *then* $s \in \mathcal{T}_\bullet^{\mathsf{V}}$ *and* $t\, (\blacktriangleright^{\mathsf{V}})^= s^{-\mathsf{V}}$.
CBS	*If* $t \rightsquigarrow^{\mathsf{S}} s$ *then* $t^{\mathsf{S}} \rightsquigarrow^* s^{\mathsf{S}}$.	*If* $t^{\mathsf{S}} \rightsquigarrow s$ *then* $s \in \mathcal{T}_\bullet^{\mathsf{S}}$ *and* $t\, (\rightsquigarrow^{\mathsf{S}})^= s^{-\mathsf{S}}$.

In the CBV *case, completeness only holds for an extended relation* $\blacktriangleright^{\mathsf{V}}$*, defined as* $\rightsquigarrow^{\mathsf{V}}$ *but adding an inverse garbage collection rule that derives* $t \blacktriangleright^{\mathsf{V}} t[x/\mathsf{v}^+]$ *if* $x \notin \mathsf{fv}(t)$.

7 Embedding the Bang-Calculus

This section presents an embedding of the Bang-calculus into $\lambda^{!\bullet}$. Our presentation of the Bang-calculus uses explicit substitutions and reduction at a distance. It is based on the formulation of Bucciarelli et al [12] but where one-shot execution of explicit substitutions (the $\mathtt{s!}$ rule in Sec. 2 of [12]) is replaced by explicit replacement (rules $\to_{\mathsf{ls!}}$ and $\to_{\mathsf{gc!}}$ below), in unison with the presentation of our other calculi in this work. A further simplification will be introduced by dropping the rule for dereliction, which does not change its expressive power, as explained below.

Syntax and Reduction Semantics of the Bang-calculus The set of *Bang terms* is denoted by $\mathcal{T}_{\mathsf{B}^{\mathsf{der}}}$ and given by:

$$t, s, \ldots ::= x \mid \lambda x.\, t \mid t\, s \mid !t \mid \mathtt{der}(t) \mid t[x/s]$$

We denote with $\mathcal{T}_{\mathsf{B}}$ the set of *simplified* Bang terms; where terms of the form $\mathtt{der}(t)$ are disallowed. *Contexts* $(\mathtt{C}, \mathtt{D}, \ldots)$ and *substitution contexts* $(\mathtt{L}, \mathtt{K}, \ldots)$ are defined as expected, as well as the notions of free variables of a term $(\mathsf{fv}(t))$, domain of a substitution context $(\mathsf{dom}(\mathtt{L}))$, plugging a term into a context allowing variable-capture $(\mathtt{C}\langle t\rangle$ and $t\mathtt{L})$ and avoiding variable-capture $(\mathtt{C}\langle\!\langle t\rangle\!\rangle)$.

There are four rewriting rules, closed by arbitrary contexts:

$$\begin{array}{rlrl} (\lambda x.\, t)\mathtt{L}\; s & \to_{\mathsf{db}}\; t[x/s]\mathtt{L} & \mathtt{C}\langle\!\langle x\rangle\!\rangle[x/(!s)\mathtt{L}] & \to_{\mathsf{ls!}}\; \mathtt{C}\langle\!\langle s\rangle\!\rangle[x/!s]\mathtt{L} \\ t[x/(!s)\mathtt{L}] & \to_{\mathsf{gc!}}\; t\mathtt{L}\ (\text{if } x \notin \mathsf{fv}(t)) & \mathtt{der}((!t)\mathtt{L}) & \to_{\mathsf{d!}}\; t\mathtt{L} \end{array}$$

Full Bang-reduction is given by the union $\to_{\mathsf{B}^{\mathsf{der}}} := \to_{\mathsf{db}} \cup \to_{\mathsf{ls!}} \cup \to_{\mathsf{gc!}} \cup \to_{\mathsf{d!}}$. Simplified Bang-reduction over simplified Bang terms $\to_{\mathsf{B}} \subseteq \mathcal{T}_{\mathsf{B}} \times \mathcal{T}_{\mathsf{B}}$ is given by $\to_{\mathsf{B}} := \to_{\mathsf{db}} \cup \to_{\mathsf{ls!}} \cup \to_{\mathsf{gc!}}$. Working in the simplified fragment (without the $\mathtt{der}(-)$ constructor) does not result in any loss of expressivity. Indeed:

Proposition 7.1 (Simplified Bang simulation [PROOF IN PROP. G.1]). *$\to_{\mathsf{B}^{\mathsf{der}}}$ and $\to_{\mathsf{B}}$ simulate each other.*

Proof. The key to this result is the fact that $\mathtt{der}((!t)\mathtt{L})$ can be understood as a shorthand for $x[x/(!t)\mathtt{L}]$. The rule $\mathsf{d!}$ in the Bang-calculus may be simulated by $\mathsf{ls!}$.

In the sequel, by *Bang-calculus* we will mean $\to_{\mathsf{B}}$ reduction over simplified Bang terms.

Typed Bang-calculus A typing system for the Bang-calculus is defined as follows. The set of *types* is given by:

$$A, B ::= \alpha \mid !A \mid !A \to B$$

Typing environments $(\Gamma, \Delta, \ldots)$ are partial functions assigning variables to types *prefixed with* !, written $x_1 : !A_1, \ldots, x_n : !A_n$. *Typing judgments* are of the form $\Gamma \vdash t : A$ and defined by:

$$\frac{}{\Gamma, x : !A \vdash x : A}\,\mathtt{b\text{-}var} \quad \frac{\Gamma, x : !A \vdash t : B}{\Gamma \vdash \lambda x.\, t : !A \to B}\,\mathtt{b\text{-}abs} \quad \frac{\Gamma \vdash t : !A \to B \qquad \Gamma \vdash s : !A}{\Gamma \vdash t\, s : B}\,\mathtt{b\text{-}app}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash !t : !A}\,\mathtt{b\text{-}prom} \quad \frac{\Gamma, x : !A \vdash t : B \qquad \Gamma \vdash s : !A}{\Gamma \vdash t[x/s] : B}\,\mathtt{b\text{-}es}$$

Embedding the Bang-calculus in $\lambda^{!\bullet}$ The Bang translation $(\cdot)^{\mathsf{B}}$ is defined on types and terms by:

$$\alpha^{\mathsf{B}} := \alpha \qquad (!A)^{\mathsf{B}} := !\bullet A^{\mathsf{B}} \qquad (!A \to B)^{\mathsf{B}} := !\bullet A^{\mathsf{B}} \multimap B^{\mathsf{B}}$$

$$\begin{array}{lll} x^{\mathsf{B}} := \mathtt{req}(x) & (\lambda x.\, t)^{\mathsf{B}} := \lambda a.\, t^{\mathsf{B}}[x/a] & (t\ s)^{\mathsf{B}} := t^{\mathsf{B}}\ s^{\mathsf{B}} \\ (!t)^{\mathsf{B}} := !\bullet t^{\mathsf{B}} & t[x/s]^{\mathsf{B}} := t^{\mathsf{B}}[x/s^{\mathsf{B}}] & \end{array}$$

The translation is extended to typing environments: $(x_1 : !A_1, \ldots, x_n : !A_n)^{\mathsf{B}} := x_1 : A_1^{\mathsf{B}}, \ldots, x_n : A_n^{\mathsf{B}}$, and judgments: $(\Gamma \vdash t : A)^{\mathsf{B}} := \Gamma^{\mathsf{B}}; \cdot \vdash t^{\mathsf{B}} : A^{\mathsf{B}}$.

Proposition 7.2 (Bang typing [PROOF IN PROP. G.2]). *If $\Gamma \vdash t : A$ then $\Gamma^{\mathsf{B}}; \cdot \vdash t^{\mathsf{B}} : A^{\mathsf{B}}$.*

Lemma 7.1 (Bang simulation). *If $t \to_{\mathsf{B}} s$ then $t^{\mathsf{B}} \to_{\bullet}^{+} s^{\mathsf{B}}$.*

Proof. By induction on the derivation of $t \to_{\mathsf{B}} s$. The interesting cases are when there is a db, $\mathsf{ls!}$, or $\mathsf{gc!}$ step at the root. If $(\lambda x.\, t)\mathsf{L}\ s \to_{\mathsf{db}} t[x/s]\mathsf{L}$, then:

$$((\lambda x.\, t)\mathsf{L}\ s)^{\mathsf{B}} = (\lambda a.\, t^{\mathsf{B}}[x/a])\mathsf{L}^{\mathsf{B}}\ s^{\mathsf{B}} \to_{\bullet\mathsf{db}} t^{\mathsf{B}}[x/s^{\mathsf{B}}]\mathsf{L}^{\mathsf{B}} = (t[x/s]\mathsf{L})^{\mathsf{B}}$$

If $\mathsf{C}\langle\langle x\rangle\rangle[x/(!s)\mathsf{L}] \to_{\mathsf{ls!}} \mathsf{C}\langle\langle s\rangle\rangle[x/!s]\mathsf{L}$, then:

$$\begin{aligned} \mathsf{C}\langle\langle x\rangle\rangle[x/(!s)\mathsf{L}]^{\mathsf{B}} = \mathsf{C}^{\mathsf{B}}\langle\langle \mathtt{req}(x)\rangle\rangle[x/(!\bullet s^{\mathsf{B}})\mathsf{L}^{\mathsf{B}}] &\to_{\bullet\mathsf{ls}} \mathsf{C}^{\mathsf{B}}\langle\langle \mathtt{req}(\bullet s^{\mathsf{B}})\rangle\rangle[x/!\bullet s^{\mathsf{B}}]\mathsf{L}^{\mathsf{B}} \\ &\to_{\bullet\mathsf{req}} \mathsf{C}^{\mathsf{B}}\langle\langle s^{\mathsf{B}}\rangle\rangle[x/!\bullet s^{\mathsf{B}}]\mathsf{L}^{\mathsf{B}} = (\mathsf{C}\langle\langle s\rangle\rangle[x/s]\mathsf{L})^{\mathsf{B}} \end{aligned}$$

If $t[x/(!s)\mathsf{L}] \to_{\mathsf{gc!}} t\mathsf{L}$ with $x \notin \mathsf{fv}(t)$, then $t[x/(!s)\mathsf{L}]^{\mathsf{B}} = t^{\mathsf{B}}[x/(!\bullet s^{\mathsf{B}})\mathsf{L}^{\mathsf{B}}] \to_{\bullet\mathsf{gc}} t^{\mathsf{B}}\mathsf{L}^{\mathsf{B}}$. Note that $x \notin \mathsf{fv}(t^{\mathsf{B}})$ because $\mathsf{fv}(t^{\mathsf{B}}) = \mathsf{fv}(t)$. ∎

For completeness, we define an **inverse Bang translation**. First, we define a subset $\mathcal{T}_{\bullet}^{\mathsf{B}} \subseteq \mathcal{T}_{\bullet}$, containing the closure by $\to_{\bullet}$-reduction of the image of $(\cdot)^{\mathsf{B}}$:

$$\underline{t}, \underline{s}, \ldots ::= \mathtt{req}(u) \mid \mathtt{req}(\bullet \underline{t}) \mid \lambda a.\, \underline{t}[u/a] \mid \underline{t}\ \underline{s} \mid !\bullet \underline{t} \mid \underline{t}[u/\underline{s}]$$

where a is assumed to be fresh in $\lambda a.\, \underline{t}[u/a]$. The *inverse Bang translation* is a function $\cdot^{-\mathsf{B}} : \mathcal{T}_{\bullet}^{\mathsf{B}} \to \mathcal{T}_{\mathsf{LSC}}$ defined as follows, by induction on the derivation of a term with the (unambiguous) grammar above:

$$\begin{array}{lll} \mathtt{req}(x)^{-\mathsf{B}} := x & \mathtt{req}(\bullet \underline{t})^{-\mathsf{B}} := \underline{t}^{-\mathsf{B}} & (\lambda a.\, \underline{t}[x/a])^{-\mathsf{B}} := \lambda x.\, \underline{t}^{-\mathsf{B}} \\ (\underline{t}\ \underline{s})^{-\mathsf{B}} := \underline{t}^{-\mathsf{B}}\ \underline{s}^{-\mathsf{B}} & (!\bullet \underline{s})^{-\mathsf{B}} := !\underline{s}^{-\mathsf{B}} & \underline{t}[x/\underline{s}]^{-\mathsf{B}} := \underline{t}^{-\mathsf{B}}[x/\underline{s}^{-\mathsf{B}}] \end{array}$$

It is easy to check that $(\cdot)^{-\mathsf{B}}$ is the left-inverse of $(\cdot)^{\mathsf{B}}$.

Lemma 7.2 (Inverse Bang simulation [PROOF IN LEMMA G.7]). *Let $\underline{t} \in \mathcal{T}_{\mathsf{B}}$ and $s \in \mathcal{T}_{\bullet}$. If $\underline{t} \to_{\bullet} s$ then $s \in \mathcal{T}_{\mathsf{B}}$ and $\underline{t}^{-\mathsf{B}} \to_{\mathsf{B}}^{=} s^{-\mathsf{B}}$.*

Theorem 7.1 (Sound and complete Bang embedding). *Let $t, s \in \mathcal{T}_{\mathsf{B}}$. Then $t \to_{\mathsf{B}}^{*} s$ if and only if $t^{\mathsf{B}} \to_{\bullet}^{*} s^{\mathsf{B}}$. Moreover, t is in $\to_{\mathsf{B}}$-normal form iff t^{B} is in $\to_{\bullet}$-normal form.*

Remark 7.1. Composing our CBV and CBN translations with the $(\cdot)^{\mathsf{B}}$ translation of above and then performing dereliction unfolding (*i.e.* replacing $\mathtt{der}(t)$ with $u[u/t]$) we obtain the call-by-name and call-by-value translations of [17] with one minor difference in the CBV case: [17] translates variables to $!x$, whereas the above composition would translate it to $!\bullet\mathtt{req}(x)$, that is, x is replaced with its $\eta_{\bullet}$-expansion. Unfortunately, this translation $(\cdot)^{\mathsf{V+}}$ fails to be complete (*cf.* Rem. 5.2).

8 Related Work and Conclusions

Related Work. The seminal work [23] is the first work to have related Girard's embeddings of intuitionistic logic into LL with evaluation mechanisms[7]. Call-by-push-value (CBPV) [20,21] is a calculus that distinguishes *values* from *computations* and allows to subsume both the CBV and CBN evaluation mechanisms. Ehrhard [14] studied the connection between CBPV [20,21] and LL, producing a calculus which was later modified to become the Bang-calculus [15]. CBV and CBN translations to the Bang-calculus were studied in [15]. Soundness and completeness of these translations with respect to reduction was proved by Guerrieri and Manzonetto [17] for a slightly different notion of reduction for the Bang-calculus than that of [15]. The CBV translation does not preserve normal forms; an amended translation that does was studied in [12,13]. Intuitionistic truth in terms of classical provability underlies Gödel's embedding of intuitionistic logic into (classical) S4. In [28], the authors consider a program similar to that of CBPV but where that target language is a modal lambda calculus. Promotion and dereliction are recast as boxing and unboxing and CBV and CBN are described in terms of a so called *call-by-box* evaluation mechanism [28].

Conclusions. This work introduces MSCLL, a Sharing Linear Logic. It arises from splitting each exponential modality ($!/?$) into a sharing modality ($!/?$) and a cloning modality ($\bullet/\circ$). MSCLL is conservative over MELL and enjoys cut-elimination. The usual embeddings of intuitionistic logic into LL can be restated in the setting of $\lambda^{!\bullet}$, a Sharing Linear λ-calculus derived from MSCLL. The decomposition of the of-course modality allows us to define an embedding of intuitionistic logic into $\lambda^{!\bullet}$, corresponding to a *call-by-need* λ-calculus CBS. The following table summarizes the, sound and complete, embeddings studied in Section 5:

	$A \to B$	x	$\lambda x.\, t$	$t\, s$	$t[x/s]$
CBN, $(\cdot)^{\mathsf{N}}$	$!\bullet A^{\mathsf{N}} \to B^{\mathsf{N}}$	$\mathtt{req}(x)$	$\lambda a.\, t^{\mathsf{N}}[x/a]$	$t^{\mathsf{N}}\, !\bullet s^{\mathsf{N}}$	$t^{\mathsf{N}}[x/!\bullet s^{\mathsf{N}}]$
CBV, $(\cdot)^{\mathsf{V}}$	$!\bullet A^{\mathsf{V}} \to !\bullet B^{\mathsf{V}}$	$!x$	$!\bullet\lambda a.\, t^{\mathsf{V}}[x/a]$	$\mathtt{req}(u)[u/t^{\mathsf{V}}]\, s^{\mathsf{V}}$	$t^{\mathsf{V}}[x/s^{\mathsf{V}}]$
CBS, $(\cdot)^{\mathsf{S}}$	$!\bullet A^{\mathsf{S}} \to \bullet B^{\mathsf{S}}$	x	$\bullet\lambda a.\, t^{\mathsf{S}}[x/a]$	$\mathtt{req}(t^{\mathsf{S}})\, !s^{\mathsf{S}}$	$t^{\mathsf{S}}[x/!s^{\mathsf{S}}]$

A weak evaluation mechanism can be defined for $\lambda^{!\bullet}$ that simulates weak evaluation in the original calculi in a sound and complete way. Moreover, MSCLL also admits a sound and complete embedding of the Bang-calculus .

There are several avenues worth pursuing. First, developing an appropriate notion of proof nets and semantics for MSCLL, which perhaps would help clarify the somewhat intriguing interaction between the sharing and access modalities. Second, studying operational properties of the $\lambda^{!\bullet}$-calculus such as standardization (as developed for LSC [3]) and solvability. Additionally, one can consider extending weak evaluation in $\lambda^{!\bullet}$ to *strong* evaluation, to simulate strong CBN/CBV/CBS evaluation. Also, our use of multiple exponentials is reminiscent of subexponentials [26], where instead of one pair of of-course and why-not modalities one introduces a family of them, each of which cannot be proven equivalent to any other. Further work is required to determine if there is a rigorous connection with subexponentials.

[7] Although the paper mentions some other authors that had already hinted at this.

It should be noted that our original motivation to study MSCLL was to try to provide a unified logical account of CBN, CBV, and CBNd. In [23], an attempt was made at embedding CBNd in a linear λ-calculus, but the target language had to be changed to become *affine*, allowing weakening of arbitrary propositions.

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A Appendix: Preliminary Notions

A.1 Simple Results about Abstract Translations

Proposition A.1 (Conditions for soundness [proof of Prop. 2.1]). *If $x_1 \to_{\mathcal{X}} x_2$ implies $T(x_1) \to^*_{\mathcal{Y}} T(x_2)$ for every $x_1, x_2 \in X$, then T is sound.*

Proof. By induction on the number of steps in a reduction sequence $x_1 \to^*_{\mathcal{X}} x_2$, it is immediate to conclude that $T(x_1) \to^*_{\mathcal{Y}} T(x_2)$.

Theorem A.1 (Conditions for completeness [proof of Thm. 2.1]). *Let $Y' \subseteq Y$ and let $T^{-1} : Y' \to X$ be a function. Suppose that T^{-1} is the left-inverse of T,* i.e. *for all $x \in X$ we have that $T(x) \in Y'$ and $T^{-1}(T(x)) = x$. Suppose moreover that T^{-1} simulates reduction,* i.e. *for all $y_1 \in Y'$ and $y_2 \in Y$ such that $y_1 \to_{\mathcal{Y}} y_2$, we have that $y_2 \in Y'$ and $T^{-1}(y_1) \to^*_{\mathcal{X}} T^{-1}(y_2)$. Then T is complete.*

Proof. First we claim that for all $y_1 \in Y'$ and $y_2 \in Y$ such that $y_1 \to^*_{\mathcal{Y}} y_2$ we have that $y_2 \in Y'$ and $T^{-1}(y_1) \to^*_{\mathcal{X}} T^{-1}(y_2)$. Indeed, suppose that $y_1 \to^n_{\mathcal{Y}} y_2$ in n steps. We proceed by induction on n. The base case ($n = 0$) is immediate, so suppose that $n > 0$ and that $y_1 \to_{\mathcal{Y}} y'_1 \to^{n-1}_{\mathcal{Y}} y_2$. Since T^{-1} simulates reduction, we have that $y'_1 \in Y'$ and that $T^{-1}(y_1) \to^*_{\mathcal{X}} T^{-1}(y'_1)$. Moreover, by IH, we have that $y_2 \in Y'$ and $T^{-1}(y'_1) \to^*_{\mathcal{X}} T^{-1}(y_2)$. Composing the reduction sequences we obtain $T^{-1}(y_1) \to^*_{\mathcal{X}} T^{-1}(y_2)$, as required.

Now suppose that $T(x_1) \to^*_{\mathcal{Y}} T(x_2)$. Since T^{-1} is the left-inverse of T, we know that $T(x_1) \in Y'$ and $T(x_2) \in Y'$. By the previous claim, $T^{-1}(T(x_1)) \to^*_{\mathcal{X}} T^{-1}(T(x_2))$. Finally, since T^{-1} is the left-inverse of T, we have that $x_1 = T^{-1}(T(x_1)) \to^*_{\mathcal{X}} T^{-1}(T(x_2)) = x_2$, as required.

B Appendix: Linear Logic with Restricted Access

Definition B.1 (Mix rule).

$$\frac{\vdash \Gamma, !A \qquad \vdash \Delta, (?A^{\perp})^n \qquad n \geq 0}{\vdash \Gamma, \Delta}\,\texttt{mix}$$

where $(A)^n$ stands for $A, \ldots, A$ (n times).

Definition B.2 (Size, degree, height).

- *An instance of a rule in a derivation is said to be* cut-like *if it is an instance of the* `cut` *rule or of the* `mix` *rule.*
- *In a cut-like instance of a rule, the* eliminated formula *is A for instances of* `cut` *and $!A$ for instances of* `mix`.
- *The* size *of a formula A is written $|A|$ and defined as follows:*

$$\begin{aligned} |\alpha| = |\overline{\alpha}| &:= 1 \\ |A \otimes B| = |A \mathbin{⅋} B| &:= 1 + |A| + |B| \\ |!A| = |?A| = |\bullet A| = |\circ A| &:= 1 + |A| \end{aligned}$$

- *The* degree *of a cut-like instance of a rule is the size of the eliminated formula.*
- *We write $\pi \rhd^d \Gamma$ if π is a derivation of the judgment $\vdash \Gamma$ such that every cut-like instance of a rule is of degree at most d. We write $\vdash^d \Gamma$ if there is a derivation π such that $\pi \rhd^d \Gamma$.*
- *The* height *of a derivation π is written $h(\pi)$ and defined as usual, regarding the derivation as a finite tree.*

Definition B.3 (Principal formula). *Given an instance of an inference rule, we say that a formula in the conclusion is a* principal formula *according to the following criteria:*

1. $\mathtt{ax}$, $\mathtt{cut}$, *and* $\mathtt{mix}$*: there are no principal formulae in the conclusion of these rules.*
2. $\otimes$*: the instance of $A \otimes B$ is principal.*
3. $⅋$*: the instance of $A ⅋ B$ is principal.*
4. $?\mathtt{w}$ *and* $?\mathtt{c}$*: the instance of $?A$ is principal.*
5. $?\circ\mathtt{d}$*: the instance of $?\circ A$ is principal.*
6. $!\mathtt{p}$*: all the formulae in the conclusion are principal.*
7. $\bullet$*: the instance of $\bullet A$ is principal.*
8. $\circ$*: the instance of $\circ A$ is principal.*

Note that, following Lincoln [22, Section 2.6], we consider all the formulae in the conclusion of the $!\mathtt{p}$ rule to be principal. This is just a convenience in nomenclature to simplify the presentation of the cut-elimination proof.

Furthermore, we distinguish between commutative *and* principal *cut-like instances of rules:*

1. *A cut-like instance of a rule is* left-principal *if the eliminated formula is principal in the first premise of the rule. A cut-like instance of a rule is* right-principal *if the negation of the eliminated formula is principal in the second premise of the rule. A cut-like instance of a rule is* principal *if it is both left* and *right-principal.*
2. *A cut-like instance of a rule is* left-commutative *if it is not left-principal, it is* right-commutative *if it is not right-principal, and it is* commutative *if it is not principal,* i.e. *if it is either left* or *right-commutative.*

Lemma B.1 (Generalized structural rules). *The following generalized structural rules admissible, using only instances of structural rules ($?\mathtt{w}$ and $?\mathtt{c}$):*

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?\Delta}\,?\mathtt{w}^* \quad \frac{\vdash \Gamma, ?\Delta, ?\Delta}{\vdash \Gamma, ?\Delta}\,?\mathtt{c}^* \quad \frac{\vdash \Gamma, (?A)^n \qquad n \geq 0}{\vdash \Gamma, ?A}\,?\mathtt{c}_n$$

Proof. Rules $?\mathtt{w}^*$ and $?\mathtt{c}^*$ are straightforward by induction on Δ. Rule $?\mathtt{c}_n$ is straightforward by induction on n.

Lemma B.2 (Empty mix lemma). *If $\vdash^d \Gamma, !A$ and $\vdash^d \Delta$ then $\vdash^d \Gamma, \Delta$.*

Proof. Let $\pi_1 \rhd^d \Gamma, !A$ and $\pi_2 \rhd^d \Delta$. We proceed by induction on π_1:

1. $\texttt{ax}$: Then Γ must be of the form $?A^\perp$. The situation is:

$$\dfrac{}{\vdash^d ?A^\perp, !A}\;\texttt{ax}$$

So it suffices to take:

$$\dfrac{\begin{array}{c}\pi_2\\ \vdash^d \Delta\end{array}}{\vdash^d ?A^\perp, \Delta}\;?\texttt{w}$$

2. $\texttt{cut}$: We consider two subcases, depending on whether $!A$ occurs in the first premise of the $\texttt{cut}$ rule, or in the second premise of the $\texttt{cut}$ rule.
   2.1 If $!A$ occurs in the first premise of the $\texttt{cut}$ rule, then Γ is of the form $\Gamma = \Gamma_1, \Gamma_2$ and there exists a formula B such that $|B| \leq d$ and:

$$\dfrac{\begin{array}{c}\pi_{11}\\ \vdash^d \Gamma_1, B, !A\end{array} \quad \begin{array}{c}\pi_{12}\\ \vdash^d \Gamma_2, B^\perp\end{array}}{\vdash^d \Gamma_1, \Gamma_2, !A}\;\texttt{cut}$$

   So by IH on π_{11} we can take:

$$\dfrac{\dfrac{\text{IH}}{\vdash^d \Gamma_1, B, \Delta} \quad \begin{array}{c}\pi_{12}\\ \vdash^d \Gamma_2, B^\perp\end{array}}{\vdash^d \Gamma_1, \Gamma_2, \Delta}\;\texttt{cut}$$

   2.2 Symmetric to the previous case (2.1).
3. $\texttt{mix}$: We consider two subcases, depending on whether $!A$ occurs in the first premise of the $\texttt{mix}$ rule, or in the second premise of the $\texttt{mix}$ rule.
   3.1 If $!A$ occurs in the first premise of the $\texttt{mix}$ rule, then Γ is of the form $\Gamma = \Gamma_1, \Gamma_2$ and there exists a formula B such that $|!B| \leq d$ and an $n \geq 0$ such that:

$$\dfrac{\begin{array}{c}\pi_{11}\\ \vdash^d \Gamma_1, !B, !A\end{array} \quad \begin{array}{c}\pi_{12}\\ \vdash^d \Gamma_2, (?B^\perp)^n\end{array}}{\vdash^d \Gamma_1, \Gamma_2, !A}\;\texttt{mix}$$

   So by IH on π_{11} we can take:

$$\dfrac{\dfrac{\text{IH}}{\vdash^d \Gamma_1, !B, \Delta} \quad \begin{array}{c}\pi_{12}\\ \vdash^d \Gamma_2, (?B^\perp)^n\end{array}}{\vdash^d \Gamma_1, \Gamma_2, \Delta}\;\texttt{mix}$$

   3.2 Symmetric to the previous case (3.1).
4. $\otimes$: We consider two subcases, depending on whether $!A$ occurs in the first premise of the $\otimes$ rule, or in the second premise of the $\otimes$ rule.
   4.1 If $!A$ occurs in the first premise of the $\otimes$ rule, then Γ is of the form $\Gamma = \Gamma_1, \Gamma_2, B \otimes C$ and:

$$\dfrac{\begin{array}{c}\pi_{11}\\ \vdash^d \Gamma_1, B, !A\end{array} \quad \begin{array}{c}\pi_{12}\\ \vdash^d \Gamma_2, C\end{array}}{\vdash^d \Gamma_1, \Gamma_2, B \otimes C, !A}\;\otimes$$

So by IH on π_{11} we can take:

$$\dfrac{\dfrac{\text{IH}}{\vdash^d \Gamma_1, B, \Delta} \quad \begin{array}{c}\pi_{12}\\ \vdash^d \Gamma_2, C\end{array}}{\vdash^d \Gamma_1, \Gamma_2, B \otimes C, \Delta}\ \otimes$$

4.2 Symmetric to the previous case (4.1).

5. $\parr$: Then Γ is of the form $\Gamma = \Gamma', B \parr C$ and the situation is:

$$\dfrac{\begin{array}{c}\pi'_1\\ \vdash^d \Gamma', B, C, !A\end{array}}{\vdash^d \Gamma', B \parr C, !A}\ \parr$$

So by IH on π'_1 we can take:

$$\dfrac{\dfrac{\text{IH}}{\vdash^d \Gamma', B, C, \Delta}}{\vdash^d \Gamma', B \parr C, \Delta}\ \parr$$

6. $!\mathtt{p}$: Then Γ is of the form $\Gamma = ?\Gamma'$ and the situation is:

$$\dfrac{\begin{array}{c}\pi'_1\\ \vdash^d ?\Gamma', A\end{array}}{\vdash^d ?\Gamma', !A}\ !\mathtt{p}$$

So it suffices to take:

$$\dfrac{\vdash^d \Delta}{\vdash^d ?\Gamma', \Delta}\ ?\mathtt{w}^*$$

7. $?\mathtt{w}$: Then Γ is of the form $\Gamma = \Gamma', ?B$ and the situation is:

$$\dfrac{\begin{array}{c}\pi'_1\\ \vdash^d \Gamma', !A\end{array}}{\vdash^d \Gamma', ?B, !A}\ ?\mathtt{w}$$

So by IH on π'_1 we can take:

$$\dfrac{\dfrac{\text{IH}}{\vdash^d \Gamma', \Delta}}{\vdash^d \Gamma', ?B, \Delta}\ ?\mathtt{w}$$

8. $?\mathtt{c}$: Then Γ is of the form $\Gamma = \Gamma', ?B$ and the situation is:

$$\dfrac{\begin{array}{c}\pi'_1\\ \vdash^d \Gamma', ?B, ?B, !A\end{array}}{\vdash^d \Gamma', ?B, !A}\ ?\mathtt{c}$$

So by IH on π'_1 we can take:

$$\dfrac{\begin{array}{c}\pi'_1\\ \vdash^d \Gamma', ?B, ?B, \Delta\end{array}}{\vdash^d \Gamma', ?B, \Delta}\ ?\mathtt{c}$$

9. $?\circ\mathtt{d}$: Then Γ is of the form $\Gamma = \Gamma', ?\circ B$ and the situation is:

$$\dfrac{\begin{matrix}\pi_1' \\ \vdash^d \Gamma', \circ B, !A\end{matrix}}{\vdash^d \Gamma', ?\circ B, !A}\ ?\circ\mathtt{d}$$

So by IH on π_1' we can take:

$$\dfrac{\begin{matrix}\pi_1' \\ \vdash^d \Gamma', \circ B, \Delta\end{matrix}}{\vdash^d \Gamma', ?\circ B, \Delta}\ ?\circ\mathtt{d}$$

10. $\bullet$: Similar to case 9.
11. $\circ$: Similar to case 9.

Lemma B.3 (Cut/mix flattening). *The following rules are admissible:*

1. **Cut-flattening.**
$$\dfrac{\vdash^d \Gamma, A \qquad \vdash^d \Delta, A^{\perp} \qquad |A| = d+1}{\vdash^d \Gamma, \Delta}\ \mathtt{cut}^d$$

2. **Mix-flattening.**
$$\dfrac{\vdash^d \Gamma, !A \qquad \vdash^d \Delta, (?A^{\perp})^n \qquad |!A| = d+1}{\vdash^d \Gamma, \Delta}\ \mathtt{mix}^d$$

Proof. In each of the two rules of the statement, let us write π_1 for the derivation of the first premise and π_2 for the derivation of the second premise. We prove the two items simultaneously, by induction on $h(\pi_1) + h(\pi_2)$. We proceed by case analysis on the last rule used to construct the derivations π_1 and π_2. As usual, in the case analysis, we distinguish between commutative and principal cases. Note that, for cut-flattening, we only study the left-commutative cases, since the right-commutative cases are symmetric. Moreover, for mix-flattening, in right-commutative cases we may assume that the $\mathtt{mix}^d$ rule we are trying to derive is left-principal.

1. *Cut-flattening, left-commutative case,* $\mathtt{ax}$. Then we have that $\pi_1 \rhd^d \Gamma, A$ is constructed using the $\mathtt{ax}$ rule, so $\Gamma = A^{\perp}$. The situation is:

$$\dfrac{\dfrac{}{\vdash^d A^{\perp}, A}\ \mathtt{ax} \qquad \begin{matrix}\pi_2 \\ \vdash^d \Delta, A^{\perp}\end{matrix}}{\vdash A^{\perp}, \Delta}\ \mathtt{cut}^d$$

So we can take π_2 itself.

2. *Cut-flattening, left-commutative case,* $\mathtt{cut}$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $\mathtt{cut}$ rule. We consider two subcases, depending on whether the eliminated formula A occurs in the first premise of the $\mathtt{cut}$ rule or in the second premise of the $\mathtt{cut}$ rule:

2.1 If A occurs in the first premise of the $\mathtt{cut}$ rule then Γ must be of the form Γ_1, Γ_2, and the situation is as follows, where $|B| \leq d$:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, B, A} \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, B^\perp}}{\vdash^d \Gamma_1, \Gamma_2, A}\ \mathtt{cut} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash \Gamma_1, \Gamma_2, \Delta}\ \mathtt{cut}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, B, A} \quad \overset{\pi_{2}}{\vdash^d \Delta, A^\perp}}{\vdash^d \Gamma_1, \Delta, B}\ \mathtt{cut}^d \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, B^\perp}}{\vdash^d \Gamma_1, \Gamma_2, \Delta}\ \mathtt{cut}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

2.2 If A occurs in the second premise of the $\mathtt{cut}$ rule, the proof is symmetric to the previous case (2.1).

3. *Cut-flattening, left-commutative case,* $\mathtt{mix}$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $\mathtt{mix}$ rule. We consider two subcases, depending on whether the eliminated formula A occurs in the first premise of the $\mathtt{mix}$ rule or in the second premise of the $\mathtt{mix}$ rule:

3.1 If A occurs in the first premise of the $\mathtt{mix}$ rule, then Γ must be of the form Γ_1, Γ_2, and the situation is as follows, where $|!B| \leq d$ and $n \geq 0$:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, A, !B} \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, (?B^\perp)^n}}{\vdash^d \Gamma_1, \Gamma_2, A}\ \mathtt{mix} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash \Gamma_1, \Gamma_2, \Delta}\ \mathtt{cut}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, A, !B} \quad \overset{\pi_{2}}{\vdash^d \Delta, A^\perp}}{\vdash^d \Gamma_1, !B, \Delta}\ \mathtt{cut}^d \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, (?B^\perp)^n}}{\vdash^d \Gamma_1, \Gamma_2, \Delta}\ \mathtt{mix}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

3.2 If A occurs in the second premise of the $\mathtt{mix}$rule, then Γ must be of the form Γ_1, Γ_2 and the situation is as follows, where $|!B| \leq d$ and $n \geq 0$:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, !B} \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, A, (?B^\perp)^n}}{\vdash^d \Gamma_1, \Gamma_2, A}\ \mathtt{mix} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash \Gamma_1, \Gamma_2, \Delta}\ \mathtt{cut}^d$$

Resorting to the IH, we can take:

$$\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, !B} \quad \dfrac{\overset{\pi_{12}}{\vdash^d \Gamma_2, (?B^\perp)^n, A} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash^d \Gamma_2, \Delta, (?B^\perp)^n}\,\texttt{cut}^d}{\vdash^d \Gamma_1, \Gamma_2, \Delta}\,\texttt{mix}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

4. *Cut-flattening, left-commutative case,* $\otimes$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $\otimes$ rule. We consider two subcases, depending on whether the eliminated formula A occurs in the first premise of the $\otimes$ rule or in the second premise of the $\otimes$ rule:
   4.1 If A occurs in the first premise of the $\otimes$ rule, then the situation is:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, B, A} \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, C}}{\vdash^d \Gamma_1, \Gamma_2, B \otimes C, A}\,\otimes \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash \Gamma_1, \Gamma_2, B \otimes C, \Delta}\,\texttt{cut}^d$$

   Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, B, A} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash^d \Gamma_1, B, \Delta}\,\texttt{cut}^d \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, C}}{\vdash^d \Gamma_1, \Gamma_2, B \otimes C, \Delta}\,\otimes$$

   To apply the IH, note that sum of the heights of the subderivations decreases.
   4.2 If A occurs on the second premise of the $\otimes$ rule, the proof is symmetric to the previous case (4.1).
5. *Cut-flattening, left-commutative case,* $⅋$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $⅋$ rule. The situation is:

$$\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d \Gamma', B, C, A}}{\vdash^d \Gamma', B ⅋ C, A}\,⅋ \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash \Gamma', \Delta, B ⅋ C}\,\texttt{cut}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d \Gamma', B, C, A} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash^d \Gamma', \Delta, B, C}\,\texttt{cut}^d}{\vdash^d \Gamma', \Delta, B ⅋ C}\,⅋$$

To apply the IH, note that sum of the heights of the subderivations decreases.

6. *Cut-flattening, left-commutative case,* $!\texttt{p}$. We argue that this case is impossible. Indeed, $\pi_1 \rhd^d \Gamma, A$ must be constructed using the $!\texttt{p}$ rule, but all the formulae in the conclusion of $!\texttt{p}$ are principal by definition.

7. *Cut-flattening, left-commutative case,* $?\mathtt{w}$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $?\mathtt{w}$ rule. Note that Γ must be of the form $\Gamma', ?B$. The situation is:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \Gamma', A}}{\vdash^d \Gamma', ?B, A}\,?\mathtt{w} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash \Gamma', \Delta, ?B}\,\mathtt{cut}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \Gamma', A} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash^d \Gamma', \Delta}\,\mathtt{cut}^d}{\vdash^d \Gamma', \Delta, ?B}\,?\mathtt{w}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

8. *Cut-flattening, left-commutative case,* $?\mathtt{c}$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $?\mathtt{c}$ rule. Note that Γ must be of the form $\Gamma', ?B$. The situation is:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \Gamma', A, ?B, ?B}}{\vdash^d \Gamma', A, ?B}\,?\mathtt{c} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash \Gamma', \Delta, ?B}\,\mathtt{cut}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \Gamma', A, ?B, ?B} \quad \overset{\pi_2}{\vdash^d \Delta, A^\perp}}{\vdash^d \Gamma', \Delta, ?B, ?B}\,\mathtt{cut}^d}{\vdash^d \Gamma', \Delta, ?B}\,?\mathtt{c}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

9. *Cut-flattening, left-commutative case,* $?\circ\mathtt{d}$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $?\circ\mathtt{d}$ rule. Note that Γ must be of the form $\Gamma', ?\circ B$. The situation is:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \Gamma', \circ B}}{\vdash^d \Gamma', ?\circ B}\,?\circ\mathtt{d} \quad \overset{\pi_2}{\vdash^d \Delta}}{\vdash \Gamma', ?\circ B, \Delta}\,\mathtt{cut}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \Gamma', \circ B} \quad \overset{\pi_2}{\vdash^d \Delta}}{\vdash^d \Gamma', \circ B, \Delta}\,\mathtt{cut}^d}{\vdash^d \Gamma', ?\circ B, \Delta}\,?\circ\mathtt{d}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

10. *Cut-flattening, left-commutative case,* $\bullet$. Similar to case 9.
11. *Cut-flattening, left-commutative case,* $\circ$. Similar to case 9.
12. *Cut-flattening, principal case,* $\otimes$ / $\parr$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $\otimes$ rule and $\pi_2 \rhd^d \Delta, A^\perp$ is constructed using the $\parr$ rule. Note that Γ must be of the form Γ_1, Γ_2 and A must be of the form $B \otimes C$. The situation is:

$$\dfrac{\dfrac{\dfrac{\pi_{11}}{\vdash^d \Gamma_1, B} \quad \dfrac{\pi_{12}}{\vdash^d \Gamma_2, C}}{\vdash^d \Gamma_1, \Gamma_2, B \otimes C}\otimes \quad \dfrac{\dfrac{\pi_2'}{\vdash^d \Delta, B^\perp, C^\perp}}{\vdash^d \Delta, B^\perp \parr C^\perp}\parr}{\vdash \Gamma_1, \Gamma_2, \Delta}\,\mathtt{cut}^d$$

Then we can take:

$$\dfrac{\dfrac{\pi_{11}}{\vdash^d \Gamma_1, B} \quad \dfrac{\dfrac{\pi_{12}}{\vdash^d \Gamma_2, C} \quad \dfrac{\pi_2'}{\vdash^d \Delta, B^\perp, C^\perp}}{\vdash^d \Gamma_2, \Delta, B^\perp}\,\mathtt{cut}}{\vdash^d \Gamma_1, \Gamma_2, \Delta}\,\mathtt{cut}$$

Note that we do not need to apply the IH, as the sizes of the eliminated formulae strictly decrease.
13. *Cut-flattening, principal case,* $!\mathtt{p}$ / $!\mathtt{p}$. Then $\pi_1 \rhd^d \Gamma, A$ and $\pi_2 \rhd^d \Delta, A^\perp$ are both constructed using the $!\mathtt{p}$ rule. Since the conclusion of the $!\mathtt{p}$ rule is of the form $?\Sigma, !C$, we consider two subcases, depending on whether A is of the form $!A_1$ or of the form $?A_1$:
    13.1 If A is of the form $A = !A_1$. Then Γ must be of the form $?\Gamma'$. Moreover, note that $A^\perp = (!A_1)^\perp = ?A_1^\perp$ and, since $A^\perp$ is part of the conclusion of π_2, which is also constructed using the $!\mathtt{p}$ rule, we know that $A^\perp$ must be of the form $?A_1^\perp$. Moreover, since $\vdash \Delta, ?A_1^\perp$ is the conclusion of an instance of the $!\mathtt{p}$ rule, Δ must be of the form $\Delta = ?\Delta', !B$. In summary, the situation is:

$$\dfrac{\dfrac{\dfrac{\pi_1'}{\vdash^d ?\Gamma', A_1}}{\vdash^d ?\Gamma', !A_1}\,!\mathtt{p} \quad \dfrac{\dfrac{\pi_2'}{\vdash^d ?\Delta', B, ?A_1^\perp}}{\vdash^d ?\Delta', !B, ?A_1^\perp}\,!\mathtt{p}}{\vdash ?\Gamma', ?\Delta', !B}\,\mathtt{cut}^d$$

    Resorting to the IH, we can take:

$$\dfrac{\dfrac{\dfrac{\dfrac{\pi_1'}{\vdash^d ?\Gamma', A_1}}{\vdash^d ?\Gamma', \bullet A_1}\,!\mathtt{p} \quad \dfrac{\pi_2'}{\vdash^d ?\Delta', B, ?A_1^\perp}}{\vdash^d ?\Gamma', ?\Delta', B}\,\mathtt{cut}^d}{\vdash^d ?\Gamma', ?\Delta', !B}\,!\mathtt{p}$$

    To apply the IH, note that sum of the heights of the subderivations decreases.
    13.2 If A is of the form $A = ?A_1$, then since $\vdash \Gamma, A$ and $\vdash \Delta, A^\perp$ are both conclusions of instances of the $!\mathtt{p}$ rule, we have that Γ must be of the form $\Gamma = ?\Gamma', !B$, and Δ must be of the form $\Delta = ?\Delta'$, and the proof is symmetric to the previous case (13.1).

14. *Cut-flattening, principal case,* $!\mathtt{p}$ / $?\mathtt{w}$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $!\mathtt{p}$ rule and $\pi_1 \rhd^d \Delta, A^\perp$ is constructed using the $?\mathtt{w}$ rule. Note that, since $A^\perp$ is the principal formula of the $?\mathtt{w}$ rule, $A^\perp$ must be of the form $A^\perp = ?A_1$. Then $A = !A_1^\perp$ and, since $\vdash \Gamma, A$ is the conclusion of an instance of the $!\mathtt{p}$ rule, we have that Γ is of the form $\Gamma = ?\Gamma'$. The situation is:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1'\\ \vdash^d ?\Gamma', A_1^\perp\end{array}}{\vdash^d ?\Gamma', !A_1^\perp}\,!\mathtt{p} \qquad \dfrac{\begin{array}{c}\pi_2'\\ \vdash^d \Delta\end{array}}{\vdash^d \Delta, ?A_1}\,?\mathtt{w}}{\vdash ?\Gamma', \Delta}\,\mathtt{cut}^d$$

Hence we can take:

$$\dfrac{\begin{array}{c}\pi_2'\\ \vdash^d \Delta\end{array}}{\vdash^d ?\Gamma', \Delta}\,?\mathtt{w}^*$$

15. *Cut-flattening, principal case,* $!\mathtt{p}$ / $?\mathtt{c}$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $!\mathtt{p}$ rule and $\pi_2 \rhd^d \Delta, A^\perp$ is constructed using the $?\mathtt{c}$ rule. Note that, since $A^\perp$ is the principal formula of the $?\mathtt{c}$ rule, $A^\perp$ must be of the form $A^\perp = ?A_1$. Then $A = !A_1^\perp$ and, since $\vdash \Gamma, A$ is the conclusion of an instance of the $!\mathtt{p}$ rule, we have that Γ is of the form $\Gamma = ?\Gamma'$. The situation is:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1'\\ \vdash^d ?\Gamma', A_1^\perp\end{array}}{\vdash^d ?\Gamma', !A_1^\perp}\,!\mathtt{p} \qquad \dfrac{\begin{array}{c}\pi_2'\\ \vdash^d \Delta, ?A_1, ?A_1\end{array}}{\vdash^d \Delta, ?A_1}\,?\mathtt{c}}{\vdash ?\Gamma', \Delta}\,\mathtt{cut}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1'\\ \vdash^d ?\Gamma', A_1^\perp\end{array}}{\vdash^d ?\Gamma', !A_1^\perp}\,!\mathtt{p} \qquad \begin{array}{c}\pi_2'\\ \vdash^d \Delta, (?A_1)^2\end{array}}{\vdash^d ?\Gamma', \Delta}\,\mathtt{mix}^d$$

To apply the IH, note that sum of the heights of the subderivations decreases.

16. *Cut-flattening, principal case,* $!\mathtt{p}$ / $?\circ\mathtt{d}$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $!\mathtt{p}$ rule and $\pi_2 \rhd^d \Delta, A^\perp$ is constructed using the $?\circ\mathtt{d}$ rule. Note that, since $A^\perp$ is the principal formula of the $?\circ\mathtt{d}$ rule, $A^\perp$ must be of the form $A^\perp = ?\circ A_1$. Then $A = !\bullet A_1^\perp$ and, since $\vdash \Gamma, A$ is the conclusion of an instance of the $!\mathtt{p}$ rule, we have that Γ is of the form $\Gamma = ?\Gamma'$. The situation is:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1'\\ \vdash^d ?\Gamma', \bullet A_1^\perp\end{array}}{\vdash^d ?\Gamma', !\bullet A_1^\perp}\,!\mathtt{p} \qquad \dfrac{\begin{array}{c}\pi_2'\\ \vdash^d \Delta, \circ A_1\end{array}}{\vdash^d \Delta, ?\circ A_1}\,?\circ\mathtt{d}}{\vdash ?\Gamma', \Delta}\,\mathtt{cut}^d$$

Then we can take:

$$\dfrac{\begin{array}{c}\pi_1'\\ \vdash^d ?\Gamma', \bullet A_1^\perp\end{array} \quad \begin{array}{c}\pi_2'\\ \vdash^d \Delta, \circ A_1\end{array}}{\vdash^d ?\Gamma', \Delta}\,\mathtt{cut}$$

Note that we do not need to apply the IH, as the size of the eliminated formula strictly decreases.

17. *Cut-flattening, principal case,* $\bullet/\circ$. Then $\pi_1 \rhd^d \Gamma, A$ is constructed using the $\bullet$ rule, and $\pi_2 \rhd^d \Delta, A^\perp$ is constructed using the $\circ$ rule. Since A is the principal formula of the $\bullet$ rule we have that A must be of the form $A = \bullet A_1$ and $A^\perp = \circ A_1^\perp$. The situation is:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1'\\ \vdash^d \Gamma, A_1\end{array}}{\vdash^d \Gamma, \bullet A_1}\,\bullet \qquad \dfrac{\begin{array}{c}\pi_2'\\ \vdash^d \Delta, A_1^\perp\end{array}}{\vdash^d \Delta, \circ A_1^\perp}\,\circ}{\vdash \Gamma, \Delta}\,\mathtt{cut}^d$$

Then we can take:

$$\dfrac{\begin{array}{c}\pi_1'\\ \vdash^d \Gamma, A_1\end{array} \quad \begin{array}{c}\pi_2'\\ \vdash^d \Delta, A_1^\perp\end{array}}{\vdash^d \Gamma, \Delta}\,\mathtt{cut}$$

Note that we do not need to apply the IH, as the size of the eliminated formula strictly decreases.

18. *Cut-flattening, remaining principal cases.* The remaining principal cases are symmetric to already covered cases: the principal $⅋/\otimes$ case is symmetric to case 12; the principal $?\mathtt{w}/!\mathtt{p}$ case is symmetric to case 14; the principal $?\mathtt{c}/!\mathtt{p}$ case is symmetric to case 15; the principal $?\circ\mathtt{d}/!\mathtt{p}$ case is symmetric to case 16; and the principal $\circ/\bullet$ case is symmetric to case 17.

19. *Mix-flattening, left-commutative case,* $\mathtt{ax}$. Then $\pi_1 \rhd^d \Gamma, !A$ is constructed using the $\mathtt{ax}$ rule, so $\Gamma = ?A^\perp$. The situation is:

$$\dfrac{\dfrac{}{\vdash^d ?A^\perp, !A}\,\mathtt{ax} \qquad \begin{array}{c}\pi_2\\ \vdash^d \Delta, (?A^\perp)^n\end{array}}{\vdash ?A^\perp, \Delta}\,\mathtt{mix}^d$$

Then we can take:

$$\dfrac{\begin{array}{c}\pi_2\\ \vdash^d \Delta, (?A^\perp)^n\end{array}}{\vdash^d ?A^\perp, \Delta}\,?\mathtt{c}_n$$

20. *Mix-flattening, right-commutative case,* $\mathtt{ax}$. Then $\pi_2 \rhd^d \Delta, (?A^\perp)^n$ is constructed using the $\mathtt{ax}$ rule, so the number of formulae in $\Delta, (?A^\perp)^n$ is exactly 2. In particular, $n \leq 2$. Moreover, n cannot be exactly 2 because the conclusion of the $\mathtt{ax}$ rule cannot be of the form $?A^\perp, ?A^\perp$. Therefore n is either 0 or 1, and we consider two subcases:

    20.1 If $n = 0$, then $\Delta = B, B^\perp$ and the situation is:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdash^d \Gamma, !A\end{array} \qquad \dfrac{}{\vdash^d B, B^\perp}\,\mathtt{ax}}{\vdash \Gamma, B, B^\perp}\,\mathtt{mix}^d$$

Then by Lem. B.2 we conclude that $\vdash^d \Gamma, B, B^\perp$.

20.2 If $n = 1$, then $\Delta = !A$ and the situation is:

$$\dfrac{\overset{\pi_1}{\vdash^d \Gamma, !A} \quad \dfrac{}{\vdash^d !A, ?A^\perp}\,\mathtt{ax}}{\vdash \Gamma, !A}\,\mathtt{mix}^d$$

So we can take π_1 itself.

21. *Mix-flattening, left-commutative case,* `cut`. Then $\pi_1 \rhd^d \Gamma, !A$ is constructed using `cut`. We consider two subcases, depending on whether the eliminated formula $!A$ occurs in the first premise of the `cut` rule or in the second premise of the `cut` rule:

21.1 If $!A$ occurs in the first premise of the `cut` rule then Γ must be of the form Γ_1, Γ_2 and the situation is as follows, where $|B| \le d$:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, !A, B} \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, B^\perp}}{\vdash^d \Gamma_1, \Gamma_2, !A}\,\mathtt{cut} \quad \overset{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash \Gamma_1, \Gamma_2, \Delta}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, B, !A} \quad \overset{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash^d \Gamma_1, \Delta, B}\,\mathtt{mix}^d \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, B^\perp}}{\vdash^d \Gamma_1, \Gamma_2, \Delta}\,\mathtt{cut}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

21.2 Symmetric to the previous case (21.1).

22. *Mix-flattening, right-commutative case,* `cut`. Then $\pi_2 \rhd^d \Delta, (?A^\perp)^n$ is constructed using the `cut` rule. Moreover, we may assume that the instance of $\mathtt{mix}^d$ we are constructing is left-principal, since otherwise it would fall into one of the left-commutative cases, so $\pi_1 \rhd^d \Gamma, !A$ must necessarily be constructed using the `!p` rule and Γ must be of the form $\Gamma = ?\Gamma'$. Then the situation is as follows, where $\Delta = \Delta_1, \Delta_2$ and $n = n_1 + n_2$ and $|B| \le d$:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\,\mathtt{!p} \quad \dfrac{\overset{\pi_{21}}{\vdash^d \Delta_1, (?A^\perp)^{n_1}, B} \quad \overset{\pi_{22}}{\vdash^d \Delta_2, (?A^\perp)^{n_2}, B^\perp}}{\vdash^d \Delta_1, \Delta_2, (?A^\perp)^n}\,\mathtt{cut}}{\vdash ?\Gamma', \Delta_1, \Delta_2}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\rho_1}{\vdash^d ?\Gamma', \Delta_1, B} \quad \overset{\rho_2}{\vdash^d ?\Gamma', \Delta_2, B^\perp}}{\vdash^d ?\Gamma', ?\Gamma', \Delta_1, \Delta_2}\,\mathtt{cut}}{\vdash^d ?\Gamma', \Delta_1, \Delta_2}\,?\mathtt{c}^*$$

where:

$$\rho_1 := \left(\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\,!\mathtt{p} \quad \overset{\pi_{21}}{\vdash^d \Delta_1, B, (?A^\perp)^{n_1}}}{\vdash^d ?\Gamma', \Delta_1, B}\,\mathtt{mix}^d \right)$$

$$\rho_2 := \left(\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\,!\mathtt{p} \quad \overset{\pi_{22}}{\vdash^d \Delta_2, (?A^\perp)^{n_2}, B^\perp}}{\vdash^d ?\Gamma', \Delta_2, B^\perp}\,\mathtt{mix}^d \right)$$

To apply the IH, note that sum of the heights of the subderivations decreases.

23. *Mix-flattening, left-commutative case,* `mix`. Then $\pi_1 \rhd^d \Gamma, !A$ is constructed using the `mix` rule. We consider two subcases, depending on whether the eliminated formula $!A$ occurs in the first premise of the `mix` rule, or in the second premise of the `mix` rule:

23.1 If $!A$ occurs in the first premise of the `mix` rule then Γ must be of the form Γ_1, Γ_2 and the situation is as follows, where $|!B| \leq d$ and $m \geq 0$:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, !A, !B} \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, (?B^\perp)^m}}{\vdash^d \Gamma_1, \Gamma_2, !A}\,\mathtt{mix} \quad \overset{\pi_2}{\vdash^d \Delta, (?\neg A)^n}}{\vdash \Gamma_1, \Gamma_2, \Delta}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, !B, !A} \quad \overset{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash^d \Gamma_1, \Delta, !B}\,\mathtt{mix}^d \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, (?B^\perp)^m}}{\vdash^d \Gamma_1, \Gamma_2, \Delta}\,\mathtt{mix}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

23.2 If $!A$ occurs in the second premise of the `mix` rule then Γ must be of the form Γ_1, Γ_2 and is as follows, where $|!B| \leq d$ and $m \geq 0$:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, !B} \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, !A, (?B^\perp)^m}}{\vdash^d \Gamma_1, \Gamma_2, !A}\,\mathtt{mix} \quad \overset{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash \Gamma_1, \Gamma_2, \Delta}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, !B} \quad \dfrac{\overset{\pi_{12}}{\vdash^d \Gamma_2, (?B^\perp)^m, !A} \quad \overset{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash^d \Gamma_2, \Delta, (?B^\perp)^m}\,\mathtt{mix}^d}{\vdash^d \Gamma_1, \Gamma_2, \Delta}\,\mathtt{mix}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

24. *Mix-flattening, right-commutative case,* $\mathtt{mix}$. Then $\pi_2 \rhd^d \Delta, (?A^\perp)^n$ is constructed using the $\mathtt{mix}$ rule. Moreover, we may assume that the instance of $\mathtt{mix}^d$ we are constructing is left-principal, since otherwise it would fall into one of the left-commutative cases, so $\pi_1 \rhd^d \Gamma, !A$ must necessarily be constructed using the $!\mathtt{p}$ rule and Γ must be of the form $\Gamma = ?\Gamma'$. Then the situation is as follows, where $\Delta = \Delta_1, \Delta_2$ and $n = n_1 + n_2$ and $|!B| \leq d$ and $m \geq 0$:

$$\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\,!\mathtt{p} \qquad \dfrac{\overset{\pi_{21}}{\vdash^d \Delta_1, (?A^\perp)^{n_1}, !B} \quad \overset{\pi_{22}}{\vdash^d \Delta_2, (?A^\perp)^{n_2}, (?B^\perp)^m}}{\vdash^d \Delta_1, \Delta_2, (?A^\perp)^n}\,\mathtt{mix}}{\vdash ?\Gamma', \Delta_1, \Delta_2}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\rho_1}{\vdash^d ?\Gamma', \Delta_1, !B} \quad \overset{\rho_2}{\vdash^d ?\Gamma', \Delta_2, (?B^\perp)^m}}{\vdash^d ?\Gamma', ?\Gamma', \Delta_1, \Delta_2}\,\mathtt{mix}}{\vdash^d ?\Gamma', \Delta_1, \Delta_2}\,?\mathtt{c}^*$$

where:

$$\rho_1 := \left(\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\,!\mathtt{p} \quad \overset{\pi_{21}}{\vdash^d \Delta_1, !B, (?A^\perp)^{n_1}}}{\vdash^d ?\Gamma', \Delta_1, !B}\,\mathtt{mix}^d \right)$$

$$\rho_2 := \left(\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\,!\mathtt{p} \quad \overset{\pi_{22}}{\vdash^d \Delta_2, (?A^\perp)^{n_2}, (?B^\perp)^m}}{\vdash^d ?\Gamma', \Delta_2, (?B^\perp)^m}\,\mathtt{mix}^d \right)$$

To apply the IH, note that sum of the heights of the subderivations decreases.

25. *Mix-flattening, left-commutative case,* $\otimes$. Then $\pi_1 \rhd^d \Gamma, !A$ is constructed using the $\otimes$ rule. We consider two subcases, depending on whether the eliminated formula $!A$ occurs in the first premise of the $\otimes$ rule or in the second premise of the $\otimes$ rule:

25.1 If $!A$ occurs in the first premise of the $\otimes$ rule, then Γ must be of the form $\Gamma_1, \Gamma_2, B \otimes C$ and the situation is as follows:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, B, !A} \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, C}}{\vdash^d \Gamma_1, \Gamma_2, B \otimes C, !A}\,\otimes \quad \overset{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash \Gamma_1, \Gamma_2, B \otimes C, \Delta}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_{11}}{\vdash^d \Gamma_1, B, !A} \quad \overset{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash^d \Gamma_1, B, \Delta}\,\mathtt{mix}^d \quad \overset{\pi_{12}}{\vdash^d \Gamma_2, C}}{\vdash^d \Gamma_1, \Gamma_2, B \otimes C, \Delta}\,\otimes$$

To apply the IH, note that sum of the heights of the subderivations decreases.

25.2 Symmetric to the previous case (25.1).

26. *Mix-flattening, right-commutative case,* $\otimes$. Then $\pi_2 \rhd^d \Delta, (?A^\perp)^n$ is constructed using the $\otimes$ rule. Moreover, we may assume that the instance of $\mathtt{mix}^d$ we are constructing is left-principal, since otherwise it would fall into one of the left-commutative cases, so $\pi_1 \rhd^d \Gamma, !A$ must necessarily be constructed using the $!\mathtt{p}$ rule and Γ must be of the form $\Gamma = ?\Gamma'$. Then the situation is as follows, where $\Delta = \Delta_1, \Delta_2, B \otimes C$ and $n = n_1 + n_2$:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d ?\Gamma, A}}{\vdash^d ?\Gamma, !A}\,!\mathtt{p} \quad \dfrac{\overset{\pi_{21}}{\vdash^d \Delta_1, B, (?A^\perp)^{n_1}} \quad \overset{\pi_{22}}{\vdash^d \Delta_2, C, (?A^\perp)^{n_2}}}{\vdash^d \Delta_1, \Delta_2, B \otimes C, (?A^\perp)^n}\,\otimes}{\vdash ?\Gamma, \Delta_1, \Delta_2, B \otimes C}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\rho_1}{\vdash^d ?\Gamma, \Delta_1, B} \quad \overset{\rho_2}{\vdash^d ?\Gamma, \Delta_2, C}}{\vdash^d ?\Gamma, ?\Gamma, \Delta_1, \Delta_2, B \otimes C}\,\otimes}{\vdash^d ?\Gamma, \Delta_1, \Delta_2, B \otimes C}\,?\mathtt{c}^*$$

where:

$$\rho_1 := \left(\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d ?\Gamma, A}}{\vdash^d ?\Gamma, !A}\,!\mathtt{p} \quad \overset{\pi_{21}}{\vdash^d \Delta_1, B, (?A^\perp)^{n_1}}}{\vdash^d ?\Gamma, \Delta_1, B}\,\mathtt{mix}^d \right)$$

$$\rho_2 := \left(\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d ?\Gamma, A}}{\vdash^d ?\Gamma, !A}\,!\mathtt{p} \quad \overset{\pi_{22}}{\vdash^d \Delta_2, C, (?A^\perp)^{n_2}}}{\vdash^d ?\Gamma, \Delta_2, C}\,\mathtt{mix}^d \right)$$

To apply the IH, note that sum of the heights of the subderivations decreases.

27. *Mix-flattening, left-commutative case,* $⅋$. Then $\pi_1 \rhd^d \Gamma, !A$ is constructed using the $⅋$ rule. Then Γ must be of the form $\Gamma = \Gamma', B ⅋ C$ and the situation is:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \Gamma', B, C, !A}}{\vdash^d \Gamma', B ⅋ C, !A}\,⅋ \quad \overset{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash \Gamma', B ⅋ C, \Delta}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \Gamma', B, C, !A} \quad \overset{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash^d \Gamma', B, C, \Delta}\,\mathtt{mix}^d}{\vdash^d \Gamma', B ⅋ C, \Delta}\,⅋$$

To apply the IH, note that sum of the heights of the subderivations decreases.

28. *Mix-flattening, right-commutative case,* $\parr$. Then $\pi_2 \rhd^d \varDelta, (?A^\perp)^n$ is constructed using the $\parr$ rule. Then $\varDelta$ must be of the form $\varDelta = \varDelta', B \parr C$ and the situation is:

$$\dfrac{\overset{\pi_1}{\vdash^d \varGamma, !A} \qquad \dfrac{\overset{\pi_2'}{\vdash^d \varDelta', B, C, (?A^\perp)^n}}{\vdash^d \varDelta', B \parr C, (?A^\perp)^n}\,\parr}{\vdash \varGamma, \varDelta', B \parr C}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_1}{\vdash^d \varGamma, !A} \quad \overset{\pi_2'}{\vdash^d \varDelta', B, C, (?A^\perp)^n}}{\vdash^d \varGamma, \varDelta', B, C}\,\mathtt{mix}^d}{\vdash^d \varGamma, \varDelta', B \parr C}\,\parr$$

To apply the IH, note that sum of the heights of the subderivations decreases.

29. *Mix-flattening, left-commutative case,* $!\mathtt{p}$. We argue that this case is impossible. Indeed, $\pi_1 \rhd^{\varGamma, !A}$ must be constructed using the $!\mathtt{p}$ rule, but all the formulae in the conclusion of $!\mathtt{p}$ are principal by definition.

30. *Mix-flattening, right-commutative case,* $!\mathtt{p}$. Then $\pi_2 \rhd^d \varDelta, (?A^\perp)^n$ must be constructed using the $!\mathtt{p}$ rule, and it is such that $?A^\perp$ is not a principal formula in the conclusion of the $!\mathtt{p}$ rule. Recall that all the formulae in the conclusion of the $!\mathtt{p}$ rule are principal, so this case is only possible when $n = 0$. Then, since $\vdash \varDelta$ is the conclusion of an instance of the $!\mathtt{p}$ rule, we have that $\varDelta$ must be of the form $\varDelta = ?\varDelta', !B$. Then the situation is as follows:

$$\dfrac{\overset{\pi_1}{\vdash^d \varGamma, !A} \qquad \dfrac{\overset{\pi_2'}{\vdash^d ?\varDelta', B}}{\vdash^d ?\varDelta', !B}\,!\mathtt{p}}{\vdash \varGamma, ?\varDelta', !B}\,\mathtt{mix}^d$$

Then by Lem. B.2 we conclude that $\vdash^d \varGamma, ?\varDelta', !B$, as required.

31. *Mix-flattening, left-commutative case,* $?\mathtt{w}$. Then $\pi_1 \rhd^d \varGamma, !A$ must be constructed using the $?\mathtt{w}$ rule. Then $\varGamma$ must be of the form $\varGamma = \varGamma', ?B$ and the situation is:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \varGamma', !A}}{\vdash^d \varGamma', ?B, !A}\,?\mathtt{w} \qquad \overset{\pi_2}{\vdash^d \varDelta, (?A^\perp)^n}}{\vdash \varGamma', ?B, \varDelta}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d \varGamma', !A} \quad \overset{\pi_2}{\vdash^d \varDelta, (?A^\perp)^n}}{\vdash^d \varGamma', \varDelta}\,\mathtt{mix}^d}{\vdash^d \varGamma', ?B, \varDelta}\,?\mathtt{w}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

32. *Mix-flattening, right-commutative case,* $?\mathtt{w}$. Similar to the previous case (31).
33. *Mix-flattening, left-commutative case,* $?\mathtt{c}$. Then $\pi_1 \rhd^d \Gamma, !A$ must be constructed using the $?\mathtt{c}$ rule. Then Γ must be of the form $\Gamma = \Gamma', ?B$ and the situation is:

$$\dfrac{\dfrac{\dfrac{\pi_1'}{\vdash^d \Gamma', ?B, ?B, !A}}{\vdash^d \Gamma', ?B, !A}\,?\mathtt{c} \quad \dfrac{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash \Gamma', ?B, \Delta}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\dfrac{\pi_1'}{\vdash^d \Gamma', ?B, ?B, !A} \quad \dfrac{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash^d \Gamma', ?B, ?B, \Delta}\,\mathtt{mix}^d}{\vdash^d \Gamma', ?B, \Delta}\,?\mathtt{c}$$

To apply the IH, note that sum of the heights of the subderivations decreases.
34. *Mix-flattening, right-commutative case,* $?\mathtt{c}$. Similar to the previous case (33).
35. *Mix-flattening, left-commutative case,* $?\circ\mathtt{d}$. Then $\pi_1 \rhd^d \Gamma, !A$ must be constructed using the $?\circ\mathtt{d}$ rule. Then Γ must be of the form $\Gamma = \Gamma', ?\circ B$ and the situation is:

$$\dfrac{\dfrac{\dfrac{\pi_1'}{\vdash^d \Gamma', \circ B, !A}}{\vdash^d \Gamma', ?\circ B, !A}\,?\circ\mathtt{d} \quad \dfrac{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash \Gamma', ?\circ B, \Delta}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\dfrac{\pi_1'}{\vdash^d \Gamma', \circ B, !A} \quad \dfrac{\pi_2}{\vdash^d \Delta, (?A^\perp)^n}}{\vdash^d \Gamma', \circ B, \Delta}\,\mathtt{mix}^d}{\vdash^d \Gamma', ?\circ B, \Delta}\,?\circ\mathtt{d}$$

To apply the IH, note that sum of the heights of the subderivations decreases.
36. *Mix-flattening, right-commutative case,* $?\circ\mathtt{d}$. Similar to the previous case (35).
37. *Mix-flattening, left-commutative case,* $\bullet$. Similar to case 35.
38. *Mix-flattening, right-commutative case,* $\bullet$. Similar to case 35.
39. *Mix-flattening, left-commutative case,* $\circ$. Similar to case 35.
40. *Mix-flattening, right-commutative case,* $\circ$. Similar to case 35.
41. *Mix-flattening, principal case,* $!\mathtt{p}$ / $!\mathtt{p}$. Then $\pi_1 \rhd^d \Gamma, !A$ and $\pi_2 \rhd^d \Delta, (?A^\perp)^n$ are both constructed using the $!\mathtt{p}$ rule. Since $\vdash \Gamma, !A$ is the conclusion of an instance of the $!\mathtt{p}$ rule, Γ must be of the form $\Gamma = ?\Gamma'$. Moreover, since $\vdash \Delta, (?A^\perp)^n$ is the conclusion of an instance of the $!\mathtt{p}$ rule, with $?A^\perp$ as a principal formula, Δ must be of the form $\Delta = ?\Delta', !B$. If $n = 0$, it suffices to resort to Lem. B.2 to conclude that $\vdash^d \Gamma, ?\Delta', !B$. If $n > 0$, the situation is:

$$\dfrac{\dfrac{\dfrac{\pi_1'}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\,!\mathtt{p} \quad \dfrac{\dfrac{\pi_2'}{\vdash^d ?\Delta', B, (?A^\perp)^n}}{\vdash^d ?\Delta', !B, (?A^\perp)^n}\,!\mathtt{p}}{\vdash ?\Gamma', ?\Delta', !B}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\ !\mathtt{p} \quad \overset{\pi'_2}{\vdash^d ?\Delta', B, (?A^\perp)^n}}{\vdash^d ?\Gamma', ?\Delta', B}\ \mathtt{mix}^d}{\vdash^d ?\Gamma', ?\Delta', !B}\ !\mathtt{p}$$

To apply the IH, note that sum of the heights of the subderivations decreases.

42. *Mix-flattening, principal case,* $!\mathtt{p}$ / $?\mathtt{w}$. Then $\pi_1 \rhd^d \Gamma, !A$ is constructed using the $!\mathtt{p}$ rule and $\pi_2 \rhd^d \Delta, (?A^\perp)^n$ is constructed using the $?\mathtt{w}$ rule. Since $\vdash \Gamma, !A$ is the conclusion of an instance of the $!\mathtt{p}$ rule, Γ must be of the form $\Gamma = ?\Gamma'$. If $n = 0$, it suffices to resort to Lem. B.2 to conclude that $\vdash^d ?\Gamma', \Delta$. If $n > 0$, the situation is:

$$\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\ !\mathtt{p} \quad \dfrac{\overset{\pi'_2}{\vdash^d ?\Gamma', (?A^\perp)^{n-1}}}{\vdash^d ?\Gamma', (?A^\perp)^n}\ ?\mathtt{w}}{\vdash ?\Gamma', \Delta}\ \mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\ !\mathtt{p} \quad \overset{\pi'_2}{\vdash^d ?\Gamma', (?A^\perp)^{n-1}}}{\vdash^d ?\Gamma', \Delta}\ \mathtt{mix}^d$$

To apply the IH, note that sum of the heights of the subderivations decreases.

43. *Mix-flattening, principal case,* $!\mathtt{p}$ / $?\mathtt{c}$. Then $\pi_1 \rhd^d \Gamma, !A$ is constructed using the $!\mathtt{p}$ rule and $\pi_2 \rhd^d \Delta, (?A^\perp)^n$ is constructed using the $?\mathtt{c}$ rule. Since $\vdash \Gamma, !A$ is the conclusion of an instance of the $!\mathtt{p}$ rule, Γ must be of the form $\Gamma = ?\Gamma'$. Then the situation is:

$$\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\ !\mathtt{p} \quad \dfrac{\overset{\pi'_2}{\vdash^d \Delta, (?A^\perp)^{n+1}}}{\vdash^d \Delta, (?A^\perp)^n}\ ?\mathtt{c}}{\vdash ?\Gamma', \Delta}\ \mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi'_1}{\vdash^d ?\Gamma', A}}{\vdash^d ?\Gamma', !A}\ !\mathtt{p} \quad \overset{\pi'_2}{\vdash^d \Delta, (?A^\perp)^{n+1}}}{\vdash^d ?\Gamma', \Delta}\ \mathtt{mix}^d$$

To apply the IH, note that sum of the heights of the subderivations decreases.

44. *Mix-flattening, principal case,* $!\mathtt{p}$ / $?{\circ}\mathtt{d}$. Then $\pi_1 \rhd^d \Gamma, !A$ is constructed using the $!\mathtt{p}$ rule and $\pi_2 \rhd^d \Delta, (?A^\perp)^n$ is constructed using the $?{\circ}\mathtt{d}$ rule. Since $\vdash \Gamma, !A$ is

the conclusion of an instance of the $!\mathtt{p}$ rule, Γ must be of the form $\Gamma = ?\Gamma'$. If $n = 0$, it suffices to resort to Lem. B.2 to conclude that $\vdash^d ?\Gamma', \Delta$. If $n > 0$, then since $\pi_2 \triangleright^d \Delta, (?A^\perp)^n$ is the conclusion of an instance of the $?\circ\mathtt{d}$ rule with $?A^\perp$ as a principal formula, $A^\perp$ must be of the form $A^\perp = \circ A_1$, so $A = \bullet A_1^\perp$. Then the situation is:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d ?\Gamma', \bullet A_1^\perp}}{\vdash^d ?\Gamma', !\bullet A_1^\perp}\,!\mathtt{p} \qquad \dfrac{\overset{\pi_2'}{\vdash^d \Delta, (?\circ A_1)^{n-1}, \circ A_1}}{\vdash^d \Delta, (?\circ A_1)^{n-1}, ?\circ A_1}\,?\circ\mathtt{d}}{\vdash ?\Gamma', \Delta}\,\mathtt{mix}^d$$

Resorting to the IH, we can take:

$$\dfrac{\dfrac{\overset{\pi_1'}{\vdash^d ?\Gamma', \bullet A_1^\perp} \qquad \dfrac{\dfrac{\overset{\pi_1'}{\vdash^d ?\Gamma', \bullet A_1^\perp}}{\vdash^d ?\Gamma', !\bullet A_1^\perp}\,!\mathtt{p} \quad \overset{\pi_2'}{\vdash^d \Delta, (?\circ A_1)^{n-1}, \circ A_1}}{\vdash^d ?\Gamma', \Delta, \circ A_1}\,\mathtt{mix}^d}{\vdash^d ?\Gamma', ?\Gamma', \Delta}\,\mathtt{cut}}{\vdash^d ?\Gamma', \Delta}\,?\mathtt{c}^*$$

To apply the IH, note that sum of the heights of the subderivations decreases. Moreover, note that the last instance of the $\mathtt{cut}$ rule used in the derivation is between $\bullet A_1^\perp$ and $\circ A_1$ and $|\bullet A_1^\perp| = \circ A_1 = d$.

45. *Mix-flattening, remaining principal cases.* The remaining principal cases are impossible, since the principal formula on the left must be of the form $!A$. This includes the cases $\otimes/⅋$, $⅋/\otimes$, $?\mathtt{w}/!\mathtt{p}$, $?\mathtt{c}/!\mathtt{p}$, $?\circ\mathtt{d}/!\mathtt{p}$, $\bullet/\circ$, and $\circ/\bullet$.

Theorem B.1 (Cut elimination [PROOF OF THM. 3.2]**).** *If $\vdash \Gamma$ is provable in* MSCLL*, then there is a derivation of $\vdash \Gamma$ without instances of the* $\mathtt{cut}$ *rule.*

Proof. More in general, we show that if $\vdash \Gamma$ is provable in MSCLL+$\mathtt{mix}$, then there is a derivation of $\vdash \Gamma$ without instances of the $\mathtt{cut}$ and $\mathtt{mix}$ rules.

Let π be a derivation of some depth, say d, *i.e.* $\pi \triangleright^d \Gamma$. We proceed by induction on d. If $d = 0$, observe that π has no instances of the $\mathtt{cut}$ and $\mathtt{mix}$ rules. For $d > 0$, it suffices to note that, in general, $\vdash^d \Gamma$ implies $\vdash^{d-1} \Gamma$. To see this, proceed by induction on the derivation of $\vdash^d \Gamma$. Most cases are straightforward by resorting to the IH, except for the $\mathtt{cut}$ and $\mathtt{mix}$ cases, in which it suffices to apply the cut/mix flattening lemma (Lem. B.3) when the size of the eliminated formula is d.

C Appendix: A Sharing Linear λ-Calculus

C.1 Logical Soundness

We extend the translation to environments and judgments:

$$\begin{aligned}(u_1 : A_1, \dots, u_n : A_n)^\star &:= A_1^\star, \dots, A_n^\star \\ (a_1 : A_1, \dots, a_n : A_n)^\star &:= A_1^\star, \dots, A_n^\star \\ (\Delta; \Gamma \vdash t : A)^\star &:= \vdash ?\circ(\Delta^{\star\perp}), \Gamma^{\star\perp}, A^\star\end{aligned}$$

Proposition C.1 (Logical Soundness of $\lambda^{!\bullet}$ [proof of Prop. 4.1]). *If $\Delta; \Gamma \vdash t : A$ holds in $\lambda^{!\bullet}$, then $\vdash ?\circ(\Delta^{\star\perp}), \Gamma^{\star\perp}, A^\star$ holds in* MSCLL.

Proof. We prove that if $\Delta; \Gamma \vdash t : A$ holds in $\lambda^{!\bullet}$, then $\vdash ?\circ(\Delta^{\star\perp}), \Gamma^{\star\perp}, A^\star$ holds in MSCLL by induction on the derivation of $\Delta; \Gamma \vdash t : A$:

1. `lvar`: Let $\Delta; a : A \vdash a : A$. Then:

$$\dfrac{\dfrac{}{\vdash A^{\star\perp}, A^\star}\ \mathtt{ax}}{\vdash ?\circ\Delta^{\star\perp}, A^{\star\perp}, A^\star}\ ?\mathtt{w}^*$$

2. `uvar`: Let $\Delta, u : A; \cdot \vdash u : \bullet A$. Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A^{\star\perp}, A^\star}\ \mathtt{ax}}{\vdash ?\circ A^{\star\perp}, A^\star}\ ?\circ\mathtt{d}}{\vdash ?\circ A^{\star\perp}, \bullet A^\star}\ \bullet}{\vdash ?\circ\Delta^{\star\perp}, ?\circ A^{\star\perp}, \bullet A^\star}\ ?\mathtt{w}^*$$

3. `abs`: Let $\Delta; \Gamma \vdash \lambda a. t : A \multimap B$ be derived from $\Delta; \Gamma, a : A \vdash t : B$. Then:

$$\dfrac{\dfrac{\text{IH}}{\vdash ?\circ\Delta^{\star\perp}, \Gamma^{\star\perp}, A^{\star\perp}, B^\star}}{\vdash ?\circ\Delta^{\star\perp}, \Gamma^{\star\perp}, A^{\star\perp} \parr B^\star}\ \parr$$

4. `app`: Let $\Delta; \Gamma_1, \Gamma_2 \vdash t\, s : B$ be derived from $\Delta; \Gamma_1 \vdash t : A \multimap B$ and $\Delta; \Gamma_2 \vdash s : B$. Then:

$$\dfrac{\dfrac{\dfrac{\text{IH}}{\vdash ?\circ\Delta^{\star\perp}, \Gamma_1^{\star\perp}, A^{\star\perp} \parr B^\star} \quad \dfrac{\dfrac{\text{IH}}{\vdash ?\circ\Delta^{\star\perp}, \Gamma_2^{\star\perp}, A^\star} \quad \dfrac{}{\vdash B^\star, B^{\star\perp}}\ \mathtt{ax}}{\vdash ?\circ\Delta^{\star\perp}, \Gamma_2^{\star\perp}, B^\star, A^\star \otimes B^{\star\perp}}\ \otimes}{\vdash ?\circ\Delta^{\star\perp}, ?\circ\Delta^{\star\perp}, \Gamma_1^{\star\perp}, \Gamma_2^{\star\perp}, B^\star}\ \mathtt{cut}}{\vdash ?\circ\Delta^{\star\perp}, \Gamma_1^{\star\perp}, \Gamma_2^{\star\perp}, B^\star}\ ?\mathtt{c}^*$$

5. `grant`: Let $\Delta; \Gamma \vdash \bullet t : \bullet A$ be derived from $\Delta; \Gamma \vdash t : A$. Then:

$$\dfrac{\vdash ?\circ\Delta^{\star\perp}, \Gamma^{\star\perp}, A^\star}{\vdash ?\circ\Delta^{\star\perp}, \Gamma^{\star\perp}, \bullet A^\star}\ \bullet$$

6. `request`: Let $\Delta; \Gamma \vdash \mathtt{req}(t) : A$ be derived from $\Delta; \Gamma \vdash t : \bullet A$. Then:

$$\dfrac{\dfrac{\text{IH}}{\vdash ?\circ\Delta^{\star\perp}, \Gamma^{\star\perp}, \bullet A^\star} \quad \dfrac{\dfrac{}{\vdash A^{\star\perp}, A^\star}\ \mathtt{ax}}{\vdash \circ A^{\star\perp}, A^\star}\ \circ}{\vdash ?\circ\Delta^{\star\perp}, \Gamma^{\star\perp}, A^\star}\ \mathtt{cut}$$

7. $\mathtt{prom}$: Let $\Delta; \cdot \vdash !t : !A$ be derived from $\Delta; \cdot \vdash t : A$. Then:

$$\dfrac{\dfrac{\text{IH}}{\vdash ?\circ\Delta^{\star\perp}, A^{\star}}}{\vdash ?\circ\Delta^{\star\perp}, !A^{\star}}\ \mathtt{!p}$$

8. $\mathtt{sub}$: Let $\Delta; \Gamma_1, \Gamma_2 \vdash t[u/s] : B$ be derived from $\Delta, u : A; \Gamma_1 \vdash t : B$ and $\Delta; \Gamma_2 \vdash s : !\bullet A$. Then:

$$\dfrac{\dfrac{\dfrac{\text{IH}}{\vdash ?\circ\Delta^{\star\perp}, ?\circ A^{\star\perp}, \Gamma_1^{\star\perp}, B^{\star}} \quad \dfrac{\text{IH}}{\vdash ?\circ\Delta^{\star\perp}, \Gamma_2^{\star\perp}, !\bullet A^{\star}}}{\vdash ?\circ\Delta^{\star\perp}, ?\circ\Delta^{\star\perp}, \Gamma_1^{\star\perp}, \Gamma_2^{\star\perp}, B^{\star}}\ \mathtt{cut}}{\vdash ?\circ\Delta^{\star\perp}, \Gamma_1^{\star\perp}, \Gamma_2^{\star\perp}, B^{\star}}\ ?\mathtt{c}^{*}$$

C.2 Confluence

We structure this section as follows. First, we recall Melliès notion of orthogonal axiomatic rewriting system. We then introduce the labeled $\lambda^{!\bullet}$-calculus, a tool through which we define a notion of residual for $\lambda^{!\bullet}$-calculus. Then we prove two important properties of labeled reduction, namely, finite developments and semantic orthogonality. Finally, we show how to model the $\lambda^{!\bullet}$-calculus as an axiomatic rewrite system and how the above mentioned axioms hold, which entails confluence.

D Orthogonal axiomatic rewriting systems

We recall here the axiomatic rewriting framework due to Melliès [24]. We shall use this framework in Section 4 to prove that the Sharing Linear λ-Calculus is confluent, up to a congruence.

An *axiomatic rewriting system* (abbreviated AxRS) is a 5-uple $(\mathtt{X}, \mathsf{S}, \mathsf{src}, \mathsf{tgt}, \cdot[\![\cdot]\!]\cdot)$ where $\mathtt{X}$ is a set whose elements are called *objects*, S is a set whose elements are called *steps*, $\mathsf{src}, \mathsf{tgt} : \mathsf{S} \to \mathtt{X}$ are functions providing the *source* and the *target* of each step, and $\cdot[\![\cdot]\!]\cdot \subseteq \mathsf{S} \times \mathsf{S} \times \mathsf{S}$ is a ternary *residual relation*, such that $\mathfrak{r}[\![\mathfrak{s}]\!]\mathfrak{r}'$ may hold only if $\mathsf{src}(\mathfrak{r}) = \mathsf{src}(\mathfrak{s})$ and $\mathsf{src}(\mathfrak{r}') = \mathsf{tgt}(\mathfrak{s})$. A *reduction* is either the empty reduction that starts an ends on an object $x \in \mathtt{X}$, written ϵ_x, or a sequence of $n \geq 1$ steps $\mathfrak{r}_1 \ldots \mathfrak{r}_n$ that are *composable*, *i.e.* $\mathsf{tgt}(\mathfrak{r}_i) = \mathsf{src}(\mathfrak{r}_{i+1})$ for all $i \in 1..n-1$. The src and tgt functions are extended to reductions as expected. The residual function is extended to reductions by declaring that $\mathfrak{r}[\![\epsilon_{\mathfrak{r}}]\!]\mathfrak{r}$, and that the binary relation $[\![\mathfrak{s}_1 \ldots \mathfrak{s}_n]\!]$ is the composition $[\![\mathfrak{s}_1]\!] \circ \ldots \circ [\![\mathfrak{s}_n]\!]$.

A *multistep* is either the empty multistep starting on an object $x \in \mathtt{X}$, written $\varnothing_x$, or a finite non-empty set of steps $M \subseteq \mathsf{S}$ all with the same source. The notion of *development* is defined as follows: the empty reduction ϵ_x is a development of the empty multistep $\varnothing_x$, and a reduction $\mathfrak{r}_1 \ldots \mathfrak{r}_n$ is a development of a multistep M if for all $i \in 1..n$ there exists a step $\mathfrak{s} \in M$ such that $\mathfrak{s}[\![\mathfrak{r}_1 \ldots \mathfrak{r}_{i-1}]\!]\mathfrak{r}_i$. A *complete development* of a multistep $\mathfrak{m}$ is a development after which no residuals of steps in $\mathfrak{m}$ remain.

An AxRS is *orthogonal* if it verifies the four following properties:

1. Auto-Erasure. For all $\mathfrak{r} \in \mathsf{S}$ there is no $\mathfrak{s} \in \mathsf{S}$ such that $\mathfrak{r}[\![\mathfrak{r}]\!]\mathfrak{s}$.

2. Finite Residuals. The set $\{\mathfrak{r}' \mid \mathfrak{r}[\![\mathfrak{s}]\!]\mathfrak{r}'\}$ is finite for all $\mathfrak{r}, \mathfrak{s} \in \mathsf{S}$.
3. Finite Developments. If $M \subseteq \mathsf{S}$ is a finite non-empty set of steps of the same source, all developments of M are finite, *i.e.* there are no infinite sequences $\mathfrak{r}_1\mathfrak{r}_2\ldots$ such that for all $i \in \mathbb{N}$ there exists a step $\mathfrak{s} \in M$ such that $\mathfrak{s}[\![\mathfrak{r}_1 \ldots \mathfrak{r}_{i-1}]\!]\mathfrak{r}_i$.
4. Semantic Orthogonality. If $\mathfrak{r}, \mathfrak{s} \in \mathsf{S}$ are steps of the same source, ρ is a complete development of $\{\mathfrak{r}' \mid \mathfrak{r}[\![\mathfrak{s}]\!]\mathfrak{r}'\}$ and σ is a complete development of $\{\mathfrak{s}' \mid \mathfrak{s}[\![\mathfrak{r}]\!]\mathfrak{s}'\}$, then the compositions $\mathfrak{r}\sigma$ and $\mathfrak{s}\rho$ have the same target and $[\![\mathfrak{r}\sigma]\!] = [\![\mathfrak{s}\rho]\!]$ are equal as binary relations.

Theorem D.1. *Orthogonal AxRSs are confluent. (See [24, Theorem 2.4]).*

Confluence of orthogonal AxRSs holds in a very strong sense. In fact, confluence diagrams are closed with complete developments of the relative projections of the reductions, which entails that these diagrams are *pushouts* in the categorical sense.

The Labeled $\lambda^{!\bullet}$-calculus

Definition D.1 (The $\lambda^{\mathcal{L}}_{\bullet}$-calculus). *The set of* labeled $\lambda^{!\bullet}$-terms $\mathsf{T}^{\mathcal{L}}_{\bullet}$ *—or just* labeled terms*— is given by the following grammar:*

$$
\begin{array}{lll}
t, s, \ldots ::= & a & \textit{linear variable} \\
 & \mid u & \textit{unrestricted variable} \\
 & \mid u^{\alpha} & \textit{unrestricted variable} \\
 & \mid \lambda a.\, t & \textit{abstraction} \\
 & \mid \lambda a^{\alpha}.\, t & \textit{abstraction} \\
 & \mid t\, s & \textit{application} \\
 & \mid \bullet t & \textit{grant} \\
 & \mid \mathtt{req}(t) & \textit{request} \\
 & \mid \mathtt{req}^{\alpha}(t) & \textit{request} \\
 & \mid !t & \textit{promotion} \\
 & \mid t[u/s] & \textit{substitution} \\
 & \mid t[u^{\alpha}/s] & \textit{substitution}
\end{array}
$$

The set of labeled $\lambda^{!\bullet}$-contexts $\mathsf{Ctxs}^{\mathcal{L}}_{\bullet}$ *—or just* labeled contexts*— is defined as follows:*

$$
\begin{array}{l}
\mathsf{C} ::= \Box \mid \lambda a.\, \mathsf{C} \mid \lambda a^{\alpha}.\, \mathsf{C} \mid \mathsf{C}\, t \mid t\, \mathsf{C} \mid \bullet\mathsf{C} \mid \mathtt{req}(\mathsf{C}) \mid \mathtt{req}^{\alpha}(\mathsf{C}) \mid !\mathsf{C} \mid \mathsf{C}[u/t] \mid \mathsf{C}[u^{\alpha}/t] \mid t[u/\mathsf{C}] \mid t[u^{\alpha}/\mathsf{C}] \\
\mathsf{L} ::= \Box \mid \mathsf{L}[u/t] \mid \mathsf{L}[u^{\alpha}/t]
\end{array}
$$

We write $u^{(\alpha)}$ for a variable u which may or may not be labeled with label α. We write x for both linear and (possibly labeled) unrestricted variables. Similar with $\mathtt{req}^{(\alpha)}(t)$ and $t[u^{(\alpha)}/s]$. Also, $\mathsf{fv}(t)$ are the free variables of t, disregarding labels. For example, $\mathsf{fv}(u\,a) = \mathsf{fv}(u^{\alpha}\,a) = \{u, a\}$. We write $\mathsf{fv}^{\mathcal{L}}(t)$ for the set of free labeled variables. For example, $\mathsf{fv}^{\mathcal{L}}(u\,a) = \varnothing$ and $\mathsf{fv}^{\mathcal{L}}(u^{\alpha}\,a) = \{u\}$. Linear substitution over labeled terms $t\{a := s\}$ is defined as expected since linear variables are not decorated with labels. We write t° to denote the term obtained from t by removing all its labels. Thus for example $(\lambda a^{\alpha}.\,(a\,\mathtt{req}^{\beta}(u)))^{\circ} = \lambda a.\,(a\,\mathtt{req}(u))$. A term t can be labeled in different ways, leading to different variants *of t. We say that t is a variant of s iff $t^{\circ} = s^{\circ}$. In particular, t is a variant of itself.*

Definition D.2 (Well-labeled terms). *The set of* well-labeled $\lambda^{!\bullet}$-terms $\mathsf{T}_{\bullet}^{\mathcal{WL}}$ *—or just* well-labeled terms*— are those labeled terms $t \in \mathsf{T}_{\bullet}^{\mathcal{L}}$ for which the predicate $t\,\mathsf{wl}$ holds, defined as follows:*

$$\frac{}{a\,\mathsf{wl}} \qquad \frac{}{u^{(\alpha)}\,\mathsf{wl}}$$

$$\frac{t\,\mathsf{wl}}{\lambda a.\, t\,\mathsf{wl}} \qquad \frac{t\,\mathsf{wl} \quad s\,\mathsf{wl}}{t\,s\,\mathsf{wl}} \qquad \frac{(\lambda a.\, t)\mathtt{L}\,\mathsf{wl} \quad s\,\mathsf{wl}}{(\lambda a^{\alpha}.\, t)\mathtt{L}\, s\,\mathsf{wl}}$$

$$\frac{t\,\mathsf{wl}}{\bullet t\,\mathsf{wl}} \qquad \frac{t\,\mathsf{wl}}{\mathtt{req}(t)\,\mathsf{wl}} \qquad \frac{(\bullet t)\mathtt{L}\,\mathsf{wl}}{\mathtt{req}^{\alpha}((\bullet t)\mathtt{L})\,\mathsf{wl}} \qquad \frac{t\,\mathsf{wl}}{!t\,\mathsf{wl}}$$

$$\frac{t\,\mathsf{wl} \quad (!(\bullet s)\mathtt{L}_1)\mathtt{L}_2\,\mathsf{wl} \quad u \in \mathsf{fv}^{\mathcal{L}}(t)}{t[u/(!(\bullet s)\mathtt{L}_1)\mathtt{L}_2]\,\mathsf{wl}} \qquad \frac{t\,\mathsf{wl} \quad (!s)\mathtt{L}\,\mathsf{wl} \quad u \notin \mathsf{fv}(t)}{t[u^{\alpha}/(!s)\mathtt{L}]\,\mathsf{wl}} \qquad \frac{t\,\mathsf{wl} \quad s\,\mathsf{wl} \quad u \notin \mathsf{fv}^{\mathcal{L}}(t)}{t[u/s]\,\mathsf{wl}}$$

Definition D.3 (Labeled reduction). *Labeled reduction at the root is defined on labeled terms as follows:*

$$\begin{array}{rcll}
(\lambda a^{\alpha}.\, t)\mathtt{L}\, s & \mapsto^{\alpha}_{\bullet\mathsf{db}} & t\{a := s\}\mathtt{L} & \textit{if } \mathsf{fv}(s) \cap \mathsf{dom}(\mathtt{L}) = \varnothing \\
\mathtt{req}^{\alpha}((\bullet t)\mathtt{L}) & \mapsto^{\alpha}_{\bullet\mathsf{req}} & t\mathtt{L} & \\
\mathtt{C}\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t)\mathtt{L}_1)\mathtt{L}_2] & \mapsto^{\alpha}_{\bullet\mathsf{ls}} & \mathtt{C}\langle\!\langle(\bullet t)\mathtt{L}_1\rangle\!\rangle[u/!(\bullet t)\mathtt{L}_1]\mathtt{L}_2 & \textit{if } u \notin \mathsf{fv}(t\mathtt{L}_1) \textit{ and } \mathsf{fv}(\mathtt{C}) \cap \mathsf{dom}(\mathtt{L}_1\mathtt{L}_2) = \varnothing \\
t[u^{\alpha}/(!s)\mathtt{L}] & \mapsto^{\alpha}_{\bullet\mathsf{gc}} & t\mathtt{L} & \textit{if } u \notin \mathsf{fv}(t)
\end{array}$$

*Note that the side conditions of the $\mapsto^{\alpha}_{\bullet\mathsf{db}}$ and $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$ rules can always be met by α-renaming. We define the α-*labeled R step relation $\to^{\alpha}_{R} := \mathtt{C}\langle\mapsto^{\alpha}_{R}\rangle$ *for each $R \in \{\bullet\mathsf{db}, \bullet\mathsf{req}, \bullet\mathsf{ls}, \bullet\mathsf{gc}\}$, where $\mathtt{C}\langle\mapsto^{\alpha}_{R}\rangle$ denotes the closure of $\mapsto^{\alpha}_{R}$ by compatibility under arbitrary labeled contexts. An α-step ($\mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \ldots$) is a tuple of one of the following forms:*

- $\langle \mathtt{C}, (\lambda a^{\alpha}.\, t)\mathtt{L}\, s\rangle$ *with* $\mathsf{fv}(s) \cap \mathsf{dom}(\mathtt{L}) = \varnothing$*;*
- $\langle \mathtt{C}, \mathtt{req}^{\alpha}((\bullet t)\mathtt{L})\rangle$*;*
- $\langle \mathtt{C}, \mathtt{D}, \mathtt{D}\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t)\mathtt{L}_1)\mathtt{L}_2]\rangle$ *with if* $u \notin \mathsf{fv}(t\mathtt{L}_1)$ *and* $\mathsf{fv}(\mathtt{D}) \cap \mathsf{dom}(\mathtt{L}_1\mathtt{L}_2) = \varnothing$*;*
- $\langle \mathtt{C}, t[u^{\alpha}/(!s)\mathtt{L}]\rangle$ *with* $u \notin \mathsf{fv}(t)$*.*

Its anchor *is the variable decorated with the label α, if the step is in $\{\bullet\mathsf{db}, \bullet\mathsf{ls}, \bullet\mathsf{gc}\}$ or the occurrence of '*$\mathtt{open}$*' that is decorated with the label α, if the step is a $\bullet\mathsf{req}$-step. Its source and target are defined as expected. For example, the source and target of the step $\langle \mathtt{C}, (\lambda a^{\alpha}.\, t)\mathtt{L}\, s\rangle$ is $\mathtt{C}\langle(\lambda a^{\alpha}.\, t)\mathtt{L}\, s\rangle$ and $\mathtt{C}\langle t\{a := s\}\mathtt{L}\rangle$, resp. We write $\mathsf{steps}_{\alpha}(t)$ for the set of α-labeled steps in t.*

The α-step *reduction relation $\to^{\alpha}_{\bullet}$ is defined as the union of the previous relations, that is, $\to^{\alpha}_{\bullet} := \to^{\alpha}_{\bullet\mathsf{db}} \cup \to^{\alpha}_{\bullet\mathsf{req}} \cup \to^{\alpha}_{\bullet\mathsf{ls}} \cup \to^{\alpha}_{\bullet\mathsf{gc}}$.*

Definition D.4 (Lifting of a step). *Let $t \in \mathsf{T}_{\bullet}^{\mathcal{L}}$ be a labeled term and $\mathfrak{r} \in \mathsf{steps}(t)$ a step in t. The $\mathfrak{r} - \alpha$-lift of t, written $\mathsf{lift}(t, \mathfrak{r}, \alpha)$, is the variant of t resulting from assigning label α to the anchor of $\mathfrak{r}$. For example, if $\mathfrak{r} = \langle \mathtt{C}, (\lambda a^{(\beta)}.\, t_1)\mathtt{L}\, t_2\rangle$, then $\mathsf{lift}(t, \mathfrak{r}, \alpha) = \mathtt{C}\langle(\lambda a^{\alpha}.\, t_1)\mathtt{L}\, t_2\rangle$.*

Lemma D.1. *Let $t \in \mathsf{T}_{\bullet}^{\mathcal{WL}}$ and $t \to^{\alpha}_{\bullet} s$ implies $s \in \mathsf{T}_{\bullet}^{\mathcal{WL}}$.*

Finite Developments and Semantic Orthogonality

Definition D.5 (Variable Multiplicity). *The multiplicity of a variable x in a well-labeled term t, denoted $\#_x(t)$, is defined as follows:*

$$
\begin{array}{rcll}
\#_a(a) &:=& 1 & \\
\#_a(b) &:=& 0 & \text{if } a \neq b \\
\#_u(u^\alpha) &:=& 1 & \\
\#_u(u) &:=& 0 & \\
\#_u(v^{(\alpha)}) &:=& 0 & \text{if } u \neq v \\
\#_x(\lambda a.\, t) &:=& \#_x(t) & \\
\#_x((\lambda a^\alpha.\, t)\mathtt{L}\; s) &:=& \#_x((\lambda a.\, t)\mathtt{L}) + \#_x(s) + \#_a(t) \times \#_x(s) & \\
\#_x(t\; s) &:=& \#_x(t) + \#_x(s) & \text{if } t\; s \text{ not a redex} \\
\#_x(\bullet t) &:=& \#_x(t) & \\
\#_x(\mathtt{req}^{(\alpha)}(t)) &:=& \#_x(t) & \\
\#_x(!t) &:=& \#_x(t) & \\
\#_x(t[u^{(\alpha)}/s]) &:=& \#_x(t) + \#_x(s) + \#_u(t) \times \#_x(s) &
\end{array}
$$

$$
\begin{array}{rcl}
\#^\phi_x(\Box) &:=& \phi(x) \\
\#^\phi_x(\mathtt{L}[u^{(\alpha)}/s]) &:=& \#^\phi_x(\mathtt{L}) + \#_x(s) + \#^\phi_u(\mathtt{L}) \times \#_x(s)
\end{array}
$$

Definition D.6 (Labeled Redex Multiplicity). *The multiplicity of labeled redexes in a well-labeled term t, denoted $\#(t)$, is defined as follows:*

$$
\begin{array}{rcll}
\#(x) &:=& 0 & \\
\#(\lambda a.\, t) &:=& \#(t) & \\
\#((\lambda a^\alpha.\, t)\mathtt{L}\; s) &:=& 1 + \#((\lambda a.\, t)\mathtt{L}) + \#(s) + \#_a(t) \times \#(s) & \\
\#(t\; s) &:=& \#(t) + \#(s) & \text{if } t\; s \text{ not a redex} \\
\#(\bullet t) &:=& \#(t) & \\
\#(\mathtt{req}(t)) &:=& \#(t) & \\
\#(\mathtt{req}^\alpha((\bullet t)\mathtt{L})) &:=& 1 + \#((\bullet t)\mathtt{L}) & \\
\#(!t) &:=& \#(t) & \\
\#(t[u^\alpha/s]) &:=& 1 + \#(t) + \#(s) & \\
\#(t[u/s]) &:=& \#(t) + \#(s) + \#_u(t) \times \#(s) + \#_u(t) &
\end{array}
$$

$$
\begin{array}{rcl}
\#^\phi(\Box) &:=& 0 \\
\#^\phi(\mathtt{L}[u^\alpha/s]) &:=& 1 + \#^\phi(\mathtt{L}) + \#(s) \\
\#^\phi(\mathtt{L}[u/s]) &:=& \#^\phi(\mathtt{L}) + \#(s) + \#^\phi_u(\mathtt{L}) \times \#(s) + \#^\phi_u(\mathtt{L})
\end{array}
$$

Lemma D.2. *Let $t, s \in \mathsf{T}^{\mathcal{L}}_\bullet$. Then $t\{a := s\} = t$ if $a \notin \mathsf{fv}(t)$.*

Proof. By induction on t.

Lemma D.3. $\#_x(t) = 0$ *if $x \notin \mathsf{fv}(t)$.*

Proof. By induction on t.

Below we use the notation $\#_\bullet(t)$ for the function that maps a variable x to $\#_x(t)$.

Lemma D.4. $\#_x(t\mathtt{L}) = \#_x^{\#_\bullet(t)}(\mathtt{L})$

Proof. By induction on $\mathtt{L}$.

1. $\mathtt{L} = \Box$. Immediate since $\#_x^{\#_\bullet(t)}(\Box) = \#_x(t)$.
2. $\mathtt{L} = \mathtt{L}_1[u^{(\alpha)}/s]$.

$$\begin{array}{ll} \#_x(t\mathtt{L}_1[u^{(\alpha)}/s]) & \\ = \#_x(t\mathtt{L}_1) + \#_x(s) + \#_u(t\mathtt{L}_1) \times \#_x(s) & (Def.\ D.5) \\ = \#_x^{\#_\bullet(t)}(\mathtt{L}_1) + \#_x(s) + \#_u^{\#_\bullet(t)}(\mathtt{L}_1) \times \#_x(s) & (IH \times 2) \\ = \#_x^{\#_\bullet(t)}(\mathtt{L}_1[u^{(\alpha)}/s]) & (Def.\ D.5) \end{array}$$

Lemma D.5. *Suppose* $t \in \mathsf{T}_\bullet^{\mathcal{WL}}$ *and* $x \notin \mathsf{fv}(t) \cup \{u\}$. *Then* $\#_x(\mathtt{C}\langle\!\langle u^\alpha \rangle\!\rangle) = \#_x(\mathtt{C}\langle\!\langle t \rangle\!\rangle)$

Proof. By induction on the size of the labeled context $\mathtt{C}$.

1. $\mathtt{C} = \Box$. Immediate from Lem. D.3.
2. $\mathtt{C} = \lambda a.\, \mathtt{C}_1$ (the cases $\mathtt{C} = \bullet\mathtt{C}_1$ and $\mathtt{C} = !\mathtt{C}_1$ are similar).

$$\begin{array}{ll} \#_x(\lambda a.\, \mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle) & \\ = \#_x(\mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle) & (Def.\ D.5) \\ = \#_x(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle) & (IH) \\ = \#_x(\lambda a.\, \mathtt{C}_1\langle\!\langle t \rangle\!\rangle) & (Def.\ D.5) \end{array}$$

3. $\mathtt{C} = \lambda a^\alpha.\, \mathtt{C}_1$. Not possible since $\mathtt{C}\langle\!\langle u^\alpha \rangle\!\rangle$ is well-labeled.
4. $\mathtt{C} = s\, \mathtt{C}_1$ (the case $\mathtt{C} = \mathtt{C}_1\ s$ is similar).
   4.1. $s \neq (\lambda a^\beta.\, r)\mathtt{L}$.

$$\begin{array}{ll} \#_x(s\, \mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle) & \\ = \#_x(s) + \#_x(\mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle) & (Def.\ D.5) \\ = \#_x(s) + \#_x(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle) & (IH) \\ = \#_x(s\, \mathtt{C}_1\langle\!\langle t \rangle\!\rangle) & (Def.\ D.5) \end{array}$$

   4.2. $s = (\lambda a^\beta.\, r)\mathtt{L}$.

$$\begin{array}{ll} \#_x((\lambda a^\beta.\, r)\mathtt{L}\, \mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle) & \\ = \#_x((\lambda a.\, r)\mathtt{L}) + \#_x(\mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle) + \#_a(r) \times \#_x(\mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle) & (Def.\ D.5) \\ = \#_x((\lambda a.\, r)\mathtt{L}) + \#_x(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle) + \#_a(r) \times \#_x(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle) & (IH \times 2) \\ = \#_x((\lambda a^\beta.\, r)\mathtt{L}\, \mathtt{C}_1\langle\!\langle t \rangle\!\rangle) & (Def.\ D.5) \end{array}$$

5. $\mathtt{C} = \mathtt{C}_1\ s$. We consider two cases depending on whether $\mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle$ is a labeled abstraction or not.
   5.1. $\mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle = (\lambda a^\beta.\, r)\mathtt{L}$. Then one of the following two cases hold:
      5.1.1. $\mathtt{C} = (\lambda a^\beta.\, \mathtt{C}_1)\mathtt{L}\ s$

$$\begin{array}{ll} \#_x((\lambda a^\beta.\, \mathtt{C}_{11}\langle\!\langle u^\alpha \rangle\!\rangle)\mathtt{L}\ s) & \\ = \#_x((\lambda a.\, \mathtt{C}_{11}\langle\!\langle u^\alpha \rangle\!\rangle)\mathtt{L}) + \#_x(s) + \#_a(\mathtt{C}_{11}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#_x(s) & (Def.\ D.5) \\ = \#_x((\lambda a.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}) + \#_x(s) + \#_a(\mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle) \times \#_x(s) & (IH \times 2) \\ = \#_x((\lambda a^\beta.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}\ s) & (Def.\ D.5) \end{array}$$

5.1.2. $\mathtt{C} = (\lambda a^{\beta}.\, r)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_{11}]\mathtt{L}_2\ s$

$$
\begin{array}{ll}
\quad \#_x((\lambda a^{\beta}.\, r)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_{11}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{L}_2\ s) & \\
= \#_x((\lambda a.\, r)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_{11}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{L}_2) + \#_x(s) + \#_a(r) \times \#_x(s) & (Def.\ D.5) \\
= \#_x^{\#_\bullet((\lambda a.\, r)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_{11}\langle\!\langle u^{\alpha}\rangle\!\rangle])}(\mathtt{L}_2) + \#_x(s) + \#_a(r) \times \#_x(s) & (Lem.\ D.4) \\
= \#_x^{\#_\bullet((\lambda a.\, r)\mathtt{L}_1)+\#_\bullet(\mathtt{C}_{11}\langle\!\langle u^{\alpha}\rangle\!\rangle)+\#_v((\lambda a.\, r)\mathtt{L}_1)\times\#_\bullet(\mathtt{C}_{11}\langle\!\langle u^{\alpha}\rangle\!\rangle)}(\mathtt{L}_2) + \#_x(s) + \#_a(r) \times \#_x(s) & (Def.\ D.5) \\
= \#_x^{\#_\bullet((\lambda a.\, r)\mathtt{L}_1)+\#_\bullet(\mathtt{C}_{11}\langle\!\langle t\rangle\!\rangle)+\#_v((\lambda a.\, r)\mathtt{L}_1)\times\#_\bullet(\mathtt{C}_{11}\langle\!\langle t\rangle\!\rangle)}(\mathtt{L}_2) + \#_x(s) + \#_a(r) \times \#_x(s) & (IH \times 2) \\
= \#_x^{\#_\bullet((\lambda a.\, r)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_{11}\langle\!\langle t\rangle\!\rangle])}(\mathtt{L}_2) + \#_x(s) + \#_a(r) \times \#_x(s) & (Def.\ D.5) \\
= \#_x((\lambda a.\, r)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_{11}\langle\!\langle t\rangle\!\rangle]\mathtt{L}_2) + \#_x(s) + \#_a(r) \times \#_x(s) & (Lem.\ D.4) \\
= \#_x((\lambda a^{\beta}.\, r)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_{11}\langle\!\langle t\rangle\!\rangle]\mathtt{L}_2\ s) & (Def.\ D.5)
\end{array}
$$

5.2. $\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle \neq (\lambda a^{\beta}.\, r)\mathtt{L}$. Then also $\mathtt{C}_1\langle\!\langle t\rangle\!\rangle \neq (\lambda a^{\beta}.\, \mathtt{C}_1)\mathtt{L}$ since t well-labeled. Thus we have:

$$
\begin{array}{ll}
\quad \#_x(\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle\ s) & \\
= \#_x(\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle) + \#_x(s) & (Def.\ D.5) \\
= \#_x(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle) + \#_x(s) & (IH) \\
= \#_x(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle\ s) & (Def.\ D.5)
\end{array}
$$

6. $\mathtt{C} = \mathtt{req}^{(\beta)}(\mathtt{C}_1)$. We need to consider multiple cases since out induction proceeds on well-formed labeled contexts.

6.1. β is not present.

$$
\begin{array}{ll}
\quad \#_x(\mathtt{req}(\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle)) & \\
= \#_x(\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle) & (Def.\ D.5) \\
= \#_x(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle) & (IH) \\
= \#_x(\mathtt{req}(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle)) & (Def.\ D.5)
\end{array}
$$

6.2. β is present. Note that $\mathtt{C}_1 = \Box$ and $\mathtt{C}_1 = \Box\mathtt{L}_1$ (with $\mathtt{L} = \mathtt{L}_1\mathtt{L}_2$) is not possible since $\mathtt{C}\langle\!\langle u^{\alpha}\rangle\!\rangle$ is well-labeled. The remaining cases are:

6.2.1. $\mathtt{C}_1 = (\bullet\mathtt{C}_2)\mathtt{L}$.

$$
\begin{array}{ll}
\quad \#_x(\mathtt{req}((\bullet\mathtt{C}_2\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L})) & \\
= \#_x((\bullet\mathtt{C}_2\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L}) & (Def.\ D.5) \\
= \#_x^{\#_\bullet(\bullet\mathtt{C}_2\langle\!\langle u^{\alpha}\rangle\!\rangle)}(\mathtt{L}) & (Lem.\ D.4) \\
= \#_x^{\#_\bullet(\bullet\mathtt{C}_2\langle\!\langle t\rangle\!\rangle)}(\mathtt{L}) & (IH) \\
= \#_x((\bullet\mathtt{C}_2\langle\!\langle t\rangle\!\rangle)\mathtt{L}) & (Lem.\ D.4) \\
= \#_x(\mathtt{req}((\bullet\mathtt{C}_2\langle\!\langle t\rangle\!\rangle)\mathtt{L})) & (Def.\ D.5)
\end{array}
$$

6.2.2. $\mathtt{C}_1 = (\bullet s)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_2]\mathtt{L}_2$

$$
\begin{array}{ll}
\#_x(\mathtt{req}((\bullet s)\mathsf{L}_1[v^{(\gamma)}/\mathsf{C}_2\langle\!\langle u^\alpha\rangle\!\rangle]\mathsf{L}_2)) & \\
= \#_x((\bullet s)\mathsf{L}_1[v^{(\gamma)}/\mathsf{C}_2\langle\!\langle u^\alpha\rangle\!\rangle]\mathsf{L}_2) & (Def.\ D.5) \\
= \#_x^{\#_\bullet((\bullet s)\mathsf{L}_1[v^{(\gamma)}/\mathsf{C}_2\langle\!\langle u^\alpha\rangle\!\rangle])}(\mathsf{L}_2) & (Lem.\ D.4) \\
= \#_x^{\#_\bullet((\bullet s)\mathsf{L}_1)+\#_\bullet(\mathsf{C}_2\langle\!\langle u^\alpha\rangle\!\rangle)+\#_v((\bullet s)\mathsf{L}_1)\times\#_\bullet(\mathsf{C}_2\langle\!\langle u^\alpha\rangle\!\rangle)}(\mathsf{L}_2) & (Def.\ D.5) \\
= \#_x^{\#_\bullet((\bullet s)\mathsf{L}_1)+\#_\bullet(\mathsf{C}_2\langle\!\langle t\rangle\!\rangle)+\#_v((\bullet s)\mathsf{L}_1)\times\#_\bullet(\mathsf{C}_2\langle\!\langle t\rangle\!\rangle)}(\mathsf{L}_2) & (IH\times 2) \\
= \#_x^{\#_\bullet((\bullet s)\mathsf{L}_1[v^{(\gamma)}/\mathsf{C}_2\langle\!\langle t\rangle\!\rangle])}(\mathsf{L}_2) & (Def.\ D.5) \\
= \#_x((\bullet s)\mathsf{L}_1[v^{(\gamma)}/\mathsf{C}_2\langle\!\langle t\rangle\!\rangle]\mathsf{L}_2) & (Lem.\ D.4) \\
= \#_x(\mathtt{req}((\bullet s)\mathsf{L}_1[v^{(\gamma)}/\mathsf{C}_2\langle\!\langle t\rangle\!\rangle]\mathsf{L}_2)) & (Def.\ D.5)
\end{array}
$$

7. $\mathsf{C} = \mathsf{C}_1[w^{(\beta)}/s]$

$$
\begin{array}{ll}
\#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[w^{(\beta)}/s]) & \\
= \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_x(s) + \#_w(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle)\times\#_x(s) & (Def.\ D.5) \\
= \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_x(s) + \#_w(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle)\times\#_x(s) & (IH\times 2) \\
= \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[w^{(\beta)}/s]) & (Def.\ D.5)
\end{array}
$$

8. $\mathsf{C} = s[w^{(\beta)}/\mathsf{C}_1]$

$$
\begin{array}{ll}
\#_x(s[w^{(\beta)}/\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle]) & \\
= \#_x(s) + \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_w(s)\times\#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) & (Def.\ D.5) \\
= \#_x(s) + \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_w(s)\times\#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) & (IH\times 2) \\
= \#_x(s[w^{(\beta)}/\mathsf{C}_1\langle\!\langle t\rangle\!\rangle]) & (Def.\ D.5)
\end{array}
$$

Lemma D.6. *Let $t, s \in \mathsf{T}_\bullet^{\mathcal{WL}}$. Suppose $x \notin \mathsf{fv}(s) \cup \{a\}$.*

1. *Then $\#_x(t\{a := s\}) = \#_x(t)$.*
2. *Suppose also $\mathsf{dom}(\mathsf{L}) \cap \mathsf{fv}(s) = \varnothing$. Then $\#_x^{\#_\bullet(t\{a:=s\})}(\mathsf{L}) = \#_x^{\#_\bullet(t)}(\mathsf{L})$.*

Proof. The first item is by induction on t.

1. $t = b$. If $b \neq a$, then $b\{a := s\} = b$ and we conclude immediately. If $b = a$, then

$$
\begin{array}{ll}
\#_x(a\{a := s\}) & \\
= \#_x(s) & \\
= 0 & (x \notin \mathsf{fv}(s), Lem.\ D.2) \\
= \#_x(a) & (x \neq a)
\end{array}
$$

2. $t = u^{(\alpha)}$. Immediate since $u^{(\alpha)}\{a := s\} = u^{(\alpha)}$
3. $t = \lambda b.\, r$ (the cases $t = t_1\, t_2$ and $t = \bullet t_1$ and $t = !t_1$ and $t = \mathtt{req}^{(\alpha)}(t_1)$ are similar).

$$
\begin{array}{ll}
\#_x((\lambda b.\, r)\{a := s\}) & \\
= \#_x(\lambda b.\, r\{a := s\}) & \\
= \#_x(r\{a := s\}) & (Def.\ D.5) \\
= \#_x(r) & (IH) \\
= \#_x(\lambda b.\, r) & (Def.\ D.5)
\end{array}
$$

4. $t = (\lambda b^{\alpha}.\, t_1)\mathtt{L}\, t_2$

$$
\begin{array}{ll}
\#_x(((\lambda b^{\alpha}.\, t_1)\mathtt{L}\, t_2)\{a := s\}) & \\
= \#_x((\lambda b^{\alpha}.\, t_1\{a := s\})\mathtt{L}\{a := s\}\, t_2\{a := s\}) & \\
= \#_x((\lambda b.\, t_1\{a := s\})\mathtt{L}\{a := s\}) + \#_x(t_2\{a := s\}) + \#_b(t_1\{a := s\}) \times \#_x(t_2\{a := s\}) & (Def.\ D.5) \\
= \#_x(((\lambda b.\, t_1)\mathtt{L})\{a := s\}) + \#_x(t_2\{a := s\}) + \#_b(t_1\{a := s\}) \times \#_x(t_2\{a := s\}) & \\
= \#_x((\lambda b.\, t_1)\mathtt{L}) + \#_x(t_2) + \#_b(t_1) \times \#_x(t_2) & (IH \times 4) \\
= \#_x((\lambda b^{\alpha}.\, t_1)\mathtt{L}\, t_2) & (Def.\ D.5)
\end{array}
$$

5. $t = t_1[u^{(\alpha)}/t_2]$

$$
\begin{array}{ll}
\#_x(t_1[u^{(\alpha)}/t_2]\{a := s\}) & \\
= \#_x(t_1\{a := s\}[u^{(\alpha)}/t_2\{a := s\}]) & \\
= \#_x(t_1\{a := s\}) + \#_x(t_2\{a := s\}) + \#_u(t_1\{a := s\}) \times \#_x(t_2\{a := s\}) & \\
= \#_x(t_1) + \#_x(t_2) + \#_u(t_1) \times \#_x(t_2) & (IH \times 4) \\
= \#_x(t_1[u^{(\alpha)}/t_2]) & (Def.\ D.5)
\end{array}
$$

The second item is by induction on $\mathtt{L}$ and uses the first item.

1. $\mathtt{L} = \Box$.

$$
\begin{array}{ll}
\#_x^{\#_\bullet(t\{a:=s\})}(\Box) & \\
= \#_x(t\{a := s\}) & (Def.\ D.5) \\
= \#_x(t) & (\text{item 1}) \\
= \#_x^{\#_\bullet(t)}(\Box) & (Def.\ D.5)
\end{array}
$$

2. $\mathtt{L} = \mathtt{L}_1[u^{(\alpha)}/r]$.

$$
\begin{array}{ll}
\#_x^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1[u^{(\alpha)}/r]) & \\
= \#_x^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1) + \#_x(r) + \#_u^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1) \times \#_x(r) & (Def.\ D.5) \\
= \#_x^{\#_\bullet(t)}(\mathtt{L}_1) + \#_x(r) + \#_u^{\#_\bullet(t)}(\mathtt{L}_1) \times \#_x(r) & (IH \times 2) \\
= \#_x^{\#_\bullet(t)}(\mathtt{L}_1[u^{(\alpha)}/r]) & (Def.\ D.5)
\end{array}
$$

Lemma D.7. *Suppose $x \neq a$. Then $\#_x(t) + \#_a(t) \times \#_x(s) = \#_x(t\{a := s\})$*

Proof. By induction on t.

1. $t = b$. If $b \neq a$, then

$$
\begin{array}{l}
\#_x(b) + \#_a(b) \times \#_x(s) \\
= \#_x(b) \\
= \#_x(b\{a := s\})
\end{array}
$$

If $b = a$, then

$$
\begin{array}{ll}
\#_x(b) + \#_a(b) \times \#_x(s) & \\
= \#_x(s) & (x \neq a) \\
= \#_x(b\{a := s\}) &
\end{array}
$$

2. $t = u^{(\alpha)}$

$$\begin{aligned} & \#_x(u^{(\alpha)}) + \#_a(u^{(\alpha)}) \times \#_x(s) \\ = & \#_x(u^{(\alpha)}) \\ = & \#_x(u^{(\alpha)}\{a := s\}) \end{aligned}$$

3. $t = \lambda b.\, r$.

$$\begin{aligned} & \#_x(\lambda b.\, r) + \#_a(\lambda b.\, r) \times \#_x(s) & \\ = & \#_x(r) + \#_a(r) \times \#_x(s) & (Def.\ D.5) \\ = & \#_x(r\{a := s\}) & (IH) \\ = & \#_x(\lambda b.\, (r\{a := s\})) & (Def.\ D.5) \\ = & \#_x((\lambda b.\, r)\{a := s\}) & \end{aligned}$$

4. $t = t_1\, t_2$.

$$\begin{aligned} & \#_x(t_1\, t_2) + \#_a(t_1\, t_2) \times \#_x(s) & \\ = & \#_x(t_1) + \#_x(t_2) + (\#_a(t_1) + \#_a(t_2)) \times \#_x(s) & (Def.\ D.5) \\ = & \#_x(t_1) + \#_x(t_2) + \#_a(t_1) \times \#_x(s) + \#_a(t_2) \times \#_x(s) & \\ = & \#_x(t_1\{a := s\}) + \#_x(t_2\{a := s\}) & (IH \times 2) \\ = & \#_x(t_1\{a := s\}\, t_2\{a := s\}) & (Def.\ D.5) \\ = & \#_x((t_1\, t_2)\{a := s\}) & \end{aligned}$$

5. $t = (\lambda b^{\alpha}.\, t_1)\mathtt{L}\, t_2$.

$$\begin{aligned} & \#_x((\lambda b^{\alpha}.\, t_1)\mathtt{L}\, t_2) + \#_a((\lambda b^{\alpha}.\, t_1)\mathtt{L}\, t_2) \times \#_x(s) & \\ = & \#_x((\lambda b.\, t_1)\mathtt{L}) + \#_x(t_2) + \#_b(t_1) \times \#_x(t_2) + \#_a((\lambda b^{\alpha}.\, t_1)\mathtt{L}\, t_2) \times \#_x(s) & (Def.\ D.5) \\ = & \#_x((\lambda b.\, t_1)\mathtt{L}) + \#_x(t_2) + \#_b(t_1) \times \#_x(t_2) + (\#_a((\lambda b.\, t_1)\mathtt{L}) + \#_a(t_2) + \#_b(t_1) \times \#_a(t_2)) \times \#_x(s) & (Def.\ D.5) \\ = & \#_x(((\lambda b.\, t_1)\mathtt{L})\{a := s\}) + \#_x(t_2) + \#_b(t_1) \times \#_x(t_2) + (\#_a(t_2) + \#_b(t_1) \times \#_a(t_2)) \times \#_x(s) & (IH) \\ = & \#_x(((\lambda b.\, t_1)\mathtt{L})\{a := s\}) + \#_x(t_2\{a := s\}) + \#_b(t_1) \times \#_x(t_2) + \#_b(t_1) \times \#_a(t_2) \times \#_x(s) & (IH) \\ = & \#_x((\lambda b.\, t_1\{a := s\})\mathtt{L}\{a := s\}) + \#_x(t_2\{a := s\}) + \#_b(t_1) \times \#_x(t_2\{a := s\}) & (Def.\ D.5) \\ = & \#_x((\lambda b.\, t_1\{a := s\})\mathtt{L}\{a := s\}) + \#_x(t_2\{a := s\}) + \#_b(t_1\{a := s\}) \times \#_x(t_2\{a := s\}) & (Lem.\ D.6(1)) \\ = & \#_x((\lambda b^{\alpha}.\, t_1\{a := s\})\mathtt{L}\{a := s\}\, t_2\{a := s\}) & (Def.\ D.5) \\ = & \#_x(((\lambda b^{\alpha}.\, t_1)\mathtt{L}\, t_2)\{a := s\}) & (Def.\ D.5) \end{aligned}$$

6. $t = \bullet t_1$.

$$\begin{aligned} & \#_x(\bullet t_1) + \#_a(\bullet t_1) \times \#_x(s) & \\ = & \#_x(t_1) + \#_a(t_1) \times \#_x(s) & (Def.\ D.5) \\ = & \#_x(t_1\{a := s\}) & (IH) \\ = & \#_x(\bullet t_1\{a := s\}) & (Def.\ D.5) \\ = & \#_x((\bullet t_1)\{a := s\}) & \end{aligned}$$

7. $t = \mathtt{req}^{(\alpha)}(t_1)$.

$$\begin{aligned} & \#_x(\mathtt{req}^{(\alpha)}(t_1)) + \#_a(\mathtt{req}^{(\alpha)}(t_1)) \times \#_x(s) & \\ = & \#_x(t_1) + \#_a(t_1) \times \#_x(s) & (Def.\ D.5) \\ = & \#_x(t_1\{a := s\}) & (IH) \\ = & \#_x(\mathtt{req}^{(\alpha)}(t_1\{a := s\})) & (Def.\ D.5) \\ = & \#_x((\mathtt{req}^{(\alpha)}(t_1))\{a := s\}) & \end{aligned}$$

8. $t = !t_1$. Same as case $t = \bullet t_1$.
9. $t = t_1[u^{(\alpha)}/t_2]$

$$\begin{array}{ll}
\#_x(t_1[u^{(\alpha)}/t_2]) + \#_a(t_1[u^{(\alpha)}/t_2]) \times \#_x(s) & \\
= \#_x(t_1) + \#_x(t_2) + \#_u(t_1) \times \#_x(t_2) + \#_a(t_1[u^{(\alpha)}/t_2]) \times \#_x(s) & (Def.\ D.5) \\
= \#_x(t_1) + \#_x(t_2) + \#_u(t_1) \times \#_x(t_2) + (\#_a(t_1) + \#_a(t_2) + \#_u(t_1) \times \#_a(t_2)) \times \#_x(s) & (Def.\ D.5) \\
= \#_x(t_1\{a := s\}) + \#_x(t_2) + \#_u(t_1) \times \#_x(t_2) + (\#_a(t_2) + \#_u(t_1) \times \#_a(t_2)) \times \#_x(s) & (IH) \\
= \#_x(t_1\{a := s\}) + \#_x(t_2\{a := s\}) + \#_u(t_1) \times \#_x(t_2) + \#_u(t_1) \times \#_a(t_2) \times \#_x(s) & (IH) \\
= \#_x(t_1\{a := s\}) + \#_x(t_2\{a := s\}) + \#_u(t_1) \times \#_x(t_2\{a := s\}) & (IH) \\
= \#_x(t_1\{a := s\}) + \#_x(t_2\{a := s\}) + \#_u(t_1\{a := s\}) \times \#_x(t_2\{a := s\}) & (Lem.\ D.6(1)) \\
= \#_x(t_1\{a := s\}[u^{(\alpha)}/t_2\{a := s\}]) & (Def.\ D.5) \\
= \#_x(t_1[u^{(\alpha)}/t_2]\{a := s\}) &
\end{array}$$

Lemma D.8. *Suppose* $\mathsf{dom}(\mathsf{L}) \cap \mathsf{fv}(s) = \emptyset$. *Then*

$$\#_x((\lambda a.\, t)\mathsf{L}) + \#_a(t) \times \#_x(s) = \#_x^{\#_\bullet(t\{a:=s\})}(\mathsf{L})$$

Proof. By induction on L. Without loss of generality, we assume $x \neq a$.

1. $\mathsf{L} = \Box$.

$$\begin{array}{ll}
\#_x(\lambda a.\, t) + \#_a(t) \times \#_x(s) & \\
= \#_x(t) + \#_a(t) \times \#_x(s) & (Def.\ D.5) \\
= \#_x(t\{a := s\}) & (Lem.\ D.7) \\
= \#_x^{\#_\bullet(t\{a:=s\})}(\Box) & (Def.\ D.5)
\end{array}$$

2. $\mathsf{L} = \mathsf{L}_1[u^{(\alpha)}/r]$.

$$\begin{array}{ll}
\#_x((\lambda a.\, t)\mathsf{L}_1[u^{(\alpha)}/r]) + \#_a(t) \times \#_x(s) & \\
= \#_x((\lambda a.\, t)\mathsf{L}_1) + \#_x(r) + \#_u((\lambda a.\, t)\mathsf{L}_1) \times \#_x(r) + \#_a(t) \times \#_x(s) & (Def.\ D.5) \\
= \#_x^{\#_\bullet(t\{a:=s\})}(\mathsf{L}_1) + \#_x(r) + \#_u((\lambda a.\, t)\mathsf{L}_1) \times \#_x(r) & (IH) \\
= \#_x^{\#_\bullet(t\{a:=s\})}(\mathsf{L}_1) + \#_x(r) + \#_u^{\#_\bullet(\lambda a.\, t)}(\mathsf{L}_1) \times \#_x(r) & (Lem.\ D.4) \\
= \#_x^{\#_\bullet(t\{a:=s\})}(\mathsf{L}_1) + \#_x(r) + \#_u^{\#_\bullet(t)}(\mathsf{L}_1) \times \#_x(r) & (Def.\ D.5) \\
= \#_x^{\#_\bullet(t\{a:=s\})}(\mathsf{L}_1) + \#_x(r) + \#_u^{\#_\bullet(t\{a:=s\})}(\mathsf{L}_1) \times \#_x(r) & (Lem.\ D.6(2)) \\
= \#_x^{\#_\bullet(t\{a:=s\})}(\mathsf{L}_1[u^{(\alpha)}/r]) & (Def.\ D.5)
\end{array}$$

Lemma D.9. *Suppose* $\mathsf{dom}(\mathsf{L}) \cap \mathsf{fv}(t) = \emptyset$. *For any s,*

1. $\#_x(t\mathsf{L}) \leq \#_x(t) + \#_x(s\mathsf{L})$.
2. $\#(t\mathsf{L}) \leq \#(t) + \#^{\#_\bullet(s)}(\mathsf{L})$

Proof. Both items are by induction on L. For the first item we have:

1. $\mathsf{L} = \Box$. Then $\#_x(t) \leq \#_x(t) + \#_x(s)$ is immediate.

2. $\mathtt{L} = \mathtt{L}_1[u^{(\alpha)}/r]$

$$\begin{array}{ll} \#_x(t\mathtt{L}_1[u^{(\alpha)}/r]) & \\ = \#_x(t\mathtt{L}_1) + \#_x(r) + \#_u(t\mathtt{L}_1) \times \#_x(r) & (Def.\ D.5) \\ \leq \#_x(t) + \#_x(s\mathtt{L}_1) + \#_x(r) + \#_u(t\mathtt{L}_1) \times \#_x(r) & (IH) \\ \leq \#_x(t) + \#_x(s\mathtt{L}_1) + \#_x(r) + (\#_u(t) + \#_u(s\mathtt{L}_1)) \times \#_x(r) & (IH) \\ = \#_x(t) + \#_x(s\mathtt{L}_1) + \#_x(r) + \#_u(t) \times \#_x(r) + \#_u(s\mathtt{L}_1) \times \#_x(r) & (Def.\ D.5) \\ = \#_x(t) + \#_x(s\mathtt{L}) + \#_u(t) \times \#_x(r) & (Def.\ D.5) \\ = \#_x(t) + \#_x(s\mathtt{L}) & (u \notin \mathsf{fv}(t)) \end{array}$$

For the second item we proceed as follows:

1. $\mathtt{L} = \Box$. Then $\#(t) = \#(t) + 0 = \#(t) + \#^{\#_\bullet(s)}(\Box)$.
2. $\mathtt{L} = \mathtt{L}_1[u/r]$

$$\begin{array}{ll} \#(t\mathtt{L}_1[u/r]) & \\ = \#(t\mathtt{L}_1) + \#(r) + \#_u(t\mathtt{L}_1) \times \#(r) + \#_u(t\mathtt{L}_1) & (Def.\ D.5) \\ \leq \#(t) + \#^{\#_\bullet(s)}(\mathtt{L}_1) + \#(r) + \#_u(t\mathtt{L}_1) \times \#(r) + \#_u(t\mathtt{L}_1) & (IH) \\ \leq \#(t) + \#^{\#_\bullet(s)}(\mathtt{L}_1) + \#(r) + (\#_u(t) + \#_u(s\mathtt{L}_1)) \times \#(r) + \#_u(t\mathtt{L}_1) & (\text{item } 1) \\ \leq \#(t) + \#^{\#_\bullet(s)}(\mathtt{L}_1) + \#(r) + (\#_u(t) + \#_u^{\#_\bullet(s)}(\mathtt{L}_1)) \times \#(r) + \#_u(t\mathtt{L}_1) & (Lem.\ D.4) \\ = \#(t) + \#^{\#_\bullet(s)}(\mathtt{L}_1) + \#(r) + \#_u^{\#_\bullet(s)}(\mathtt{L}_1) \times \#(r) + \#_u(t\mathtt{L}_1) & (u \notin \mathsf{fv}(t)) \\ \leq \#(t) + \#^{\#_\bullet(s)}(\mathtt{L}_1) + \#(r) + \#_u^{\#_\bullet(s)}(\mathtt{L}_1) \times \#(r) + \#_u(t) + \#_u(s\mathtt{L}_1) & (\text{item } 1) \\ \leq \#(t) + \#^{\#_\bullet(s)}(\mathtt{L}_1) + \#(r) + \#_u^{\#_\bullet(s)}(\mathtt{L}_1) \times \#(r) + \#_u(s\mathtt{L}_1) & (u \notin \mathsf{fv}(t)) \\ = \#(t) + \#^{\#_\bullet(s)}(\mathtt{L}_1) + \#(r) + \#_u^{\#_\bullet(s)}(\mathtt{L}_1) \times \#(r) + \#_u^{\#_\bullet(s)}(\mathtt{L}_1) & (Lem.\ D.4) \\ = \#(t) + \#^{\#_\bullet(s)}(\mathtt{L}_1[u/r]) & (Def.\ D.5) \end{array}$$

3. $\mathtt{L} = \mathtt{L}_1[u^\alpha/r]$

$$\begin{array}{ll} \#(t\mathtt{L}_1[u^\alpha/r]) & \\ = 1 + \#(t\mathtt{L}_1) + \#(r) & (Def.\ D.5) \\ \leq \#(t) + 1 + \#^{\#_\bullet(s)}(\mathtt{L}_1) + \#(r) & (IH) \\ = \#(t) + \#^{\#_\bullet(s)}(\mathtt{L}_1[u^\alpha/r]) & (Def.\ D.5) \end{array}$$

Lemma D.10. *Let $x \neq u$ and $u \notin \mathsf{fv}(t)$. Then*

$$\#_x(\mathsf{C}\langle\langle u^\alpha \rangle\rangle) + \#_u(\mathsf{C}\langle\langle u^\alpha \rangle\rangle) \times \#_x(t) = \#_x(\mathsf{C}\langle\langle t \rangle\rangle) + \#_u(\mathsf{C}\langle\langle t \rangle\rangle) \times \#_x(t)$$

Proof. By induction on the size of the labeled context C.

1. $\mathsf{C} = \Box$.

$$\begin{array}{ll} \#_x(u^\alpha) + \#_u(u^\alpha) \times \#_x(t) & \\ = 0 + \#_x(t) & (x \neq u) \\ = \#_x(t) + 0 \times \#_x(t) & \\ = \#_x(t) + \#_u(t) \times \#_x(t) & (Lem.\ D.3) \end{array}$$

2. $\mathsf{C} = \lambda a.\, \mathsf{C}_1$

$$\begin{array}{ll} \#_x(\lambda a.\, \mathsf{C}_1\langle\langle u^\alpha \rangle\rangle) + \#_u(\lambda a.\, \mathsf{C}_1\langle\langle u^\alpha \rangle\rangle) \times \#_x(t) & \\ = \#_x(\mathsf{C}_1\langle\langle u^\alpha \rangle\rangle) + \#_u(\mathsf{C}_1\langle\langle u^\alpha \rangle\rangle) \times \#_x(t) & (Def.\ D.5) \\ = \#_x(\mathsf{C}_1\langle\langle t \rangle\rangle) + \#_u(\mathsf{C}_1\langle\langle t \rangle\rangle) \times \#_x(t) & (IH) \\ = \#_x(\lambda a.\, \mathsf{C}_1\langle\langle t \rangle\rangle) + \#_u(\lambda a.\, \mathsf{C}_1\langle\langle t \rangle\rangle) \times \#_x(t) & (Def.\ D.5) \end{array}$$

3. $\mathtt{C} = \lambda a^{\beta}.\, \mathtt{C}_1$. Not possible since $\mathtt{C}\langle\!\langle u^{\alpha} \rangle\!\rangle$ is well-labeled.
4. $\mathtt{C} = \mathtt{C}_1\ s$. We consider two cases depending on whether $\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle$ is a labeled abstraction or not.
   4.1. $\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle = (\lambda a^{\beta}.\, r)\mathtt{L}$. Then one of the following two cases hold:
      4.1.1. $\mathtt{C} = (\lambda a^{\beta}.\, \mathtt{C}_{11})\mathtt{L}\ s$. We assume without loss of generality, that $a \notin \mathsf{fv}(t)$.

$$
\begin{aligned}
& \#_x((\lambda a^{\beta}.\, \mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L}\ s) + \#_u((\lambda a^{\beta}.\, \mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L}\ s) \times \#_x(t) \\
={} & \#_x((\lambda a.\, \mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L}) + \#_x(s) + \#_a(\mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle) \times \#_x(s) + \#_u((\lambda a^{\beta}.\, \mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L}\ s) \times \#_x(t) \\
={} & \#_x((\lambda a.\, \mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L}) + \#_x(s) + \#_a(\mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle) \times \#_x(s) + (\#_u((\lambda a.\, \mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L}) + \#_u(s) + \#_a(\mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle) \times \#_u(s)) \times \\
={} & \#_x((\lambda a.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}) + \#_u((\lambda a.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}) \times \#_x(t) + \#_x(s) + \#_a(\mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle) \times \#_x(s) + (\#_u(s) + \#_a(\mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle) \times \#_u(\\
={} & \#_x((\lambda a.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}) + \#_u((\lambda a.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}) \times \#_x(t) + \#_x(s) + \#_a(\mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle) \times \#_x(s) + \#_u(s) \times \#_x(t) + \#_a(\mathtt{C}_{11}\langle\!\langle u^{\alpha} \rangle\!\rangle \\
={} & \#_x((\lambda a.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}) + \#_u((\lambda a.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}) \times \#_x(t) + \#_x(s) + \#_a(\mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle) \times \#_x(s) + \#_u(s) \times \#_x(t) + \#_a(\mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle) \times \\
={} & \#_x((\lambda a^{\beta}.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}\ s) + \#_u((\lambda a.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}) \times \#_x(t) + \#_u(s) \times \#_x(t) + \#_a(\mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle) \times \#_u(s) \times \#_x(t) \\
={} & \#_x((\lambda a^{\beta}.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}\ s) + \#_u((\lambda a^{\beta}.\, \mathtt{C}_{11}\langle\!\langle t \rangle\!\rangle)\mathtt{L}\ s) \times \#_x(t)
\end{aligned}
$$

      4.1.2. $\mathtt{C} = (\lambda a^{\beta}.\, r)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_{11}]\mathtt{L}_2\ s$. Similar to the previous case.
   4.2. $\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle \neq (\lambda a^{\beta}.\, r)\mathtt{L}$.

$$
\begin{aligned}
& \#_x(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle\ s) + \#_u(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle\ s) \times \#_x(t) \\
={} & \#_x(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle) + \#_x(s) + (\#_u(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle) + \#_u(s)) \times \#_x(t) && (Def.\ D.5) \\
={} & \#_x(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle) + \#_x(s) + \#_u(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle) \times \#_x(t) + \#_u(s) \times \#_x(t) \\
={} & \#_x(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle) + \#_u(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle) \times \#_x(t) + \#_x(s) + \#_u(s) \times \#_x(t) && (IH) \\
={} & \#_x(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle\ s) + \#_u(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle\ s) \times \#_x(t) && (Def.\ D.5)
\end{aligned}
$$

5. $\mathtt{C} = s\,\mathtt{C}_1$. Two cases are possible.
   5.1. $s = (\lambda a^{\beta}.\, r)\mathtt{L}$. Similar to previous cases above.
   5.2. $s \neq (\lambda a^{\beta}.\, r)\mathtt{L}$. Similar to previous cases above.
6. $\mathtt{C} = \bullet\mathtt{C}_1$. Similar to the case $\mathtt{C} = \lambda a.\, \mathtt{C}_1$.
7. $\mathtt{C} = \mathtt{req}^{(\beta)}(\mathtt{C}_1)$. We need to consider multiple cases since our induction proceeds on well-formed labeled contexts.
   7.1. β is not present. Similar to the case $\mathtt{C} = \lambda a.\, \mathtt{C}_1$.

$$
\begin{aligned}
& \#_x(\mathtt{req}(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle)) + \#_u(\mathtt{req}(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle)) \times \#_x(t) \\
={} & \#_x(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle) + \#_u(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle) \times \#_x(t) && (Def.\ D.5) \\
={} & \#_x(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle) + \#_u(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle) \times \#_x(t) && (IH) \\
={} & \#_x(\mathtt{req}(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle)) + \#_u(\mathtt{req}(\mathtt{C}_1\langle\!\langle t \rangle\!\rangle)) \times \#_x(t) && (Def.\ D.5)
\end{aligned}
$$

   7.2. β is present. Note that $\mathtt{C}_1 = \Box$ and $\mathtt{C}_1 = \Box\mathtt{L}_1$ (with $\mathtt{L} = \mathtt{L}_1\mathtt{L}_2$) is not possible since $\mathtt{C}\langle\!\langle u^{\alpha} \rangle\!\rangle$ is well-labeled. The remaining cases are:
      7.2.1. $\mathtt{C}_1 = (\bullet\mathtt{C}_2)\mathtt{L}$.

$$
\begin{aligned}
& \#_x(\mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L})) + \#_u(\mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L})) \times \#_x(t) \\
={} & \#_x((\bullet\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L}) + \#_u((\bullet\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle)\mathtt{L}) \times \#_x(t) && (Def.\ D.5) \\
={} & \#_x^{\#_\bullet(\bullet\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle)}(\mathtt{L}) + \#_u^{\#_\bullet(\bullet\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle)}(\mathtt{L}) \times \#_x(t) && (Lem.\ D.4) \\
={} & \#_x^{\#_\bullet(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle)}(\mathtt{L}) + \#_u^{\#_\bullet(\mathtt{C}_1\langle\!\langle u^{\alpha} \rangle\!\rangle)}(\mathtt{L}) \times \#_x(t) && (Def.\ D.5) \\
={} & \#_x((\bullet\mathtt{C}_1\langle\!\langle t \rangle\!\rangle)\mathtt{L}) + \#_u((\bullet\mathtt{C}_1\langle\!\langle t \rangle\!\rangle)\mathtt{L}) \times \#_x(t) && (IH) \\
={} & \#_x(\mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle t \rangle\!\rangle)\mathtt{L})) + \#_u(\mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle t \rangle\!\rangle)\mathtt{L})) \times \#_x(t) && (Def.\ D.5)
\end{aligned}
$$

7.2.2. $\mathsf{C}_1 = (\bullet s)\mathsf{L}_1[v^{(\gamma)}/\mathsf{C}_2]\mathsf{L}_2$. Similar to the previous case.

8. $\mathsf{C} = !\mathsf{C}_1$. Similar to the case $\mathsf{C} = \lambda a.\,\mathsf{C}_1$.
9. $\mathsf{C} = \mathsf{C}_1[v^{(\beta)}/s]$

$$\begin{aligned}
&\#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v^{(\beta)}/s]) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v^{(\beta)}/s]) \times \#_x(t) \\
&= \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_x(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_x(s) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v^{(\beta)}/s]) \times \#_x(t) && (Def.\ D.5) \\
&= \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_x(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_x(s) + (\#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \times \#_x(t) && (Def.\ D.5) \\
&= \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + \#_x(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_x(s) + (\#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \times \#_x(t) && (IH) \\
&= \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + \#_x(s) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(s) + (\#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \times \#_x(t) && (Lem.\ D.5, \star) \\
&= \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^{(\beta)}/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + (\#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \times \#_x(t) && (Def.\ D.5) \\
&= \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^{(\beta)}/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + \#_u(s) \times \#_x(t) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s) \times \#_x(t) \\
&= \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^{(\beta)}/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + \#_u(s) \times \#_x(t) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_u(s) \times \#_x(t) && (Lem.\ D.5, \star) \\
&= \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^{(\beta)}/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^{(\beta)}/s]) \times \#_x(t) && (Def.\ D.5)
\end{aligned}$$

Note the use in $\star$ of the fact that $\mathsf{C}\langle\!\langle t\rangle\!\rangle$ stands for the capture-avoiding replacement of $\Box$ in C by t. In particular, $v \notin \mathsf{fv}(t)$ and $u \neq v$.

10. $\mathsf{C} = s[v^{(\beta)}/\mathsf{C}_1]$

$$\begin{aligned}
&\#_x(s[v^{(\beta)}/\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle]) + \#_u(s[v^{(\beta)}/\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle]) \times \#_x(t) \\
&= \#_x(s) + \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_v(s) \times \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s[v^{(\beta)}/\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle]) \times \#_x(t) && (Def. \\
&= \#_x(s) + \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_v(s) \times \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + (\#_u(s) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_v(s) \times \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle)) \times \#_x(t) && (Def. \\
&= \#_x(s) + \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_v(s) \times \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s) \times \#_x(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_x(t) + \#_v(s) \times \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_x(t) \\
&= \#_x(s) + \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_v(s) \times (\#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t)) + \#_u(s) \times \#_x(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_x(t) \\
&= \#_x(s) + \#_x(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_v(s) \times \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_v(s) \times \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + \#_u(s) \times \#_x(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_x(t) \\
&= \#_x(s) + \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + \#_v(s) \times \#_x(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_v(s) \times \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + \#_u(s) \times \#_x(t) \\
&= \#_x(s[v^{(\beta)}/\mathsf{C}_1\langle\!\langle t\rangle\!\rangle]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + \#_v(s) \times \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_x(t) + \#_u(s) \times \#_x(t) \\
&= \#_x(s[v^{(\beta)}/\mathsf{C}_1\langle\!\langle t\rangle\!\rangle]) + \#_u(s[v^{(\beta)}/\mathsf{C}_1\langle\!\langle t\rangle\!\rangle]) \times \#_x(t) && (Def.
\end{aligned}$$

Lemma D.11. *Suppose* $\mathsf{dom}(\mathsf{L}) \cap (\mathsf{fv}(\mathsf{C}) \cup \{u\}) = \varnothing$ *and* $x \neq u$ *and* $u \notin \mathsf{fv}(t)$.

$$\#_x(\mathsf{C}\langle\!\langle u^\alpha\rangle\!\rangle) + \#_x(t\mathsf{L}) + \#_u(\mathsf{C}\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_x(t\mathsf{L}) \geq \#_x^{\#_\bullet(\mathsf{C}\langle\!\langle t\rangle\!\rangle[u/t])}(\mathsf{L})$$

Proof. By induction on L.

1. $\mathsf{L} = \Box$.

$$\begin{aligned}
&\#_x(\mathsf{C}\langle\!\langle u^\alpha\rangle\!\rangle) + \#_x(t) + \#_u(\mathsf{C}\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_x(t) \\
&= \#_x(\mathsf{C}\langle\!\langle t\rangle\!\rangle) + \#_x(t) + \#_u(\mathsf{C}\langle\!\langle t\rangle\!\rangle) \times \#_x(t) && (Lem.\ D.10) \\
&= \#_x(\mathsf{C}\langle\!\langle t\rangle\!\rangle[u/t]) && (Def.\ D.5) \\
&= \#_x^{\#_\bullet(\mathsf{C}\langle\!\langle t\rangle\!\rangle[u/t])}(\Box) && (Def.\ D.5)
\end{aligned}$$

2. $\mathsf{L} = \mathsf{L}_1[v^{(\alpha)}/s]$

$$
\begin{aligned}
& \#_x(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#_x(t\mathsf{L}_1[v^{(\alpha)}/s]) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#_x(t\mathsf{L}_1[v^{(\alpha)}/s]) \\
= {} & \#_x(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#_x(t\mathsf{L}_1) + \#_x(s) + \#_v(t\mathsf{L}_1) \times \#_x(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#_x(t\mathsf{L}_1[v^{(\alpha)}/s]) \\
= {} & \#_x(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#_x(t\mathsf{L}_1) + \#_x(s) + \#_v(t\mathsf{L}_1) \times \#_x(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times (\#_x(t\mathsf{L}_1) + \#_x(s) + \#_v(t\mathsf{L}_1) \times \#_x(s)) \\
\geq {} & \#_x^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathsf{L}_1) + \#_x(s) + \#_v(t\mathsf{L}_1) \times \#_x(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times (\#_x(s) + \#_v(t\mathsf{L}_1) \times \#_x(s)) && (IH) \\
= {} & \#_x^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathsf{L}_1) + \#_x(s) + \#_v(t\mathsf{L}_1) \times \#_x(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#_x(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#_v(t\mathsf{L}_1) \times \#_x(s) \\
> {} & \#_x^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathsf{L}_1) + \#_x(s) + \#_v(t\mathsf{L}_1) \times \#_x(s) + \#_v(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#_x(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#_v(t\mathsf{L}_1) \times \#_x(s) && (\mathsf{dom}(\mathsf{L}) \cap (\mathsf{fv}(\mathsf{C}) \\
\geq {} & \#_x^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathsf{L}_1) + \#_x(s) + \#_v^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathsf{L}_1) \times \#_x(s) && (IH) \\
= {} & \#_x^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathsf{L}_1[v^{(\alpha)}/s]) && (Def.\ D.5)
\end{aligned}
$$

Proposition D.1. *Let $t \in \mathsf{T}_\bullet^{\mathcal{WL}}$. Then $t \to_R^\alpha s$ implies $\#_x(t) \geq \#_x(s)$*

Proof. First we consider each of the four cases for reduction at the root $t \mapsto_R^\alpha s$:

1. Suppose $t = (\lambda a^\alpha . t_1)\mathsf{L}\, t_2 \mapsto_{\bullet\mathsf{db}}^\alpha t_1\{a := t_2\}\mathsf{L} = s$ and $\mathsf{fv}(t_2) \cap \mathsf{dom}(\mathsf{L}) = \varnothing$.

$$
\begin{aligned}
& \#_x((\lambda a^\alpha . t_1)\mathsf{L}\, t_2) \\
= {} & \#_x((\lambda a. t_1)\mathsf{L}) + \#_x(t_2) + \#_a(t_1) \times \#_x(t_2) && (Def.\ D.5) \\
\geq {} & \#_x((\lambda a. t_1)\mathsf{L}) + \#_a(t_1) \times \#_x(t_2) \\
= {} & \#_x^{\#_\bullet(t_1\{a:=t_2\})}(\mathsf{L}) && (Lem.\ D.8) \\
= {} & \#_x(t_1\{a := t_2\}\mathsf{L}) && (Lem.\ D.4)
\end{aligned}
$$

2. Suppose $t = \mathtt{req}^\alpha((\bullet t_1)\mathsf{L}) \mapsto_{\bullet\mathsf{req}}^\alpha t_1\mathsf{L} = s$

$$
\begin{aligned}
& \#_x(\mathtt{req}^\alpha((\bullet t_1)\mathsf{L})) \\
= {} & \#_x((\bullet t_1)\mathsf{L}) && (Def.\ D.5) \\
= {} & \#_x(t_1\mathsf{L})
\end{aligned}
$$

3. Suppose $t = \mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle[u/(!(\bullet t_1)\mathsf{L}_1)\mathsf{L}_2] \mapsto_{\bullet\mathsf{ls}}^\alpha \mathsf{C}\langle\!\langle(\bullet t_1)\mathsf{L}_1\rangle\!\rangle[u/!(\bullet t_1)\mathsf{L}_1]\mathsf{L}_2 = s$ and $u \notin \mathsf{fv}(t_1)$ and $\mathsf{fv}(\mathsf{C}) \cap \mathsf{dom}(\mathsf{L}_1\mathsf{L}_2) = \varnothing$.

$$
\begin{aligned}
& \#_x(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle[u/(!(\bullet t_1)\mathsf{L}_1)\mathsf{L}_2]) \\
= {} & \#_x(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#_x((!(\bullet t_1)\mathsf{L}_1)\mathsf{L}_2) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#_x((!(\bullet t_1)\mathsf{L}_1)\mathsf{L}_2) && (Def.\ D.5) \\
\geq {} & \#_x^{\#_\bullet(\mathsf{C}\langle\!\langle(\bullet t_1)\mathsf{L}_1\rangle\!\rangle[u/!(\bullet t_1)\mathsf{L}_1])}(\mathsf{L}_2) && (Lem.\ D.11) \\
= {} & \#_x(\mathsf{C}\langle\!\langle(\bullet t_1)\mathsf{L}_1\rangle\!\rangle[u/!(\bullet t_1)\mathsf{L}_1]\mathsf{L}_2) && (Lem.\ D.4)
\end{aligned}
$$

4. Suppose $t = t_1[u^\alpha/(!t_2)\mathsf{L}] \mapsto_{\bullet\mathsf{gc}}^\alpha t_1\mathsf{L} = s$ and $u \notin \mathsf{fv}(t_1)$.

$$
\begin{aligned}
& \#_x(t_1[u^\alpha/(!t_2)\mathsf{L}]) \\
= {} & \#_x(t_1) + \#_x((!t_2)\mathsf{L}) + \#_u(t_1) \times \#_x((!t_2)\mathsf{L}) && (Def.\ D.5) \\
= {} & \#_x(t_1) + \#_x((!t_2)\mathsf{L}) && (Lem.\ D.3) \\
\geq {} & \#_x(t_1\mathsf{L}) && (Lem.\ D.9(1))
\end{aligned}
$$

Next we consider internal reduction. We proceed by induction on the size of the labeled context C.

1. $\mathtt{C} = \lambda a.\,\mathtt{C}_1$. Suppose $t = \lambda a.\,\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle \to_R \lambda a.\,\mathtt{C}_1\langle\!\langle s_1\rangle\!\rangle = s$ follows from $t_1 \to_R s_1$.

$$\begin{array}{ll}
\#_x(\lambda a.\,\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle) & \\
= \#_x(\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle) & (Def.\ D.5) \\
\geq \#_x(\mathtt{C}_1\langle\!\langle s_1\rangle\!\rangle) & (IH) \\
= \#_x(\lambda a.\,\mathtt{C}_1\langle\!\langle s_1\rangle\!\rangle) & (Def.\ D.5)
\end{array}$$

2. $\mathtt{C} = \lambda a^{\beta}.\,\mathtt{C}_1$. Not possible since $t \in \mathsf{T}_{\bullet}^{\mathcal{WL}}$.
3. $\mathtt{C} = (\lambda a^{\beta}.\,\mathtt{C}_1)\mathtt{L}\,t_1$. Suppose $t = (\lambda a^{\beta}.\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle)\mathtt{L}\,t_1 \to_R (\lambda a^{\beta}.\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle)\mathtt{L}\,t_1 = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{ll}
\#_x((\lambda a^{\beta}.\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle)\mathtt{L}\,t_1) & \\
= \#_x((\lambda a.\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle)\mathtt{L}) + \#_x(t_1) + \#_a(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) \times \#_x(t_1) & (Def.\ D.5) \\
= \#_x^{\#_\bullet(\lambda a.\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle)}(\mathtt{L}) + \#_x(t_1) + \#_a(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) \times \#_x(t_1) & (Lem.\ D.4) \\
= \#_x^{\#_\bullet(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle)}(\mathtt{L}) + \#_x(t_1) + \#_a(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) \times \#_x(t_1) & (Def.\ D.5) \\
\geq \#_x^{\#_\bullet(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle)}(\mathtt{L}) + \#_x(t_1) + \#_a(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) \times \#_x(t_1) & (IH \times 2) \\
= \#_x^{\#_\bullet(\lambda a.\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle)}(\mathtt{L}) + \#_x(t_1) + \#_a(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) \times \#_x(t_1) & (Def.\ D.5) \\
= \#_x((\lambda a.\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle)\mathtt{L}) + \#_x(t_1) + \#_a(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) \times \#_x(t_1) & (Lem.\ D.4) \\
= \#_x((\lambda a^{\beta}.\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle)\mathtt{L}\,t_1) &
\end{array}$$

4. $\mathtt{C} = \Box\mathtt{L}_2\,t_1$ and $t = \mathtt{D}\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\,t_1 \to_R^{\alpha} \mathtt{D}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle[u/(!(\bullet t_3)\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\,t_1 = s$ and $\mathtt{D}\langle\!\langle u^{\alpha}\rangle\!\rangle = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1$. There are to cases depending on the location of the hole in $\mathtt{D}$.

   4.1. $\mathtt{D} = (\lambda a^{\beta}.\,\mathtt{D}_1)\mathtt{L}_1$. Let $\mathtt{L} = \mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$.

$$\begin{array}{ll}
\#_x((\lambda a^{\beta}.\,\mathtt{D}_1\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\,t_1) & \\
= \#_x((\lambda a.\,\mathtt{D}_1\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L}) + \#_x(t_1) + \#_a(\mathtt{D}_1\langle\!\langle u^{\alpha}\rangle\!\rangle) \times \#_x(t_1) & (Def.\ D.5) \\
= \#_x^{\#_\bullet(\lambda a.\,\mathtt{D}_1\langle\!\langle u^{\alpha}\rangle\!\rangle\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2])}(\mathtt{L}_2) + \#_x(t_1) + \#_a(\mathtt{D}_1\langle\!\langle u^{\alpha}\rangle\!\rangle) \times \#_x(t_1) & (Lem.\ D.4) \\
= \#_x^{\#_\bullet(\lambda a.\,\mathtt{D}_1\langle\!\langle u^{\alpha}\rangle\!\rangle\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2])}(\mathtt{L}_2) + \#_x(t_1) + \#_a(\mathtt{D}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle) \times \#_x(t_1) & (Lem.\ D.5) \\
\geq \#_x^{\#_\bullet(\lambda a.\,\mathtt{D}_1\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)]\mathtt{K}_2)}(\mathtt{L}_2) + \#_x(t_1) + \#_a(\mathtt{D}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle) \times \#_x(t_1) & (\text{root case, item 3}) \\
= \#_x((\lambda a^{\beta}.\,\mathtt{D}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2) + \#_x(t_1) + \#_a(\mathtt{D}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle) \times \#_x(t_1) & (Lem.\ D.4) \\
= \#_x((\lambda a^{\beta}.\,\mathtt{D}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\,t_1) &
\end{array}$$

   4.2. $\mathtt{D} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_{12}$. Similar to the previous case.

5. $\mathtt{C} = \Box\mathtt{L}_2\,t_1$ and $t = (\lambda a^{\beta}.\,t_2)\mathtt{L}_1[u^{\alpha}/(!t_3)\mathtt{K}]\mathtt{L}_2\,t_1 \to_R^{\alpha} (\lambda a^{\beta}.\,t_2)\mathtt{L}_1\mathtt{K}\mathtt{L}_2\,t_1 = s$.

$$\begin{array}{ll}
\#_x((\lambda a^{\beta}.\,t_2)\mathtt{L}_1[u^{\alpha}/(!t_3)\mathtt{K}]\mathtt{L}_2\,t_1) & \\
= \#_x((\lambda a.\,t_2)\mathtt{L}_1[u^{\alpha}/(!t_3)\mathtt{K}]\mathtt{L}_2) + \#_x(t_1) + \#_a(t_2) \times \#_x(t_1) & (Def.\ D.5) \\
= \#_x^{\#_\bullet((\lambda a.\,t_2)\mathtt{L}_1[u^{\alpha}/(!t_3)\mathtt{K}])}(\mathtt{L}_2) + \#_x(t_1) + \#_a(t_2) \times \#_x(t_1) & (Lem.\ D.4) \\
\geq \#_x^{\#_\bullet((\lambda a.\,t_2)\mathtt{L}_1\mathtt{K})}(\mathtt{L}_2) + \#_x(t_1) + \#_a(t_2) \times \#_x(t_1) & (\text{root case, item 4}) \\
= \#_x((\lambda a^{\beta}.\,t_2)\mathtt{L}_1\mathtt{K}\mathtt{L}_2) + \#_x(t_1) + \#_a(t_2) \times \#_x(t_1) & (Lem.\ D.4) \\
= \#_x((\lambda a^{\beta}.\,t_1)\mathtt{L}_1\mathtt{K}\mathtt{L}_2\,t_1) &
\end{array}$$

6. $\mathtt{C} = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1]\mathtt{L}_2\, t_2$. Suppose $t = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle]\mathtt{L}_2\, t_2 \to_R (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle]\mathtt{L}_2\, t_2 = s$ follows from $t_3 \to_R s_3$.

$$\begin{array}{ll}
\#_x((\lambda a^{\beta}.\, t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle]\mathtt{L}_2\, t_2) & \\
= \#_x((\lambda a.\, t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle]\mathtt{L}_2) + \#_x(t_2) + \#_a(t_1)\times\#_x(t_2) & (Def.\ D.5)\\
= \#_x^{\#_\bullet((\lambda a.\, t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle])}(\mathtt{L}_2) + \#_x(t_2) + \#_a(t_1)\times\#_x(t_2) & (Lem.\ D.4)\\
= \#_x^{\#_\bullet((\lambda a.\, t_1)\mathtt{L}_1)+\#_\bullet(\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle)+\#_v((\lambda a.\, t_1)\mathtt{L}_1)\times\#_\bullet(\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle)}(\mathtt{L}_2) + \#_x(t_2) + \#_a(t_1)\times\#_x(t_2) & (Def.\ D.5)\\
\geq \#_x^{\#_\bullet((\lambda a.\, t_1)\mathtt{L}_1)+\#_\bullet(\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle)+\#_v((\lambda a.\, t_1)\mathtt{L}_1)\times\#_\bullet(\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle)}(\mathtt{L}_2) + \#_x(t_2) + \#_a(t_1)\times\#_x(t_2) & (IH)\\
= \#_x^{\#_\bullet((\lambda a.\, t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle])}(\mathtt{L}_2) + \#_x(t_2) + \#_a(t_1)\times\#_x(t_2) & (Def.\ D.5)\\
= \#_x((\lambda a.\, t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle]\mathtt{L}_2) + \#_x(t_2) + \#_a(t_1)\times\#_x(t_2) & (Lem.\ D.4)\\
= \#_x((\lambda a^{\beta}.\, t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle]\mathtt{L}_2\, t_2) & (Def.\ D.5)
\end{array}$$

7. $\mathtt{C} = (\lambda a^{\beta}.\, t_1)\mathtt{L}\,\mathtt{C}_1$. Suppose $t = (\lambda a^{\beta}.\, t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle \to_R (\lambda a^{\beta}.\, t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{ll}
\#_x((\lambda a^{\beta}.\, t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) & \\
= \#_x((\lambda a.\, t_1)\mathtt{L}) + \#_x(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#_a(t_1)\times\#_x(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) & (Def.\ D.5)\\
\geq \#_x((\lambda a.\, t_1)\mathtt{L}) + \#_x(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#_a(t_1)\times\#_x(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) & (IH\times 2)\\
= \#_x((\lambda a^{\beta}.\, t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) & (Def.\ D.5)
\end{array}$$

8. $\mathtt{C} = \mathtt{C}_1\, t_1$. Suppose $t = \mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle\, t_1 \to_R \mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle\, t_1 = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{ll}
\#_x(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle\, t_1) & \\
= \#_x(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#_x(t_1) & (Def.\ D.5)\\
\geq \#_x(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#_x(t_1) & (IH)\\
= \#_x(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle\, t_1) & (Def.\ D.5)
\end{array}$$

9. $\mathtt{C} = t_1\,\mathtt{C}_1$. Similar to the case $\mathtt{C} = \mathtt{C}_1\, t_1$.
10. $\mathtt{C} = \bullet\mathtt{C}_1$. Similar to the case $\mathtt{C} = \lambda a.\,\mathtt{C}_1$.
11. $\mathtt{C} = \mathtt{req}^{(\beta)}(\mathtt{C}_1)$. We need to consider multiple cases since out induction proceeds on well-formed labeled contexts.
    11.1. β is not present. Suppose $t = \mathtt{req}(\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle) \to_R \mathtt{req}(\mathtt{C}_1\langle\!\langle s_1\rangle\!\rangle) = s$ follows from $t_1 \to_R s_1$. Similar to the case $\mathtt{C} = \lambda a.\,\mathtt{C}_1$.
    11.2. β is present.
        11.2.1. $\mathtt{C}_1 = (\bullet\mathtt{C}_2)\mathtt{L}$. Suppose $t = \mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle)\mathtt{L}) \to_R \mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle)\mathtt{L}) = s$ follows from $t_1 \to_R s_1$.

$$\begin{array}{ll}
\#_x((\bullet\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle)\mathtt{L}) & \\
= \#_x^{\#_\bullet(\bullet\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle)}(\mathtt{L}) & (Lem.\ D.4)\\
= \#_x^{\#_\bullet(\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle)}(\mathtt{L}) & \\
\geq \#_x^{\#_\bullet(\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle)}(\mathtt{L}) & (IH)\\
= \#_x^{\#_\bullet(\bullet\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle)}(\mathtt{L}) & \\
= \#_x((\bullet\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle)\mathtt{L}) & (Lem.\ D.4)
\end{array}$$

        11.2.2. $\mathtt{C}_1 = \Box\mathtt{L}_2$ and $t = \mathtt{req}^{\beta}((\bullet\mathtt{D}\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2) \to_R^{\alpha} \mathtt{req}^{\beta}((\bullet\mathtt{D}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_)) = s$. Then we reason as in the previous case but using the root case (item 2) instead of the IH.

11.2.3. $\mathtt{C}_1 = \Box\mathtt{L}_2$ and $t = \mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}_1[u^{\alpha}/(!t_3)\mathtt{K}]\mathtt{L}_2) \to_R^{\alpha} \mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}_1\mathtt{K}\mathtt{L}_2) = s$. Same as previous case.

11.2.4. $\mathtt{C}_1 = (\bullet t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_2]\mathtt{L}_2$. Suppose $t = \mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_2\langle\!\langle t_2\rangle\!\rangle]\mathtt{L}_2) \to_R \mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_2\langle\!\langle s_2\rangle\!\rangle]\mathtt{L}_2) = s$ follows from $t_2 \to_R s_2$. Similar to the previous case.

12. $\mathtt{C} = !\mathtt{C}_1$. Similar to the case $\mathtt{C} = \lambda a.\,\mathtt{C}_1$.
13. $\mathtt{C} = \mathtt{C}_1[v^{(\beta)}/t_1]$. Suppose $t = \mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle[v^{(\beta)}/t_1] \to_R \mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle[v^{(\beta)}/t_1] = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{lll} & \#_x(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle[v^{(\beta)}/t_1]) & \\ & \#_x(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#_x(t_1) + \#_v(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) \times \#_x(t_1) & (Def.\ D.5) \\ \geq & \#_x(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#_x(t_1) + \#_v(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) \times \#_x(t_1) & (IH \times 2) \\ = & \#_x(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle[v^{(\beta)}/t_1]) & (Def.\ D.5) \end{array}$$

14. $\mathtt{C} = t_1[v^{(\beta)}/\mathtt{C}_1]$. Suppose $t = t_1[v^{(\beta)}/\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle] \to_R t_1[v^{(\beta)}/\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle] = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{lll} & \#_x(t_1[v^{(\beta)}/\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle]) & \\ & \#_x(t_1) + \#_x(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#_v(t_1) \times \#_x(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) & (Def.\ D.5) \\ \geq & \#_x(t_1) + \#_x(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#_v(t_1) \times \#_x(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) & (IH \times 2) \\ = & \#_x(t_1[v^{(\beta)}/\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle]) & (Def.\ D.5) \end{array}$$

Lemma D.12. $\#(t) + \#_a(t) \times \#(s) = \#(t\{a := s\})$

Proof. By induction on t.

1. $t = b$. If $b \neq a$, then

$$\begin{array}{ll} & \#(b) + \#_a(b) \times \#(s) \\ = & 0 \\ = & \#(b\{a := s\}) \end{array}$$

If $b = a$, then

$$\begin{array}{ll} & \#(b) + \#_a(b) \times \#(s) \\ = & \#(s) \\ = & \#(b\{a := s\}) \end{array}$$

2. $t = u^{(\alpha)}$

$$\begin{array}{ll} & \#(u^{(\alpha)}) + \#_a(u^{(\alpha)}) \times \#(s) \\ = & 0 \\ = & \#(u^{(\alpha)}\{a := s\}) \end{array}$$

3. $t = \lambda b.\, r$.

$$\begin{array}{lll} & \#(\lambda b.\, r) + \#_a(\lambda b.\, r) \times \#(s) & \\ = & \#(r) + \#_a(r) \times \#(s) & (Def.\ D.6) \\ = & \#(r\{a := s\}) & (IH) \\ = & \#((\lambda b.\, r)\{a := s\}) & (Def.\ D.6) \end{array}$$

4. $t = t_1\, t_2$.

$$\begin{aligned}
&\#(t_1\, t_2) + \#_a(t_1\, t_2) \times \#(s) \\
&= \#(t_1) + \#(t_2) + (\#_a(t_1) + \#_x(t_2)) \times \#(s) && (Def.\ D.6) \\
&= \#(t_1) + \#(t_2) + \#_a(t_1) \times \#(s) + \#_x(t_2) \times \#(s) \\
&= \#(t_1\{a := s\}) + \#(t_2\{a := s\}) && (IH \times 2) \\
&= \#(t_1\{a := s\}\, t_2\{a := s\}) && (Def.\ D.6) \\
&= \#((t_1\, t_2)\{a := s\})
\end{aligned}$$

5. $t = (\lambda b^\alpha.\, t_1)\mathrm{L}\, t_2$.

$$\begin{aligned}
&\#((\lambda b^\alpha.\, t_1)\mathrm{L}\, t_2) + \#_a((\lambda b^\alpha.\, t_1)\mathrm{L}\, t_2) \times \#(s) \\
&= 1 + \#((\lambda b.\, t_1)\mathrm{L}) + \#(t_2) + \#_b(t_1) \times \#(t_2) + \#_a((\lambda b^\alpha.\, t_1)\mathrm{L}\, t_2) \times \#(s) && (Def.\ D.6) \\
&= 1 + \#((\lambda b.\, t_1)\mathrm{L}) + \#(t_2) + \#_b(t_1) \times \#(t_2) + (\#_a((\lambda b.\, t_1)\mathrm{L}) + \#_a(t_2) + \#_b(t_1) \times \#_a(t_2)) \times \#(s) && (Def.\ D.6) \\
&= 1 + \#(((\lambda b.\, t_1)\mathrm{L})\{a := s\}) + \#(t_2) + \#_b(t_1) \times \#(t_2) + (\#_a(t_2) + \#_b(t_1) \times \#_a(t_2)) \times \#(s) && (IH) \\
&= 1 + \#(((\lambda b.\, t_1)\mathrm{L})\{a := s\}) + \#(t_2\{a := s\}) + \#_b(t_1) \times \#(t_2) + \#_b(t_1) \times \#_a(t_2) \times \#(s) && (IH) \\
&= 1 + \#((\lambda b.\, t_1\{a := s\})\mathrm{L}\{a := s\}) + \#(t_2\{a := s\}) + \#_b(t_1) \times \#(t_2\{a := s\}) && (Def.\ D.6) \\
&= 1 + \#((\lambda b.\, t_1\{a := s\})\mathrm{L}\{a := s\}) + \#(t_2\{a := s\}) + \#_b(t_1\{a := s\}) \times \#(t_2\{a := s\}) && (Lem.\ D.6(1)) \\
&= \#((\lambda b^\alpha.\, t_1\{a := s\})\mathrm{L}\{a := s\}\, t_2\{a := s\}) && (Def.\ D.5) \\
&= \#(((\lambda b^\alpha.\, t_1)\mathrm{L}\, t_2)\{a := s\}) && (Def.\ D.6)
\end{aligned}$$

6. $t = \bullet t_1$.

$$\begin{aligned}
&\#(\bullet t_1) + \#_a(\bullet t_1) \times \#(s) \\
&= \#(t_1) + \#_a(t_1) \times \#(s) && (Def.\ D.6, Def.\ D.5) \\
&= \#(t_1\{a := s\}) && (IH) \\
&= \#(\bullet(t_1\{a := s\})) && (Def.\ D.6) \\
&= \#((\bullet t_1)\{a := s\})
\end{aligned}$$

7. $t = \mathtt{req}(t_1)$. Same as case $t = \bullet t_1$.
8. $t = \mathtt{req}^\alpha((\bullet t_1)\mathrm{L})$.

$$\begin{aligned}
&\#(\mathtt{req}^\alpha((\bullet t_1)\mathrm{L})) + \#_a(\mathtt{req}^\alpha((\bullet t_1)\mathrm{L})) \times \#(s) \\
&= 1 + \#((\bullet t_1)\mathrm{L}) + \#_a((\bullet t_1)\mathrm{L}) \times \#(s) && (Def.\ D.6, Def.\ D.5) \\
&= 1 + \#(((\bullet t_1)\mathrm{L})\{a := s\}) && (IH) \\
&= \#(\mathtt{req}^\alpha(((\bullet t_1)\mathrm{L})\{a := s\})) && (Def.\ D.6) \\
&= \#(\mathtt{req}^\alpha((\bullet t_1)\mathrm{L})\{a := s\})
\end{aligned}$$

9. $t = !t_1$. Same as case $t = \bullet t_1$.
10. $t = t_1[u/t_2]$

$$\begin{aligned}
&\#(t_1[u/t_2]) + \#_a(t_1[u/t_2]) \times \#(s) \\
&= \#(t_1) + \#(t_2) + \#_u(t_1) \times \#(t_2) + \#_u(t_1) + \#_a(t_1[u^{(\alpha)}/t_2]) \times \#(s) && (Def.\ D.6) \\
&= \#(t_1) + \#(t_2) + \#_u(t_1) \times \#(t_2) + \#_u(t_1) + (\#_a(t_1) + \#_a(t_2) + \#_u(t_1) \times \#_a(t_2)) \times \#(s) && (Def.\ D.5) \\
&= \#(t_1\{a := s\}) + \#(t_2) + \#_u(t_1) \times \#(t_2) + \#_u(t_1) + \#_u(t_1) \times \#_x(t_2) + (\#_a(t_2) + \#_u(t_1) \times \#_a(t_2)) \times \#(s) && (IH) \\
&= \#(t_1\{a := s\}) + \#(t_2\{a := s\}) + \#_u(t_1) \times \#(t_2) + \#_u(t_1) + \#_u(t_1) \times \#_a(t_2) \times \#(s) && (IH) \\
&= \#(t_1\{a := s\}) + \#(t_2\{a := s\}) + \#_u(t_1) \times \#(t_2\{a := s\}) + \#_u(t_1) && (IH) \\
&= \#(t_1\{a := s\}) + \#(t_2\{a := s\}) + \#_u(t_1\{a := s\}) \times \#(t_2\{a := s\}) + \#_u(t_1\{a := s\}) && (Lem.\ D.6(1) \times 2) \\
&= \#(t_1\{a := s\}[u/t_2\{a := s\}]) && (Def.\ D.5) \\
&= \#(t_1[u/t_2]\{a := s\})
\end{aligned}$$

11. $t = t_1[u^\alpha/t_2]$ and $u \notin \mathsf{fv}(t_1)$.

$$\begin{array}{ll}
\#(t_1[u^\alpha/t_2]) + \#_a(t_1[u^\alpha/t_2]) \times \#(s) & \\
= 1 + \#(t_1) + \#(t_2) + \#_a(t_1[u^\alpha/t_2]) \times \#(s) & (Def.\ D.6) \\
= 1 + \#(t_1) + \#(t_2) + (\#_a(t_1) + \#_a(t_2) + \#_u(t_1) \times \#_a(t_2)) \times \#(s) & (Def.\ D.5) \\
= 1 + \#(t_1\{a := s\}) + \#(t_2) + (\#_a(t_2) + \#_u(t_1) \times \#_a(t_2)) \times \#(s) & (IH) \\
= 1 + \#(t_1\{a := s\}) + \#(t_2\{a := s\}) + \#_u(t_1) \times \#_a(t_2) \times \#(s) & (IH) \\
= 1 + \#(t_1\{a := s\}) + \#(t_2\{a := s\}) & (u \notin \mathsf{fv}(t)) \\
= \#(t_1\{a := s\}[u^\alpha/t_2\{a := s\}]) & (Def.\ D.6) \\
= \#(t_1[u^{(\alpha)}/t_2]\{a := s\}) &
\end{array}$$

Lemma D.13. *Suppose* $\mathsf{fv}(s) \cap \mathsf{dom}(\mathtt{L}) = \varnothing$.

$$\#((\lambda a.\, t)\mathtt{L}) + \#_a(t) \times \#(s) = \#(t\{a := s\}) + \#^{\#_\bullet(t\{a:=s\})}(\mathtt{L})$$

Proof. By induction on L.

1. $\mathtt{L} = \Box$.

$$\begin{array}{ll}
\#(\lambda a.\, t) + \#_a(t) \times \#(s) & \\
= \#(t) + \#_a(t) \times \#(s) & (Def.\ D.6) \\
= \#(t\{a := s\}) & (Lem.\ D.12) \\
= \#(t\{a := s\}) + 0 & \\
= \#(t\{a := s\}) + \#^{\#_\bullet(t\{a:=s\})}(\Box) &
\end{array}$$

2. $\mathtt{L} = \mathtt{L}_1[u^\alpha/r]$.

$$\begin{array}{ll}
\#((\lambda a.\, t)\mathtt{L}_1[u^\alpha/r]) + \#_a(t) \times \#(s) & \\
= 1 + \#((\lambda a.\, t)\mathtt{L}_1) + \#(r) + \#_a(t) \times \#_x(s) & (Def.\ D.6) \\
= 1 + \#(t\{a := s\}) + \#_x^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1) + \#(r) & (IH) \\
= \#(t\{a := s\}) + \#^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1[u^\alpha/r]) & (Def.\ D.6)
\end{array}$$

3. $\mathtt{L} = \mathtt{L}_1[u/t]$

$$\begin{array}{ll}
\#((\lambda a.\, t)\mathtt{L}_1[u/r]) + \#_a(t) \times \#(s) & \\
= \#((\lambda a.\, t)\mathtt{L}_1) + \#(r) + \#_u((\lambda a.\, t)\mathtt{L}_1) \times \#(r) + \#_u((\lambda a.\, t)\mathtt{L}_1) + \#_a(t) \times \#(s) & (Def.\ D.6) \\
= \#(t\{a := s\}) + \#_x^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1) + \#(r) + \#_u((\lambda a.\, t)\mathtt{L}_1) \times \#(r) + \#_u((\lambda a.\, t)\mathtt{L}_1) & (IH) \\
= \#(t\{a := s\}) + \#_x^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1) + \#(r) + \#_u^{\#_\bullet(\lambda a.\, t)}(\mathtt{L}_1) \times \#(r) + \#_u((\lambda a.\, t)\mathtt{L}_1) & (Lem.\ D.4) \\
= \#(t\{a := s\}) + \#_x^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1) + \#(r) + \#_u^{\#_\bullet(t)}(\mathtt{L}_1) \times \#(r) + \#_u((\lambda a.\, t)\mathtt{L}_1) & (Def.\ D.5) \\
= \#(t\{a := s\}) + \#^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1) + \#(r) + \#_u^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1) \times \#(r) + \#_u^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1) & (Lem.\ D.6(2)) \\
= \#(t\{a := s\}) + \#^{\#_\bullet(t\{a:=s\})}(\mathtt{L}_1[u/r]) & (Def.\ D.6)
\end{array}$$

Lemma D.14. $\#(t\mathtt{L}) = \#(t) + \#^{\#_\bullet(t)}(\mathtt{L})$

Proof. By induction on L.

1. $\mathtt{L} = \Box$. Immediate since $\#(t) = \#(t) + 0 = \#(t) + \#^{\#_\bullet(t)}(\Box)$.

2. $\mathtt{L} = \mathtt{L}_1[u/s]$.

$$\begin{array}{ll} \#(t\mathtt{L}_1[u/s]) & \\ = \#(t\mathtt{L}_1) + \#(s) + \#_u(t\mathtt{L}_1) \times \#(s) + \#_u(t\mathtt{L}_1) & (Def.\ D.6) \\ = \#(t) + \#^{\#_\bullet(t)}(\mathtt{L}_1) + \#(s) + \#_u(t\mathtt{L}_1) \times \#(s) + \#_u(t\mathtt{L}_1) & (IH) \\ = \#(t) + \#^{\#_\bullet(t)}(\mathtt{L}_1) + \#(s) + \#_u^{\#_\bullet(t)}(\mathtt{L}_1) \times \#(s) + \#_u^{\#_\bullet(t)}(\mathtt{L}_1) & (Lem.\ D.4 \times 2) \\ = \#(t) + \#^{\#_\bullet(t)}(\mathtt{L}_1[u/s]) & (Def.\ D.6) \end{array}$$

3. $\mathtt{L} = \mathtt{L}_1[u^\alpha/s]$.

$$\begin{array}{ll} \#(t\mathtt{L}_1[u^\alpha/s]) & \\ = 1 + \#(t\mathtt{L}_1) + \#(s) & (Def.\ D.6) \\ = 1 + \#(t) + \#^{\#_\bullet(t)}(\mathtt{L}_1) + \#(s) & (IH) \\ = \#(t) + \#^{\#_\bullet(t)}(\mathtt{L}_1[u^\alpha/s]) & (Def.\ D.6) \end{array}$$

Lemma D.15. *Suppose $u \notin \mathsf{fv}(t)$. Then,*

$$\#(\mathtt{C}\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(\mathtt{C}\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(t) + \#_u(\mathtt{C}\langle\!\langle u^\alpha\rangle\!\rangle) > \#(\mathtt{C}\langle\!\langle t\rangle\!\rangle) + \#_u(\mathtt{C}\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathtt{C}\langle\!\langle t\rangle\!\rangle)$$

Proof. By induction on the labeled context $\mathtt{C}$.

1. $\mathtt{C} = \Box$.

$$\begin{array}{ll} \#(u^\alpha) + \#_u(u^\alpha) \times \#(t) + \#_u(u^\alpha) & \\ = 0 + \#(t) + 1 & \\ > \#(t) & \\ = \#(t) + \#_u(t) \times \#(t) + \#_u(t) & (Lem.\ D.3) \end{array}$$

2. $\mathtt{C} = \lambda a.\, \mathtt{C}_1$

$$\begin{array}{ll} \#(\lambda a.\, \mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(\lambda a.\, \mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(t) + \#_u(\lambda a.\, \mathtt{C}\langle\!\langle u^\alpha\rangle\!\rangle) & \\ = \#(\mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(\mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(t) + \#_u(\mathtt{C}\langle\!\langle u^\alpha\rangle\!\rangle) & (Def.\ D.5, Def.\ D.6) \\ > \#(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathtt{C}\langle\!\langle t\rangle\!\rangle) & (IH) \\ = \#(\lambda a.\, \mathtt{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\lambda a.\, \mathtt{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\lambda a.\, \mathtt{C}\langle\!\langle t\rangle\!\rangle) & (Def.\ D.5) \end{array}$$

3. $\mathtt{C} = (\lambda a^\beta.\, \mathtt{C}_1)\mathtt{L}\ s$
4. $\mathtt{C} = (\lambda a^\beta.\, r)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1]\mathtt{L}_2\ s$
5. $\mathtt{C} = (\lambda a^\beta.\, r)\mathtt{L}\,\mathtt{C}_1$
6. $\mathtt{C} = \mathtt{C}_1\ s$

$$\begin{array}{ll} \#(\mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle\ s) + \#_u(\mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle\ s) \times \#(t) + \#_u(\mathtt{C}\langle\!\langle u^\alpha\rangle\!\rangle\ s) & \\ = \#(\mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#(s) + (\#_u(\mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s)) \times \#(t) + \#_u(\mathtt{C}\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s) & (Def.\ D.6) \\ = \#(\mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#(s) + \#_u(\mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(t) + \#_u(s) \times \#(t) + \#_u(\mathtt{C}\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s) & \\ > \#(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathtt{C}\langle\!\langle t\rangle\!\rangle) + \#(s) + \#_u(s) \times \#(t) + \#_u(s) & (IH) \\ = \#(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle\ s) + \#_u(\mathtt{C}_1\langle\!\langle t\rangle\!\rangle\ s) \times \#(t) + \#_u(\mathtt{C}\langle\!\langle t\rangle\!\rangle\ s) & (Def.\ D.5, Def.\ D.6) \end{array}$$

7. $\mathtt{C} = s\,\mathtt{C}_1$. Similar to the case $\mathtt{C} = \mathtt{C}_1\ s$.
8. $\mathtt{C} = \bullet\mathtt{C}_1$. Similar to the case $\mathtt{C} = \lambda a.\, \mathtt{C}_1$.
9. $\mathtt{C} = \mathtt{req}^{(\beta)}(\mathtt{C}_1)$. We need to consider multiple cases since out induction proceeds on well-formed labeled contexts.

9.1. β is not present. Similar to the case $\mathsf{C} = \lambda a.\,\mathsf{C}_1$.

$$
\begin{aligned}
&\#(\mathtt{req}(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle)) + \#_u(\mathtt{req}(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle)) \times \#(t) + \#_u(\mathtt{req}(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle)) \\
&= \#(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) && (Def.\ D.5, Def.\ D.6) \\
&> \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) && (IH) \\
&= \#(\mathtt{req}(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle)) + \#_u(\mathtt{req}(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle)) \times \#(t) + \#_u(\mathtt{req}(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle)) && (Def.\ D.5, Def.\ D.6)
\end{aligned}
$$

9.2. β is present.

9.2.1. $\mathsf{C}_1 = (\bullet\mathsf{C}_2)\mathsf{L}$. Similar to the case above.

9.2.2. $\mathsf{C}_1 = (\bullet s)\mathsf{L}_1[v^{(\beta)}/\mathsf{C}_2]\mathsf{L}_2$. Similar to the case above.

10. $\mathsf{C} = !\mathsf{C}_1$. Similar to the case $\mathsf{C} = \lambda a.\,\mathsf{C}_1$.
11. $\mathsf{C} = \mathsf{C}_1[v/s]$

$$
\begin{aligned}
&\#(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v/s]) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v/s]) \\
&= \#(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v/s]) \\
&= \#(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + (\#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle \\
&= \#(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + (\#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle \\
&> \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(s) + (\#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \\
&= \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#(s) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(s) + (\#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \times \# \\
&= \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + (\#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_u(s)) \times \#(t) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s) \\
&= \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s) \\
&= \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_u(s) \\
&= \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v/s])
\end{aligned}
$$

12. $\mathsf{C} = \mathsf{C}_1[v^\beta/s]$

$$
\begin{aligned}
&\#(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v^\beta/s]) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v^\beta/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v^\beta/s]) \\
&= 1 + \#(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#(s) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v/s]) \\
&= 1 + \#(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#(s) + (\#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle[v/s]) \\
&= 1 + \#(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#(s) + (\#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s)) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u \\
&> 1 + \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(s) + (\#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u \\
&= 1 + \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#(s) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(s) + (\#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s) \\
&= \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^\beta/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + (\#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_u(s)) \times \#(t) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s) \\
&= \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^\beta/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^\beta/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle u^\alpha\rangle\!\rangle) \times \#_u(s) \\
&= \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^\beta/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^\beta/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) + \#_u(s) + \#_v(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle) \times \#_u(s) \\
&= \#(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^\beta/s]) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^\beta/s]) \times \#(t) + \#_u(\mathsf{C}_1\langle\!\langle t\rangle\!\rangle[v^\beta/s])
\end{aligned}
$$

13. $\mathsf{C} = s[v/\mathsf{C}_1]$. Similar to the previous case.
14. $\mathsf{C} = s[v^\beta/\mathsf{C}_1]$. Similar to the previous case.

Lemma D.16. *Suppose* $\mathsf{dom}(\mathsf{L}) \cap (\mathsf{fv}(\mathsf{C}) \cup \{u\}) = \varnothing$. *Then*

$$\#(\mathsf{C}\langle\!\langle u^\alpha\rangle\!\rangle) + \#(t\mathsf{L}) + \#_u(\mathsf{C}\langle\!\langle u^\alpha\rangle\!\rangle) \times \#(t\mathsf{L}) + \#_u(\mathsf{C}\langle\!\langle u^\alpha\rangle\!\rangle) > \#(\mathsf{C}\langle\!\langle t\rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t\rangle\!\rangle[u/t])}(\mathsf{L})$$

Proof. By induction on L.

1. $\mathtt{L} = \square$

$$
\begin{aligned}
& \#(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#(t) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) & \\
> {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle) + \#(t) + \#_u(\mathsf{C}\langle\!\langle t \rangle\!\rangle) \times \#(t) + \#_u(\mathsf{C}\langle\!\langle t \rangle\!\rangle) & (Lem.\ D.15) \\
= {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) & (Def.\ D.6) \\
= {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\square) & (Def.\ D.6)
\end{aligned}
$$

2. $\mathtt{L} = \mathtt{L}_1[v/s]$

$$
\begin{aligned}
& \#(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#(t\mathtt{L}_1[v/s]) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#(t\mathtt{L}_1[v/s]) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \\
= {} & \#(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#(t\mathtt{L}_1) + \#(s) + \#_v(t\mathtt{L}_1) \times \#(s) + \#_v(t\mathtt{L}_1) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times (\#(t\mathtt{L}_1) + \#(s) + \#_v(t\mathtt{L}_1) \times \#(s) + \#_v(t\mathtt{L}_1)) + \#_u(\mathsf{C} \\
> {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) + \#(s) + \#_v(t\mathtt{L}_1) \times \#(s) + \#_v(t\mathtt{L}_1) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times (\#(s) + \#_v(t\mathtt{L}_1) \times \#(s) + \#_v(t\mathtt{L}_1)) \\
\geq {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) + \#(s) + \#_v(t\mathtt{L}_1) \times \#(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times (\#(s) + \#_v(t\mathtt{L}_1) \times \#(s)) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) \\
\geq {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) + \#(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#(s) + \#_v^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) \times \#(s) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) \\
\geq {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) + \#(s) + \#_v^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) \times \#(s) + \#_v^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) \\
= {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1[v/s])
\end{aligned}
$$

We have used the fact that $\#_v(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) = 0$

3. $\mathtt{L} = \mathtt{L}_1[v^\alpha/s]$

$$
\begin{aligned}
& \#(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#(t\mathtt{L}_1[v^\alpha/s]) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#(t\mathtt{L}_1[v^\alpha/s]) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) & \\
= {} & \#(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + 1 + \#(t\mathtt{L}_1) + \#(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times (1 + \#(t\mathtt{L}_1) + \#(s)) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) & \\
= {} & \#(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + 1 + \#(t\mathtt{L}_1) + \#(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#(t\mathtt{L}_1) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) & \\
> {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) + 1 + \#(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) \times \#(s) + \#_u(\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle) & (IH) \\
> {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) + 1 + \#(s) & \\
= {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + 1 + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1) + \#(s) & \\
= {} & \#(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t]) + \#^{\#_\bullet(\mathsf{C}\langle\!\langle t \rangle\!\rangle[u/t])}(\mathtt{L}_1[v^\alpha/s]) &
\end{aligned}
$$

Proposition D.2. *$t \to_R^\alpha s$ implies $\#(t) > \#(s)$*

Proof. First we consider each of the four cases for reduction at the root $t \mapsto_R^\alpha s$:

1. Suppose $t = (\lambda a^\alpha. t_1)\mathtt{L}\, t_2 \mapsto_{\bullet\mathsf{db}}^\alpha t_1\{a := t_2\}\mathtt{L} = s$ and $\mathsf{fv}(t_2) \cap \mathsf{dom}(\mathtt{L}) = \varnothing$.

$$
\begin{aligned}
& \#((\lambda a^\alpha. t_1)\mathtt{L}\, t_2) & \\
= {} & 1 + \#((\lambda a. t_1)\mathtt{L}) + \#(t_2) + \#_a(t_1) \times \#(t_2) & (Def.\ D.6) \\
> {} & \#((\lambda a. t_1)\mathtt{L}) + \#_a(t_1) \times \#(t_2) & \\
= {} & \#(t_1\{a := t_2\}) + \#^{\#_\bullet(t_1\{a:=t_2\})}(\mathtt{L}) & (Lem.\ D.13) \\
= {} & \#(t_1\{a := t_2\}\mathtt{L}) & (Lem.\ D.14)
\end{aligned}
$$

2. Suppose $t = \mathtt{req}^\alpha((\bullet t_1)\mathtt{L}) \mapsto_{\bullet\mathsf{req}}^\alpha t_1\mathtt{L} = s$

$$
\begin{aligned}
& \#(\mathtt{req}^\alpha((\bullet t_1)\mathtt{L})) & \\
= {} & 1 + \#((\bullet t_1)\mathtt{L}) & (Def.\ D.6) \\
= {} & 1 + \#(t_1\mathtt{L}) & \\
> {} & \#(t_1\mathtt{L}) &
\end{aligned}
$$

3. Suppose $t = \mathtt{C}\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_1)\mathtt{L}_1)\mathtt{L}_2] \mapsto^{\alpha}_{\bullet\mathsf{ls}} \mathtt{C}\langle\!\langle(\bullet t_1)\mathtt{L}_1\rangle\!\rangle[u/!(\bullet t_1)\mathtt{L}_1]\mathtt{L}_2 = s$ and $u \notin \mathsf{fv}(t_1)$ and $\mathsf{fv}(\mathtt{C}) \cap \mathsf{dom}(\mathtt{L}_1\mathtt{L}_2) = \emptyset$.

$$\begin{array}{ll}
\#(\mathtt{C}\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_1)\mathtt{L}_1)\mathtt{L}_2]) & \\
= \#(\mathtt{C}\langle\!\langle u^{\alpha}\rangle\!\rangle) + \#((!(\bullet t_1)\mathtt{L}_1)\mathtt{L}_2) + \#_u(\mathtt{C}\langle\!\langle u^{\alpha}\rangle\!\rangle) \times \#((!(\bullet t_1)\mathtt{L}_1)\mathtt{L}_2) + \#_u(\mathtt{C}\langle\!\langle u^{\alpha}\rangle\!\rangle) & (Def.\ D.6) \\
> \#(\mathtt{C}\langle\!\langle(\bullet t_1)\mathtt{L}_1\rangle\!\rangle[u/!(\bullet t_1)\mathtt{L}_1]) + \#^{\#_\bullet(\mathtt{C}\langle\!\langle(\bullet t_1)\mathtt{L}_1\rangle\!\rangle[u/!(\bullet t_1)\mathtt{L}_1])}(\mathtt{L}_2) & (Lem.\ D.16) \\
= \#(\mathtt{C}\langle\!\langle(\bullet t_1)\mathtt{L}_1\rangle\!\rangle[u/!(\bullet t_1)\mathtt{L}_1]\mathtt{L}_2) & (Lem.\ D.14)
\end{array}$$

4. Suppose $t = t_1[u^{\alpha}/(!t_2)\mathtt{L}] \mapsto^{\alpha}_{\bullet\mathsf{gc}} t_1\mathtt{L} = s$ and $u \notin \mathsf{fv}(t_1)$.

$$\begin{array}{ll}
\#(t_1[u^{\alpha}/(!t_2)\mathtt{L}]) & \\
= 1 + \#(t_1) + \#((!t_2)\mathtt{L}) & (Def.\ D.6) \\
= 1 + \#(t_1) + \#(!t_2) + \#^{\#_\bullet((!t_2))}(\mathtt{L}) & (Lem.\ D.14) \\
> \#(t_1) + \#^{\#_\bullet((!t_2))}(\mathtt{L}) & \\
\geq \#(t_1\mathtt{L}) & (Lem.\ D.9(2))
\end{array}$$

Next we consider internal reduction.

1. $\mathtt{C} = \lambda a.\,\mathtt{C}_1$. Suppose $t = \lambda a.\,\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle \to_R \lambda a.\,\mathtt{C}_1\langle\!\langle s_1\rangle\!\rangle = s$ follows from $t_1 \to_R s_1$.

$$\begin{array}{ll}
\#(\lambda a.\,\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle) & \\
= \#(\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle) & (Def.\ D.6) \\
> \#(\mathtt{C}_1\langle\!\langle s_1\rangle\!\rangle) & (IH) \\
= \#(\lambda a.\,\mathtt{C}_1\langle\!\langle s_1\rangle\!\rangle) & (Def.\ D.6)
\end{array}$$

2. $\mathtt{C} = (\lambda a^{\beta}.\,\mathtt{C}_1)\mathtt{L}\,t_1$. Suppose $t = (\lambda a^{\beta}.\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle)\mathtt{L}\,t_1 \to_R (\lambda a^{\beta}.\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle)\mathtt{L}\,t_1 = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{ll}
\#((\lambda a^{\beta}.\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle)\mathtt{L}\,t_1) & \\
= 1 + \#((\lambda a.\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle)\mathtt{L}) + \#(t_1) + \#_a(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) \times \#(t_1) & (Def.\ D.6) \\
= 1 + \#(\lambda a.\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#^{\#_\bullet((\lambda a.\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle))}(\mathtt{L}) + \#(t_1) + \#_a(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) \times \#(t_1) & (Lem.\ D.14) \\
> 1 + \#(\lambda a.\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#^{\#_\bullet((\lambda a.\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle))}(\mathtt{L}) + \#(t_1) + \#_a(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) \times \#(t_1) & (IH \times 2) \\
= 1 + \#((\lambda a.\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle)\mathtt{L}) + \#(t_1) + \#_a(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) \times \#(t_1) & (Lem.\ D.14) \\
= (\lambda a^{\beta}.\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle)\mathtt{L}\,t_1 & (Def.\ D.6)
\end{array}$$

3. $\mathtt{C} = \Box\mathtt{L}_2\,t_1$ and $t = \mathtt{D}\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\,t_1 \to^{\alpha}_R \mathtt{D}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle[u/(!(\bullet t_3)\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\,t_1 = s$ and $\mathtt{D}\langle\!\langle u^{\alpha}\rangle\!\rangle = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1$. There are to cases depending on the location of the hole in $\mathtt{D}$.

   3.1. $\mathtt{D} = (\lambda a^{\beta}.\,\mathtt{D}_1)\mathtt{L}_1$. Let $\mathtt{L} = \mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$. Similar to the previous case.
   3.2. $\mathtt{D} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_{12}$. Similar to the previous case.

4. $\mathtt{C} = \Box\mathtt{L}_2\,t_1$ and $t = (\lambda a^{\beta}.\,t_2)\mathtt{L}_1[u^{\alpha}/(!t_3)\mathtt{K}]\mathtt{L}_2\,t_1 \to^{\alpha}_R (\lambda a^{\beta}.\,t_2)\mathtt{L}_1\mathtt{K}\mathtt{L}_2\,t_1 = s$. Similar to the previous case.
5. $\mathtt{C} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1]\mathtt{L}_2\,t_2$. Suppose $t = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle]\mathtt{L}_2\,t_2 \to_R (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle]\mathtt{L}_2\,t_2 = s$ follows from $t_3 \to_R s_3$.

$$
\begin{array}{ll}
\#((\lambda a^{\beta}.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle]\mathtt{L}_2\,t_2) & \\
= 1 + \#((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle]\mathtt{L}_2) + \#(t_2) + \#_a(t_1)\times\#(t_2) & (\mathit{Def.}\ D.6)\\
= 1 + \#((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle]) + \#^{\#_\bullet((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle])}(\mathtt{L}_2) + \#(t_2) + \#_a(t_1)\times\#(t_2) & (\mathit{Lem.}\ D.14)\\
\geq 1 + \#((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle t_3\rangle\!\rangle]) + \#^{\#_\bullet((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle])}(\mathtt{L}_2) + \#(t_2) + \#_a(t_1)\times\#(t_2) & (\mathit{Prop.}\ D.1)\\
> 1 + \#((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle]) + \#^{\#_\bullet((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle])}(\mathtt{L}_2) + \#(t_2) + \#_a(t_1)\times\#(t_2) & (\mathit{IH})\\
= 1 + \#((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle]) + \#^{\#_\bullet((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle])}(\mathtt{L}_2) + \#(t_2) + \#_a(t_1)\times\#(t_2) & (\mathit{Lem.}\ D.14)\\
= 1 + \#((\lambda a.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle]\mathtt{L}_2) + \#(t_2) + \#_a(t_1)\times\#(t_2) & (\mathit{Def.}\ D.6)\\
= \#((\lambda a^{\beta}.\,t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_1\langle\!\langle s_3\rangle\!\rangle]\mathtt{L}_2\,t_2) & (\mathit{Def.}\ D.5)
\end{array}
$$

6. $\mathtt{C} = (\lambda a^{\beta}.\,t_1)\mathtt{L}\,\mathtt{C}_1$. Suppose $t = (\lambda a^{\beta}.\,t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle \to_R (\lambda a^{\beta}.\,t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle = s$ follows from $t_2 \to_R s_2$.

$$
\begin{array}{ll}
\#((\lambda a^{\beta}.\,t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) & \\
= 1 + \#((\lambda a.\,t_1)\mathtt{L}) + \#(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#_a(t_1)\times\#(\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle) & (\mathit{Def.}\ D.6)\\
> 1 + \#((\lambda a.\,t_1)\mathtt{L}) + \#(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#_a(t_1)\times\#(\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) & (\mathit{IH}\times 2)\\
= \#((\lambda a^{\beta}.\,t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle) & (\mathit{Def.}\ D.6)
\end{array}
$$

7. $\mathtt{C} = (\lambda a^{\beta}.\,t_1)\mathtt{L}\,\mathtt{C}_1$. Suppose $t = (\lambda a^{\beta}.\,t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle \to_R (\lambda a^{\beta}.\,t_1)\mathtt{L}\,\mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle = s$ follows from $t_2 \to_R s_2$.
8. $\mathtt{C} = \mathtt{C}_1\,t_1$. Suppose $t = \mathtt{C}_1\langle\!\langle t_2\rangle\!\rangle\,t_1 \to_R \mathtt{C}_1\langle\!\langle s_2\rangle\!\rangle\,t_1 = s$ follows from $t_2 \to_R s_2$.
9. $\mathtt{C} = t_1\,\mathtt{C}_1$. Similar to the case $\mathtt{C} = \mathtt{C}_1\,t_1$.
10. $\mathtt{C} = \bullet\mathtt{C}_1$. Similar to the case $\mathtt{C} = \lambda a.\,\mathtt{C}_1$.
11. $\mathtt{C} = \mathtt{req}^{(\beta)}(\mathtt{C}_1)$. We need to consider multiple cases since out induction proceeds on well-formed labeled contexts.
   11.1. β is not present. Suppose $t = \mathtt{req}(\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle) \to_R \mathtt{req}(\mathtt{C}_1\langle\!\langle s_1\rangle\!\rangle) = s$ follows from $t_1 \to_R s_1$. Similar to the case $\mathtt{C} = \lambda a.\,\mathtt{C}_1$.
   11.2. β is present.
      11.2.1. $\mathtt{C}_1 = (\bullet\mathtt{C}_2)\mathtt{L}$. Suppose $t = \mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle)\mathtt{L}) \to_R \mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle)\mathtt{L}) = s$ follows from $t_1 \to_R s_1$.

$$
\begin{array}{ll}
\#(\mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle)\mathtt{L})) & \\
= 1 + \#((\bullet\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle)\mathtt{L}) & (\mathit{Def.}\ D.6)\\
= 1 + \#(\bullet\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle) + \#^{\#_\bullet((\bullet\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle))}(\mathtt{L}) & (\mathit{Lem.}\ D.14)\\
\geq 1 + \#(\bullet\mathtt{C}_2\langle\!\langle t_1\rangle\!\rangle) + \#^{\#_\bullet((\bullet\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle))}(\mathtt{L}) & (\mathit{Prop.}\ D.1)\\
> 1 + \#(\bullet\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle) + \#^{\#_\bullet((\bullet\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle))}(\mathtt{L}) & (\mathit{IH})\\
= 1 + \#((\bullet\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle)\mathtt{L}) & (\mathit{Lem.}\ D.14)\\
= \#(\mathtt{req}^{\beta}((\bullet\mathtt{C}_2\langle\!\langle s_1\rangle\!\rangle)\mathtt{L})) & (\mathit{Def.}\ D.6)
\end{array}
$$

      11.2.2. $\mathtt{C}_1 = \Box\mathtt{L}_2$ and $t = \mathtt{req}^{\beta}((\bullet\mathtt{D}\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2) \to_R^{\alpha} \mathtt{req}^{\beta}((\bullet\mathtt{D}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_3)\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2) = s$. Then we reason as in the previous case but using the root case (item 2) instead of the IH.
      11.2.3. $\mathtt{C}_1 = \Box\mathtt{L}_2$ and $t = \mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}_1[u^{\alpha}/(!t_3)\mathtt{K}]\mathtt{L}_2) \to_R^{\alpha} \mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}_1\mathtt{K}\mathtt{L}_2) = s$. Same as previous case.
      11.2.4. $\mathtt{C}_1 = (\bullet t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_2]\mathtt{L}_2$. Suppose $t = \mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_2\langle\!\langle t_2\rangle\!\rangle]\mathtt{L}_2) \to_R (\bullet t_1)\mathtt{L}_1[v^{(\gamma)}/\mathtt{C}_2\langle\!\langle s_2\rangle\!\rangle]\mathtt{L}_2 = s$ follows from $t_2 \to_R s_2$.

12. $\mathsf{C} = !\mathsf{C}_1$. Similar to the case $\mathsf{C} = \lambda a.\,\mathsf{C}_1$.
13. $\mathsf{C} = \mathsf{C}_1[v/t_1]$. Suppose $t = \mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle[v/t_1] \to_R \mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle[v/t_1] = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{ll}
\#(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle[v/t_1]) & \\
= \#(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#(t_1) + \#_v(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle) \times \#(t_1) + \#_v(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle) & (Def.\ D.6) \\
\geq \#(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#(t_1) + \#_v(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle) \times \#(t_1) + \#_v(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle) & (Prop.\ D.1) \\
> \#(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#(t_1) + \#_v(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle) \times \#(t_1) + \#_v(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle) & (IH) \\
= \#(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle[v/t_1]) & (Def.\ D.6)
\end{array}$$

14. $\mathsf{C} = \mathsf{C}_1[v^\beta/t_1]$. Suppose $t = \mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle[v^\beta/t_1] \to_R \mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle[v^\beta/t_1] = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{ll}
\#(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle[v^\beta/t_1]) & \\
= 1 + \#(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#(t_1) & (Def.\ D.6) \\
> 1 + \#(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#(t_1) & (IH) \\
= \#(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle[v^\beta/t_1]) & (Def.\ D.6)
\end{array}$$

15. $\mathsf{C} = t_1[v/\mathsf{C}_1]$. Suppose $t = t_1[v/\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle] \to_R t_1[v/\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle] = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{ll}
\#(t_1[v/\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle]) & \\
= \#(t_1) + \#(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#_v(t_1) \times \#(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle) + \#_v(t_1) & (Def.\ D.6) \\
> \#(t_1) + \#(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#_v(t_1) \times \#(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle) + \#_v(t_1) & (IH \times 2) \\
= \#(t_1[v/\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle]) & (Def.\ D.6)
\end{array}$$

16. $\mathsf{C} = t_1[v^\beta/\mathsf{C}_1]$. Suppose $t = t_1[v^\beta/\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle] \to_R t_1[v^\beta/\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle] = s$ follows from $t_2 \to_R s_2$.

$$\begin{array}{ll}
\#(t_1[v^\beta/\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle]) & \\
= 1 + \#(t_1) + \#(\mathsf{C}_1\langle\!\langle t_2\rangle\!\rangle) & (Def.\ D.6) \\
> 1 + \#(t_1) + \#(\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle) & (IH) \\
= \#(t_1[v^\beta/\mathsf{C}_1\langle\!\langle s_2\rangle\!\rangle]) & (Def.\ D.6)
\end{array}$$

Semantic Orthogonality Semantic orthogonality, namely that if $t \to^\alpha s$ and $t \to^\beta r$, then there exists p such that $s \twoheadrightarrow^\beta p$ and $r \twoheadrightarrow^\alpha p$, fails as illustrated below:

$$\begin{tikzcd}
 & t_1[v/(!\bullet t_2)\mathbb{K}] \arrow[dr, equal, "\beta"] & \\
 & & t_1[v/(!\bullet t_2)\mathbb{K}] \\
t_1[u^\alpha/(!v^\beta)[v/(!\bullet t_2)\mathbb{K}]] \arrow[uur, "\alpha"] \arrow[ddr, "\beta"'] & & \neq \\
 & & t_1[v/!\bullet t_2]\mathbb{K} \\
 & t_1[u^\alpha/(!\bullet t_2)[v/(!\bullet t_2)]]\mathbb{K} \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

This motivates the following notion of *flattening*. $\lambda^{!\bullet}$ rewriting modulo flattening does indeed satisfy semantic orthogonality.

Definition D.7 (Flattening). Flattening *is a binary relation* $\equiv \subseteq \mathsf{T}^{\mathcal{L}}_{\bullet} \times \mathsf{T}^{\mathcal{L}}_{\bullet}$ *defined by the following inference rules that prove judgements of the form* $t \equiv s$*:*

$$\frac{v \notin \mathsf{fv}(t)}{t[u^{(\alpha)}/s[v^{(\beta)}/r]] \equiv t[u^{(\alpha)}/s][v^{(\beta)}/r]} \equiv\mathtt{F}$$

$$\frac{}{a \equiv a} \equiv\mathtt{LV} \qquad \frac{}{u^{(\alpha)} \equiv u^{(\alpha)}} \equiv\mathtt{UV} \qquad \frac{s \equiv t}{t \equiv s} \equiv\mathtt{S} \qquad \frac{t \equiv r \quad r \equiv s}{t \equiv s} \equiv\mathtt{T}$$

$$\frac{t \equiv s}{\lambda a^{(\alpha)}.\, t \equiv \lambda a^{(\alpha)}.\, s} \equiv\mathtt{Abs} \qquad \frac{t_1 \equiv s_1 \quad t_2 \equiv s_2}{t_1\, t_2 \equiv s_1\, s_2} \equiv\mathtt{App}$$

$$\frac{t \equiv s}{\bullet t \equiv \bullet s} \equiv\mathtt{Sh} \qquad \frac{t \equiv s}{!t \equiv !s} \equiv\mathtt{Ofc} \qquad \frac{t \equiv s}{\mathtt{req}^{(\alpha)}(t) \equiv \mathtt{req}^{(\alpha)}(s)} \equiv\mathtt{Open}$$

$$\frac{t_1 \equiv s_1 \quad t_2 \equiv s_2}{t_1[u^{(\alpha)}/t_2] \equiv s_1[u^{(\alpha)}/s_2]} \equiv\mathtt{ES}$$

We write $t \overset{\pi}{\equiv} s$ *when* π *is a derivation of* $t \equiv s$*. We also occasionally write* $t \equiv s$*, when there exists* π *such that* $t \overset{\pi}{\equiv} s$*. Finally, we write* $t \overset{\pi}{\equiv}_1 s$ *if the rule* $\equiv\mathtt{F}$ *is used exactly once in* π*.*

Remark D.1. Suppose $t \overset{\pi}{\equiv} s$. First note that we may assume, without loss of generality, that if the rule $\equiv\mathtt{F}$ is used exactly once, transitivity is not used at all (since if $\equiv\mathtt{F}$ is not used at all in a derivation of $t \equiv s$, then $t = s$). Second, it is easy to verify that there exists $n > 0$ and $r_1, \dots r_n$ and $\pi_1, \dots, \pi_{n-1}$ such that $r_1 = t$ and $r_n = s$ and $r_1 \overset{\pi_1}{\equiv}_1 r_2, r_2 \overset{\pi_2}{\equiv}_1 r_3, \dots, r_{n-1} \overset{\pi_{n-1}}{\equiv}_1 r_n$, where $n-1$ is the number of times $\equiv\mathtt{F}$ was used in π. In particular, if $n = 1$ then $t = s$.

Remark D.2. Note that $t \equiv t$, for all $t \in \mathsf{T}^{\mathcal{L}}_{\bullet}$ (*i.e.* $\equiv$ is reflexive) as may be verified by straightforward induction on t.

Lemma D.17. *Suppose* $t \overset{\pi}{\equiv} s$*. Then* $\mathsf{fv}(t) = \mathsf{fv}(s)$*.*

Proof. By induction on π.

Lemma D.18. *Suppose* $t \overset{\pi}{\equiv} s$*.*

1. *If* $t \in \mathsf{T}^{\mathcal{WL}}_{\bullet}$*, then* $s \in \mathsf{T}^{\mathcal{WL}}_{\bullet}$*.*
2. *If* $s \in \mathsf{T}^{\mathcal{WL}}_{\bullet}$*, then* $t \in \mathsf{T}^{\mathcal{WL}}_{\bullet}$*.*

Proof. The proof is by induction on π. The only interesting case is the rule $\equiv\mathtt{F}$.

$$\frac{v \notin \mathsf{fv}(t)}{t[u^{(\alpha)}/s[v^{(\beta)}/r]] \equiv t[u^{(\alpha)}/s][v^{(\beta)}/r]} \equiv\mathtt{F}$$

If the labels are not present, the result is immediate. If the label α is present in u, then $s = (!s_1)\mathrm{L}$ and the result is also immediate since the scope of u^{α} does not change.

Suppose v is labeled with β on the left-hand side and hence $v \notin \mathsf{fv}(s)$. Then the condition $v \notin \mathsf{fv}(t)$ makes sure that $v \notin \mathsf{fv}(t[u^{(\alpha)}/s])$ in the right-hand side. Similarly, suppose v is labeled with β on the right-hand side. Then $v \notin \mathsf{fv}(t[u^{(\alpha)}/s])$ and hence, in particular, $v \notin \mathsf{fv}(s)$.

Lemma D.19. *Suppose $t \overset{\pi}{\equiv} s$.*

1. *Then $t\{a := r\} \equiv s\{a := r\}$.*
2. *Then $r\{a := t\} \equiv r\{a := s\}$.*

Proof. The first item is proved by induction on π and resorts to Rem. D.2. The second by induction on r.

Lemma D.20.

1. *Suppose $\mathtt{C}\langle\!\langle u^\alpha \rangle\!\rangle \overset{\pi}{\equiv}_1 t$. Then*
   1.1. *there exists $\mathtt{D}$ such that $t = \mathtt{D}\langle\!\langle u^\alpha \rangle\!\rangle$; and*
   1.2. *$\mathtt{C}\langle\!\langle s \rangle\!\rangle \equiv \mathtt{D}\langle\!\langle s \rangle\!\rangle$, for all s.*
2. *Similarly, if $t \overset{\pi}{\equiv}_1 \mathtt{C}\langle\!\langle u^\alpha \rangle\!\rangle$, then*
   2.1. *there exists $\mathtt{D}$ such that $t = \mathtt{D}\langle\!\langle u^\alpha \rangle\!\rangle$; and*
   2.2. *$\mathtt{C}\langle\!\langle s \rangle\!\rangle \equiv \mathtt{D}\langle\!\langle s \rangle\!\rangle$, for all s.*

Proof. By simultaneous induction on both items. We consider all possible cases for π:

1. π ends in $\equiv$`LV`. Not possible.
2. π ends in $\equiv$`F`. Then $t = t_1[v^{(\beta)}/t_2][w^{(\gamma)}/t_3]$ and $\mathtt{C}\langle\!\langle u^\alpha \rangle\!\rangle = t_1[v^{(\beta)}/t_2[w^{(\gamma)}/t_3]]$. There are three cases.
   2.1. $\mathtt{C} = \mathtt{C}_1[v^{(\beta)}/t_2[w^{(\gamma)}/t_3]]$. We set $\mathtt{D} := \mathtt{C}_1[v^{(\beta)}/t_2][w^{(\gamma)}/t_3]$ and conclude.
   2.2. $\mathtt{C} = t_1[v^{(\beta)}/\mathtt{C}_1[w^{(\gamma)}/t_3]]$. We set $\mathtt{D} := t_1[v^{(\beta)}/\mathtt{C}_1][w^{(\gamma)}/t_3]$ and conclude.
   2.3. $\mathtt{C} = t_1[v^{(\beta)}/t_2[w^{(\gamma)}/\mathtt{C}_1]]$. We set $\mathtt{D} := t_1[v^{(\beta)}/t_2][w^{(\gamma)}/\mathtt{C}_1]$ and conclude.
3. π ends in $\equiv$`UV`. Then $t = u^\alpha$ and $\mathtt{C} = \Box$. We set $\mathtt{D} := \Box$ and conclude.
4. π ends in $\equiv$`S`. We resort to the IH w.r.t. item (2).
5. π ends in $\equiv$`T`. Not possible by Rem. D.1.
6. π ends in $\equiv$`Abs` (the cases $\equiv$`Sh` $\equiv$`Ofc`, and $\equiv$`Open` are similar and omitted). Then $t = \lambda a^{(\beta)}.\, t_1$ and $\mathtt{C} = \lambda a^{(\beta)}.\, \mathtt{C}_1$ and the derivation ends in

$$\frac{t_1 \equiv_1 \mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle}{\lambda a^{(\beta)}.\, t_1 \equiv_1 \lambda a^{(\beta)}.\, \mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle} \equiv\mathtt{Abs}$$

   We conclude from the IH.
7. π ends in $\equiv$`App` (the case $\equiv$`ES` is similar and omitted). Then $t = t_1\, t_2$ and there are two cases.
   7.1. $\mathtt{C} = \mathtt{C}_1\, s_2$. There are two further cases.
      7.1.1. The derivation ends in

$$\frac{\mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle \equiv_1 t_1 \quad s_2 = t_2}{\mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle\, s_2 \equiv_1 t_1\, t_2} \equiv\mathtt{App}$$

      We conclude from the IH.

7.1.2. The derivation ends in

$$\frac{\mathsf{C}_1\langle\!\langle u^\alpha \rangle\!\rangle = t_1 \quad s_2 \equiv_1 t_2}{\mathsf{C}_1\langle\!\langle u^\alpha \rangle\!\rangle\, s_2 \equiv_1 t_1\, t_2} \equiv\mathtt{App}$$

We conclude immediately by setting $\mathtt{D}$ to $\mathsf{C}_1\, t_2$.

7.2. $\mathsf{C} = s_1\,\mathsf{C}_1$. Similar to the previous case.

Lemma D.21. *Suppose $t \overset{\pi}{\equiv}_1 s$.*

1. *Then*
   - 1.1. $t = a$ *implies* $s = a$
   - 1.2. $t = u$ *implies* $s = u$
   - 1.3. $t = u^\alpha$ *implies* $s = u^\alpha$
   - 1.4. $t = \lambda a.\, t_1$ *implies* $s = \lambda a.\, s_1$ *and* $t_1 \equiv_1 s_1$
   - 1.5. $t = \lambda a^\alpha.\, t_1$ *implies* $s = \lambda a.\, s_1$ *and* $t_1 \equiv_1 s_1$
   - 1.6. $t = t_1\, t_2$ *implies* $s = s_1\, s_2$ *and either*
     - 1.6.1. $t_1 \equiv_1 s_1$ *and* $t_2 = s_2$*; or*
     - 1.6.2. $t_1 = s_1$ *and* $t_2 \equiv_1 s_2$.
   - 1.7. $t = \bullet t_1$ *implies* $s = \bullet s_1$ *and* $t_1 \equiv_1 s_1$
   - 1.8. $t = \mathtt{req}(t_1)$ *implies* $s = \mathtt{req}(s_1)$ *and* $t_1 \equiv_1 s_1$
   - 1.9. $t = \mathtt{req}^\alpha(t_1)$ *implies* $s = \mathtt{req}^\alpha(s_1)$ *and* $t_1 \equiv_1 s_1$
   - 1.10. $t = !t_1$ *implies* $s = !s_1$ *and* $t_1 \equiv_1 s_1$
   - 1.11. $t = t_1[u^{(\alpha)}/t_2]$ *implies either*
     - 1.11.1. $s = s_1[u^{(\alpha)}/s_2]$ *and* $t_1 \equiv_1 s_1$ *and* $t_2 = s_2$*; or*
     - 1.11.2. $s = s_1[u^{(\alpha)}/s_2]$ *and* $t_1 = s_1$ *and* $t_2 \equiv_1 s_2$*; or*
     - 1.11.3. $t_2 = t_{21}[v^{(\beta)}/t_{22}]$ *and* $s = t_1[u^{(\alpha)}/t_{21}][v^{(\beta)}/t_{22}]$ *and* $v \notin \mathsf{fv}(t_1)$*; or*
     - 1.11.4. $t_1 = t_{11}[v^{(\beta)}/t_{12}]$ *and* $s = t_{11}[v^{(\beta)}/t_{21}[u^{(\alpha)}/t_2]]$ *and* $u \notin \mathsf{fv}(t_{11})$.
2. *Then*
   - 2.1. $s = a$ *implies* $t = a$
   - 2.2. $s = u$ *implies* $t = u$
   - 2.3. $s = u^\alpha$ *implies* $t = u^\alpha$
   - 2.4. $s = \lambda a.\, s_1$ *implies* $t = \lambda a.\, t_1$ *and* $s_1 \equiv_1 t_1$
   - 2.5. $s = \lambda a^\alpha.\, s_1$ *implies* $t = \lambda a.\, t_1$ *and* $s_1 \equiv_1 t_1$
   - 2.6. $s = s_1\, s_2$ *implies* $t = t_1\, t_2$ *and either 1)* $s_1 \equiv_1 t_1$ *and* $s_2 = t_2$*; or 2)* $s_1 = t_1$ *and* $s_2 \equiv_1 t_2$.
   - 2.7. $s = \bullet s_1$ *implies* $t = \bullet t_1$ *and* $s_1 \equiv_1 t_1$
   - 2.8. $s = \mathtt{req}(s_1)$ *implies* $t = \mathtt{req}(t_1)$ *and* $s_1 \equiv_1 t_1$
   - 2.9. $s = \mathtt{req}^\alpha(s_1)$ *implies* $t = \mathtt{req}^\alpha(t_1)$ *and* $s_1 \equiv_1 t_1$
   - 2.10. $s = !s_1$ *implies* $t = !t_1$ *and* $s_1 \equiv_1 t_1$
   - 2.11. $s = s_1[u^{(\alpha)}/s_2]$ *implies either*
     - 2.11.1. $t = t_1[u^{(\alpha)}/t_2]$ *and* $s_1 \equiv_1 t_1$ *and* $s_2 = t_2$*; or*
     - 2.11.2. $t = t_1[u^{(\alpha)}/t_2]$ *and* $s_1 = t_1$ *and* $s_2 \equiv_1 t_2$*; or*
     - 2.11.3. $s_2 = s_{21}[v^{(\beta)}/s_{22}]$ *and* $t = s_1[u^{(\alpha)}/s_{21}][v^{(\beta)}/s_{22}]$ *and* $v \notin \mathsf{fv}(s_1)$*; or*
     - 2.11.4. $s_1 = s_{11}[v^{(\beta)}/s_{12}]$ *and* $t = s_{11}[v^{(\beta)}/s_{21}[u^{(\alpha)}/s_2]]$ *and* $u \notin \mathsf{fv}(s_{11})$.

Proof. Both items are proved by simultaneous induction on t and s. By Rem. D.1 we may assume that $\equiv\mathtt{T}$ is not used in π.

1. Cases for t.
   1.1. $t = a$ implies $s = a$. Then π must end in $\equiv$LV, or $\equiv$S. If it ends in $\equiv$LV, we conclude immediately. If it ends in $\equiv$S, we resort to the IH on item (2).
   1.2. $t = u$ implies $s = u$. Similar to the case $t = a$.
   1.3. $t = u^\alpha$ implies $s = u^\alpha$. Similar to the case $t = a$.
   1.4. $t = \lambda a. t_1$. Then π must end in $\equiv$Abs, or $\equiv$S. If it ends in $\equiv$Abs, we conclude immediately. If it ends in $\equiv$S, we resort to the IH on item (2).
   1.5. $t = \lambda a^\alpha. t_1$. Similar to the previous case.
   1.6. $t = t_1\, t_2$. Then π must end in $\equiv$App, or $\equiv$S. If it ends in $\equiv$S, we resort to the IH on item (2). If it ends in $\equiv$App, two cases are possible. One is when π ends in:

$$\frac{t_1 \equiv_1 s_1 \quad t_2 = s_2}{t_1\, t_2 \equiv_1 s_1\, s_2} \equiv\mathtt{App}$$

   Note that by Rem. D.1, $t_2 \equiv s_2$ implies $t_2 = s_2$. The result then holds immediately. The other case is when π ends with

$$\frac{t_1 = s_1 \quad t_2 \equiv_1 s_2}{t_1\, t_2 \equiv_1 s_1\, s_2} \equiv\mathtt{App}$$

   and is treated similarly.
   1.7. $t = \bullet t_1$. Then π must end in $\equiv$Sh, or $\equiv$S. If it ends in $\equiv$Sh, we conclude immediately. If it ends in $\equiv$S, we resort to the IH on item (2).
   1.8. $t = \mathtt{req}(t_1)$. Then π must end in $\equiv$Open, or $\equiv$S. If it ends in $\equiv$Open, we conclude immediately. If it ends in $\equiv$S, we resort to the IH on item (2).
   1.9. $t = \mathtt{req}^\alpha(t_1)$. Similar to the previous case.
   1.10. $t = !t_1$. Similar to the case $t = \bullet t_1$.
   1.11. $t = t[u^{(\alpha)}/s]$. Then π must end in $\equiv$F, or $\equiv$ES or $\equiv$S. If it ends in $\equiv$S, we resort to the IH on item (2). If π must end in $\equiv$F, then 10.3.3. holds. If π must end in $\equiv$ES, then either 10.3.1. or 10.3.2. holds.
2. The cases for s are symmetric to the ones for t considered above.

Lemma D.22. $\equiv_1 \to^\alpha_\bullet \subseteq \to^\alpha_\bullet \equiv$

Proof. Suppose $t \equiv_1 s \to^\alpha_\bullet r$. We prove that there exists p such that $t \to^\alpha_\bullet p \equiv r$. We perform induction on $t \in \mathsf{T}^{\mathcal{WL}}_\bullet$.

1. $t = a$ (case $t = u^{(\alpha)}$ is similar). Then $s = a$ by Lem. D.21(1.1.). The result holds trivially since a is in $\to^\alpha_\bullet$ normal form.
2. $t = \lambda a. t_1$. Then by Lem. D.21(1.4.), $s = \lambda a. s_1$, for some s_1, with $t_1 \equiv_1 s_1$. Moreover, $s \to^\alpha_\bullet r$ must follow from $s_1 \to^\alpha_\bullet r_1$. We conclude from the IH.
3. $t = t_1\, t_1$. Then by Lem. D.21(1.6.), $s = s_1\, s_2$, for some s_1, s_2, with $t_1 \equiv_1 s_1$ and $t_2 = s_2$ or $t_2 \equiv_1 s_2$ and $t_1 = s_1$. Moreover, since $t_1 \in \mathsf{T}^{\mathcal{WL}}_\bullet$, there can be no reduction at the root and thus $s \to^\alpha_\bullet r$ must follow from either $s_1 \to^\alpha_\bullet r_1$ (in which case $r = r_1 s_2$) or $s_2 \to^\alpha_\bullet r_2$ (in which case $r = s_1 r_2$). In all four cases, we conclude from the IH or immediately. For example, if $t_1 \equiv_1 s_1$ and $s_1 \to^\alpha_\bullet r_1$ holds, then the IH gives us p_1 such that $t_1 \to^\alpha_\bullet p_1 \equiv r_1$. From the latter we conclude, $t_1\, t_2 \to^\alpha_\bullet p_1\, t_2 \equiv r_1 s_2$.

4. $t = (\lambda a^\beta.\, t_1)\mathtt{L}\, t_2$. Then by Lem. D.21(1.6.), $s = s_1\, s_2$, for some s_1, s_2, and two possible cases may arise.
   4.1. $(\lambda a^\beta.\, t_1)\mathtt{L} \equiv_1 s_1$ and $t_2 = s_2$. We now consider each possible way in which $(\lambda a^\beta.\, t_1)\mathtt{L} \equiv_1 s_1$.
      4.1.1. $s_1 = (\lambda a^\beta.\, s_{11})\mathtt{L}$ and $(\lambda a^\beta.\, t_1)\mathtt{L} \equiv_1 s_1$ follows from $t_1 \equiv_1 s_{11}$. Suppose $s = \mathtt{C}\langle g\rangle \to^\alpha_\bullet \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto^\alpha_\bullet d$. We consider each possible case for $\mathtt{C}$
         4.1.1.1. $\mathtt{C} = \Box$. Then $\alpha = \beta$ and $r = s_1\{a := t_2\}\mathtt{L}$.

$$\begin{tikzcd}
(\lambda a^\beta.\, t_1)\mathtt{L}\, t_2 \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & (\lambda a^\beta.\, s_{11})\mathtt{L}\, t_2 \arrow[d, "\alpha"] \\
t_1\{a := t_2\}\mathtt{L} \arrow[r, phantom, "\equiv"] & s_{11}\{a := t_2\}\mathtt{L}
\end{tikzcd}$$

The equivalence at the bottom follows from Lem. D.19(1).

         4.1.1.2. $\mathtt{C} = (\lambda a^\beta.\, \mathtt{C}_1)\mathtt{L}\, t_2$. We conclude from the IH on t_1 since $t_1 \equiv_1 s_{11} = \mathtt{C}_1\langle g\rangle \to^\alpha_\bullet \mathtt{C}_1\langle d\rangle = r_1$.
         4.1.1.3. $\mathtt{C} = \Box\mathtt{L}_2\, t_2$. There are two possibilities for the step $g \mapsto^\alpha_\bullet d$. It can either be a $\mapsto^\alpha_{\bullet\mathsf{ls}}$-step or a $\mapsto^\alpha_{\bullet\mathsf{gc}}$-step.
            4.1.1.3.1. The step $g \mapsto^\alpha_\bullet d$ is a $\mapsto^\alpha_{\bullet\mathsf{ls}}$-step. Then $\mathtt{L} = \mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$ and the step has the form $\mathtt{E}\langle\!\langle u^\alpha\rangle\!\rangle[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^\alpha_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2$. Moreover, from $t_1 \equiv_1 s_{11}$, also $(\lambda a^\beta.\, t_1)\mathtt{L} \equiv_1 (\lambda a^\beta.\, s_{11})\mathtt{L} = \mathtt{E}\langle\!\langle u^\alpha\rangle\!\rangle$. Therefore, by Lem. D.20(1), there exists $\mathtt{D}$ such that $t_1 = \mathtt{D}\langle\!\langle u^\alpha\rangle\!\rangle$. We consider each possible form for $\mathtt{E}$.
               4.1.1.3.1.1. $\mathtt{E} = (\lambda a^\beta.\, \mathtt{E}_1)\mathtt{L}_1$ and $\mathtt{L} = \mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$ and $a \notin \mathsf{fv}((!(\bullet t_{21})\mathtt{K}_1))$. From Lem. D.20(2), it must be the case that $\mathtt{D} = (\lambda a^\beta.\, \mathtt{D}_1)\mathtt{L}_1$, for some $\mathtt{D}_1$.

$$\begin{tikzcd}
(\lambda a^\beta.\, \mathtt{D}_1\langle\!\langle u^\alpha\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & (\lambda a^\beta.\, \mathtt{E}_1\langle\!\langle u^\alpha\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\, t_2 \arrow[d, "\alpha"] \\
(\lambda a^\beta.\, \mathtt{D}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv"] & (\lambda a^\beta.\, \mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\, t_2
\end{tikzcd}$$

The bottom equivalence holds from Lem. D.20(2).

               4.1.1.3.1.2. $\mathtt{E} = (\lambda a^\beta.\, s_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}$ and $\mathtt{L} = \mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{11}[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$. From Lem. D.20(2), it must be the case that $\mathtt{D} = (\lambda a^\beta.\, t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_{12}$, for some $\mathtt{D}_1$.

$$\begin{tikzcd}
(\lambda a^\beta.\, t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle\!\langle u^\alpha\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & (\lambda a^\beta.\, s_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle u^\alpha\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\, t_2 \arrow[d, "\alpha"] \\
(\lambda a^\beta.\, t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv"] & (\lambda a^\beta.\, s_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\, t_2
\end{tikzcd}$$

The bottom equivalence holds from Lem. D.20(2).

4.1.1.3.2. The step $g \mapsto^{\alpha}_{\bullet} d$ is a $\mapsto^{\alpha}_{\bullet\mathsf{gc}}$-step. Then it has the form $(\lambda a^{\beta}.\, s_{11})\mathtt{L}_1[u^{\alpha}/(!t_{12})\mathtt{K}] \mapsto^{\alpha}_{\bullet\mathsf{gc}} (\lambda a^{\beta}.\, s_{11})\mathtt{L}_1\mathtt{K}$ and $u \notin \mathsf{fv}((\lambda a^{\beta}.\, t_1)\mathtt{L}_1)$. We make use of Lem. D.17 and reason as follows:

$$\begin{array}{ccc}
(\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{\alpha}/(!t_{12})\mathtt{K}]\mathtt{L}_2\, t_2 & \equiv_1 & (\lambda a^{\beta}.\, s_{11})\mathtt{L}_1[u^{\alpha}/(!t_{12})\mathtt{K}]\mathtt{L}_2\, t_2 \\
\downarrow \alpha & & \downarrow \alpha \\
(\lambda a^{\beta}.\, t_1)\mathtt{L}_1\mathtt{K}\mathtt{L}_2\, t_2 & \equiv & (\lambda a^{\beta}.\, s_{11})\mathtt{L}_1\mathtt{K}\mathtt{L}_2\, t_2
\end{array}$$

4.1.1.4. $\mathtt{C} = (\lambda a^{\beta}.\, s_{11})\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1]\mathtt{L}_2\, t_2$. Then the $t \equiv_1 s$ equivalence and the $s \to^{\alpha}_{\bullet} r$ step are disjoint.

$$\begin{array}{ccc}
(\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1\langle g\rangle]\mathtt{L}_2\, t_2 & \equiv_1 & (\lambda a^{\beta}.\, s_{11})\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1\langle g\rangle]\mathtt{L}_2\, t_2 \\
\downarrow \alpha & & \downarrow \alpha \\
(\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1\langle d\rangle]\mathtt{L}_2\, t_2 & \equiv & (\lambda a^{\beta}.\, s_{11})\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1\langle d\rangle]\mathtt{L}_2\, t_2
\end{array}$$

4.1.2. $s_1 = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/t_3[v^{(\delta)}/t_4]]\mathtt{L}_2$ and $\mathtt{L} = \mathtt{L}_1[u^{(\gamma)}/t_3][v^{(\delta)}/t_4]\mathtt{L}_2$ and $v \notin \mathsf{fv}((\lambda a^{\beta}.\, t_1)\mathtt{L}_1)$. Suppose $s = \mathtt{C}\langle g\rangle \to^{\alpha}_{\bullet} \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto^{\alpha}_{\bullet} d$. We consider each possible case for $\mathtt{C}$.

4.1.2.1. $\mathtt{C} = \Box$. Then $\alpha = \beta$ and we have

$$\begin{array}{ccc}
(\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/t_3][v^{(\delta)}/t_4]\mathtt{L}_2\, t_2 & \equiv_1 & (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/t_3[v^{(\delta)}/t_4]]\mathtt{L}_2\, t_2 \\
\downarrow \alpha & & \downarrow \alpha \\
t_1\{a := t_2\}\mathtt{L}_1[u^{(\gamma)}/t_3][v^{(\delta)}/t_4]\mathtt{L}_2 & \equiv & t_1\{a := t_2\}\mathtt{L}_1[u^{(\gamma)}/t_3[v^{(\delta)}/t_4]]\mathtt{L}_2
\end{array}$$

4.1.2.2. $\mathtt{C} = (\lambda a^{\beta}.\, \mathtt{C}_1)\mathtt{L}_1[u^{(\gamma)}/t_3[v^{(\delta)}/t_4]]\mathtt{L}_2\, t_2$. Similar to the case $\mathtt{C} = \Box$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint (similar to the case $\mathtt{C} = \Box$). We conclude immediately.

4.1.2.3. $\mathtt{C} = (\lambda a^{\beta}.\, t_1)\mathtt{L}_{11}[v^{(\eta)}/\mathtt{C}_1]\mathtt{L}_{12}[u^{(\gamma)}/t_3[v^{(\delta)}/t_4]]\mathtt{L}_2\, t_2$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint (similar to the case $\mathtt{C} = \Box$). We conclude immediately.

4.1.2.4. $\mathtt{C} = \Box\mathtt{L}_2\, t_2$. There are two possibilities for the step $g \mapsto^{\alpha}_{\bullet} d$. It can either be a $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$-step or a $\mapsto^{\alpha}_{\bullet\mathsf{gc}}$-step.

4.1.2.4.1. The step $g \mapsto^{\alpha}_{\bullet} d$ is a $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$-step. Then the label γ is absent, $t_3 = (!(\bullet t_{31})\mathtt{K}_1)\mathtt{K}_2$ and thus $s = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u/(!(\bullet t_{31})\mathtt{K}_1)\mathtt{K}_2[v^{(\delta)}/t_4]]\mathtt{L}_2\, t_2$ and the step has the form $\mathtt{E}\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{31})\mathtt{K}_1)\mathtt{K}_2[v^{(\delta)}/t_4]] \mapsto^{\alpha}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{11})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet t_{31})\mathtt{K}_1]\mathtt{K}_2[v^{(\delta)}/t_4]$. We consider each possible form for $\mathtt{E}$.

4.1.2.4.1.1. $\mathtt{E} = (\lambda a^{\beta}.\, \mathtt{E}_1)\mathtt{L}_1$ and $\mathtt{L} = \mathtt{L}_1[u/(!(\bullet t_{31})\mathtt{K}_1)\mathtt{K}_2][v^{(\delta)}/t_4]\mathtt{L}_2$ and $a \notin \mathsf{fv}((!(\bullet t_{31})\mathtt{K}_1))$.

$$
\begin{tikzcd}
(\lambda a^{\beta}.\,\mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{31})\mathtt{K}_1)\mathtt{K}_2][v^{(\delta)}/t_4]\mathtt{L}_2 \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & (\lambda a^{\beta}.\,\mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{31})\mathtt{K}_1)\mathtt{K}_2[v^{(\delta)}/t_4]]\mathtt{L}_2 \arrow[d, "\alpha"] \\
(\lambda a^{\beta}.\,\mathtt{E}_1\langle\!\langle (\bullet t_{31})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{31})\mathtt{K}_1)]\mathtt{K}_2[v^{(\delta)}/t_4]\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv"] & (\lambda a^{\beta}.\,\mathtt{E}_1\langle\!\langle (\bullet t_{31})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{31})\mathtt{K}_1)]\mathtt{K}_2[v^{(\delta)}/t_4]\mathtt{L}_2\, t_2
\end{tikzcd}
$$

4.1.2.4.1.2. $\mathtt{E} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}$ and $\mathtt{L} = \mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}[u/(!(\bullet t_{31})\mathtt{K}_1)\mathtt{K}_2][v^{(\delta)}/t_4]\mathtt{L}_2$.

$$
\begin{tikzcd}
(\lambda a^{\beta}.\,t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{31})\mathtt{K}_1)\mathtt{K}_2][v^{(\delta)}/t_4]\mathtt{L}_2 \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & (\lambda a^{\beta}.\,t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{31})\mathtt{K}_1)\mathtt{K}_2[\arrow[d, "\alpha"] \\
(\lambda a^{\beta}.\,t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle (\bullet t_{31})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{31})\mathtt{K}_1)]\mathtt{K}_2[v^{(\delta)}/t_4]\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv"] & (\lambda a^{\beta}.\,t_1)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle (\bullet t_{31})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{31})\mathtt{K}_1)]\mathtt{K}
\end{tikzcd}
$$

4.1.2.4.2. The step $g \mapsto^{\alpha}_{\bullet} d$ is a $\mapsto^{\alpha}_{\bullet\mathsf{gc}}$-step. Then $\gamma = \alpha$ and $t_3 = (!t_{12})\mathtt{K}$ and the step has the form $(\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{\alpha}/(!t_{12})\mathtt{K}[v^{(\delta)}/t_4]] \mapsto^{\alpha}_{\bullet\mathsf{gc}} (\lambda a^{\beta}.\,t_1)\mathtt{L}_1\mathtt{K}[v^{(\delta)}/t_4]$ and $u \notin \mathsf{fv}((\lambda a^{\beta}.\,t_1)\mathtt{L}_1)$. We make use of Lem. D.17 and reason as follows:

$$
\begin{tikzcd}
(\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{\alpha}/(!t_{12})\mathtt{K}][v^{(\delta)}/t_4]\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & (\lambda a^{\beta}.\,s_{11})\mathtt{L}_1[u^{\alpha}/(!t_{12})\mathtt{K}[v^{(\delta)}/t_4]]\mathtt{L}_2\, t_2 \arrow[d, "\alpha"] \\
(\lambda a^{\beta}.\,t_1)\mathtt{L}_1\mathtt{K}[v^{(\delta)}/t_4]\mathtt{L}_2\, t_2 \arrow[r, phantom, "="] & (\lambda a^{\beta}.\,s_{11})\mathtt{L}_1\mathtt{K}[v^{(\delta)}/t_4]\mathtt{L}_2\, t_2
\end{tikzcd}
$$

4.1.2.5. $\mathtt{C} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\gamma)}/\square]\mathtt{L}_2\, t_2$. There are two possibilities for the step $g \mapsto^{\alpha}_{\bullet} d$. It can either be a $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$-step or a $\mapsto^{\alpha}_{\bullet\mathsf{gc}}$-step.

4.1.2.5.1. The step $g \mapsto^{\alpha}_{\bullet} d$ is a $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$-step. Then the label δ is absent, $t_4 = (!(\bullet t_{41})\mathtt{K}_1)\mathtt{K}_2$ and thus $s = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\gamma)}/t_3[v/(!(\bullet t_{41})\mathtt{K}_1)\mathtt{K}_2]]\mathtt{L}_2\, t_2$ and the step has the form $\mathtt{E}\langle\!\langle v^{\alpha}\rangle\!\rangle[v/(!(\bullet t_{41})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\alpha}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle (\bullet t_{41})\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_{41})\mathtt{K}_1]\mathtt{K}_2$. Moreover, by $(\lambda a^{\beta}.\,t_1)\mathtt{L} \equiv_1 \mathtt{E}\langle\!\langle u^{\alpha}\rangle\!\rangle$ and Lem. D.20(1), there exists $\mathtt{D}$ such that $s_1 = \mathtt{D}\langle\!\langle u^{\alpha}\rangle\!\rangle$. We consider each possible form for $\mathtt{E}$.

4.1.2.5.1.1. $\mathtt{E} = (\lambda a^{\beta}.\,\mathtt{E}_1)\mathtt{L}_1$. This case is not possible since $v \notin \mathsf{fv}((\lambda a^{\beta}.\,t_1)\mathtt{L}_1)$.

4.1.2.5.1.2. $\mathtt{E} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}$. This case is not possible since $v \notin \mathsf{fv}((\lambda a^{\beta}.\,t_1)\mathtt{L}_1)$.

4.1.2.5.1.3. $\mathtt{E} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{E}_1[v/(!(\bullet t_{41})\mathtt{K}_1)\mathtt{K}_2]]$.

$$
\begin{tikzcd}
(\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{E}_1\langle\!\langle v^{\alpha}\rangle\!\rangle][v/(!(\bullet t_{41})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{E}_1\langle\!\langle v^{\alpha}\rangle\!\rangle[v/(!(\bullet t_{41})\mathtt{K}_1)\mathtt{K}_2]]\mathtt{L}_2\, t_2 \arrow[d, "\alpha"] \\
(\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{E}_1\langle\!\langle (\bullet t_{41})\mathtt{K}_1\rangle\!\rangle][v/(!(\bullet t_{41})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv"] & (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{E}_1\langle\!\langle (\bullet t_{41})\mathtt{K}_1\rangle\!\rangle[v/(!(\bullet t_{41})\mathtt{K}_1)]\mathtt{K}_2]\mathtt{L}_2\, t_2
\end{tikzcd}
$$

Note the use of additional instances of $\equiv\mathtt{F}$ at the bottom of the diagram.

4.1.2.5.2. The step $g \mapsto^{\alpha}_{\bullet} d$ is a $\mapsto^{\alpha}_{\bullet\mathsf{gc}}$-step. Then $\gamma = \alpha$ and $t_4 = (!t_{41})\mathtt{K}$ and the step has the form $(\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\beta)}/t_3[v^{\alpha}/(!t_{41})\mathtt{K}]] \mapsto^{\alpha}_{\bullet\mathsf{gc}} (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\beta)}/t_3\mathtt{K}]$ and $v \notin \mathsf{fv}(t_3)$. Moreover, also $v \notin \mathsf{fv}((\lambda a^{\beta}.\, t_1)\mathtt{L}_1)$, by hypothesis of this case.

$$\begin{array}{ccc}
(\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\beta)}/t_3][v^{\alpha}/(!t_{41})\mathtt{K}]\mathtt{L}_2\, t_2 & \equiv_1 & (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\beta)}/t_3[v^{\alpha}/(!t_{41})\mathtt{K}]]\mathtt{L}_2\, t_2 \\
\downarrow \alpha & & \downarrow \alpha \\
(\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\beta)}/t_3]\mathtt{K}\mathtt{L}_2\, t_2 & \equiv & (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\beta)}/t_3\mathtt{K}]\mathtt{L}_2\, t_2
\end{array}$$

Note the use of additional instances of $\equiv\mathtt{F}$ at the bottom of the diagram.

4.1.2.6. $\mathtt{C} = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1[v^{(\delta)}/t_4]]\mathtt{L}_2\, t_2$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.1.2.7. $\mathtt{C} = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/t_3[v^{(\delta)}/\mathtt{C}_1]]\mathtt{L}_2\, t_2$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.1.2.8. $\mathtt{C} = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/t_3[v^{(\delta)}/t_4]]\mathtt{L}_{21}[v^{(\eta)}/\mathtt{C}_1]\mathtt{L}_{22}\, t_2$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.1.2.9. $\mathtt{C} = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/t_3[v^{(\delta)}/t_4]]\mathtt{L}_2\, \mathtt{C}_1$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.1.3. $s_1 = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/t_2][v^{(\delta)}/t_3]\mathtt{L}_2$ and $\mathtt{L} = \mathtt{L}_1[u^{(\gamma)}/t_2[v^{(\delta)}/t_3]]\mathtt{L}_2$. Symmetric to case 4.1.2..

4.1.4. $s_1 = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/s_{11}]\mathtt{L}_2$ and $\mathtt{L} = \mathtt{L}_1[u^{(\gamma)}/t_{12}]\mathtt{L}_2$ and $(\lambda a^{\beta}.\, t_1)\mathtt{L} \equiv_1 s_1$ follows from $t_{12} \equiv_1 s_{11}$. Suppose $s = \mathtt{C}\langle g\rangle \to^{\alpha}_{\bullet} \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto^{\alpha}_{\bullet} d$. We consider each possible case for $\mathtt{C}$

4.1.4.1. $\mathtt{C} = \Box$. Then $\alpha = \beta$ and we reason as follows:

$$\begin{array}{ccc}
(\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/t_{12}]\mathtt{L}_2\, t_2 & \equiv_1 & (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{(\gamma)}/s_{11}]\mathtt{L}_2\, t_2 \\
\downarrow \alpha & & \downarrow \alpha \\
t_1\{a := t_2\}\mathtt{L}_1[u^{(\gamma)}/t_{12}]\mathtt{L}_2 & \equiv & t_1\{a := t_2\}\mathtt{L}_1[u^{(\gamma)}/s_{11}]\mathtt{L}_2
\end{array}$$

4.1.4.2. $\mathtt{C} = (\lambda a^{\beta}.\, \mathtt{C}_1)\mathtt{L}_1[u^{(\gamma)}/s_{11}]\mathtt{L}_2\, t_2$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.1.4.3. $\mathtt{C} = (\lambda a^{\beta}.\, t_1)\mathtt{L}_{11}[v^{(\delta)}/\mathtt{C}_1]\mathtt{L}_{12}[u^{(\gamma)}/s_{11}]\mathtt{L}_2\, t_2$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.1.4.4. $\mathtt{C} = \Box\mathtt{L}_2\, t_2$. There are two possibilities for the step $g \mapsto^{\alpha}_{\bullet} d$. It can either be a $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$-step or a $\mapsto^{\alpha}_{\bullet\mathsf{gc}}$-step.

4.1.4.4.1. The step $g \mapsto^{\alpha}_{\bullet} d$ is a $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$-step. Then the label γ is absent, $s_{11} = (!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2$ and thus $s = (\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u/(!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\, t_2$ and the step has the form $\mathtt{E}\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\alpha}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet s_{111})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet s_{111})\mathtt{K}_1]\mathtt{K}_2$.

We next consider each possible way in which

$$t_{12} \equiv_1 (!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2$$

and then, for each of these, each possible form for $\mathtt{E}$.

4.1.4.4.1.1. $t_{12} = (!(\bullet t_{121})\mathtt{K}_1)\mathtt{K}_2$ and $t_{12} \equiv_1 (!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2$ follows from $t_{121} \equiv_1 s_{111}$. Then there are two possible forms for $\mathtt{E}$

4.1.4.4.1.1.1. $\mathtt{E} = (\lambda a^{\beta}.\, \mathtt{E}_1)\mathtt{L}_1$ and $\mathtt{L} = \mathtt{L}_1[u/(!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$ and $a \notin \mathsf{fv}((!(\bullet s_{111})\mathtt{K}_1))$.

$$\begin{tikzcd}
(\lambda a^{\beta}.\, \mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \arrow[r, phantom, "\equiv"] \arrow[d, "\alpha"] & (\lambda a^{\beta}.\, \mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \arrow[d, "\alpha"] \\
(\lambda a^{\beta}.\, \mathtt{E}_1\langle\!\langle(\bullet s_{111})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet s_{111})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv"] & (\lambda a^{\beta}.\, \mathtt{E}_1\langle\!\langle(\bullet s_{111})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet s_{111})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\, t_2
\end{tikzcd}$$

4.1.4.4.1.1.2. $\mathtt{E} = (\lambda a^{\beta}.\, t_1)\mathtt{L}_{11}[v^{(\delta)}/\mathtt{E}_1]\mathtt{L}_{12}$ and $\mathtt{L} = \mathtt{L}_{11}[v^{(\delta)}/\mathtt{E}_1]\mathtt{L}_{12}[u/(!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$.

$$\begin{tikzcd}
(\lambda a^{\beta}.\, t_1)\mathtt{L}_{11}[v^{(\delta)}/\mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \arrow[r, phantom, "\equiv"] \arrow[d, "\alpha"] & (\lambda a^{\beta}.\, t_1)\mathtt{L}_{11}[v^{(\delta)}/\mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L} \arrow[d, "\alpha"] \\
(\lambda a^{\beta}.\, t_1)\mathtt{L}_{11}[v^{(\delta)}/\mathtt{E}_1\langle\!\langle(\bullet s_{111})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet s_{111})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2\, t_2 \arrow[r, phantom, "\equiv"] & (\lambda a^{\beta}.\, t_1)\mathtt{L}_{11}[v^{(\delta)}/\mathtt{E}_1\langle\!\langle(\bullet s_{111})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet s_{111})\mathtt{K}_1)]\mathtt{K}
\end{tikzcd}$$

4.1.4.4.1.2. $t_{12} = (!(\bullet s_{111})\mathtt{K}_{11}[v^{(\delta)}/t_{121}]\mathtt{K}_{12})\mathtt{K}_2$ and $s_{11} = (!(\bullet s_{111})\mathtt{K}_{11}[v^{(\delta)}/s_{112}]\mathtt{K}_{12})\mathtt{K}_2$ and $t_{121} \equiv_1 s_{112}$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

4.1.4.4.1.3. $t_{12} = (!(\bullet s_{111})\mathtt{K}_{11}[u^{(\alpha)}/t_{121}][v^{(\delta)}/t_{122}]\mathtt{K}_{12})\mathtt{K}_2$ and $s_{11} = (!(\bullet s_{111})\mathtt{K}_{11}[u^{(\alpha)}/t_{121}[v^{(\delta)}/t_{122}]]\mathtt{K}_{12})\mathtt{K}_2$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

4.1.4.4.1.4. $t_{12} = (!(\bullet s_{111})\mathtt{K}_{11}[u^{(\alpha)}/t_{121}[v^{(\delta)}/t_{122}]]\mathtt{K}_{12})\mathtt{K}_2$ and $s_{11} = (!(\bullet s_{21})\mathtt{K}_{11}[u^{(\alpha)}/t_{121}][v^{(\delta)}/t_{122}]\mathtt{K}_{12})\mathtt{K}_2$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

4.1.4.4.1.5. $t_{12} = (!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_{21}[v^{(\delta)}/t_{121}]\mathtt{K}_{12}$ and $s_{11} = (!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_{21}[v^{(\delta)}/s_{112}]\mathtt{K}_{12}$ and $t_{121} \equiv_1 s_{112}$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

4.1.4.4.1.6. $t_{12} = (!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_{21}[u^{(\alpha)}/t_{121}][v^{(\delta)}/t_{122}]\mathtt{K}_{22}$ and $s_{11} = (!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_{21}[u^{(\alpha)}/t_{121}[v^{(\delta)}/t_{122}]]\mathtt{K}_{22}$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

4.1.4.4.1.7. $t_{12} = (!(\bullet s_{111})\mathtt{K}_1)\mathtt{K}_{21}[u^{(\alpha)}/t_{121}[v^{(\delta)}/t_{122}]]\mathtt{K}_{22}$ and $s_{11} = (!(\bullet s_{21})\mathtt{K}_1)\mathtt{K}_{21}[u^{(\alpha)}/t_{121}][v^{(\delta)}/t_{122}]\mathtt{K}_{22}$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

4.1.4.4.2. The step $g \mapsto_{\bullet}^{\alpha} d$ is a $\mapsto_{\bullet\mathsf{gc}}^{\alpha}$-step. Then $\gamma = \alpha$ and $s_{11} = (!s_{111})\mathtt{K}$ and the step has the form $(\lambda a^{\beta}.\, t_1)\mathtt{L}_1[u^{\alpha}/(!s_{111})\mathtt{K}] \mapsto_{\bullet\mathsf{gc}}^{\alpha} (\lambda a^{\beta}.\, t_1)\mathtt{L}_1\mathtt{K}$ and $u \notin \mathsf{fv}((\lambda a^{\beta}.\, t_1)\mathtt{L}_1)$. We make use of Lem. D.17 and reason as follows:

$$\begin{array}{ccc}
(\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{\alpha}/(!s_{111})\mathtt{K}]\mathtt{L}_2\,t_2 & \equiv_1 & (\lambda a^{\beta}.\,s_{11})\mathtt{L}_1[u^{\alpha}/(!s_{111})\mathtt{K}]\mathtt{L}_2\,t_2 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\
(\lambda a^{\beta}.\,t_1)\mathtt{L}_1\mathtt{K}\mathtt{L}_2\,t_2 & \equiv & (\lambda a^{\beta}.\,s_{11})\mathtt{L}_1\mathtt{K}\mathtt{L}_2\,t_2
\end{array}$$

4.1.4.5. $\mathtt{C} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1]\mathtt{L}_2\,t_2$. We conclude from the IH.

4.1.4.6. $\mathtt{C} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\gamma)}/s_{11}]\mathtt{L}_{21}[v^{(\delta)}/\mathtt{C}_1]\mathtt{L}_{22}\,t_2$. The $\equiv_1$-step and the $\rightarrow^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.2. $(\lambda a^{\beta}.\,t_1)\mathtt{L} = s_1$ and $t_2 \equiv_1 s_2$. That is to say, $s = (\lambda a^{\beta}.\,t_1)\mathtt{L}\,s_2$ and $t = (\lambda a^{\beta}.\,t_1)\mathtt{L}\,t_2 \equiv_1 s$ follows from $t_2 \equiv_1 s_2$. Suppose $s = \mathtt{C}\langle g\rangle \rightarrow^{\alpha}_{\bullet} \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto^{\alpha}_{\bullet} d$. We consider each possible case for $\mathtt{C}$.

4.2.1. $\mathtt{C} = \Box$. Then $\alpha = \beta$ and $r = s_1\{a := t_2\}\mathtt{L}$.

$$\begin{array}{ccc}
(\lambda a^{\beta}.\,t_1)\mathtt{L}\,t_2 & \equiv_1 & (\lambda a^{\beta}.\,t_1)\mathtt{L}\,s_2 \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\
t_1\{a := t_2\}\mathtt{L} & \equiv & t_1\{a := s_2\}\mathtt{L}
\end{array}$$

The equivalence at the bottom follows from Lem. D.19(2).

4.2.2. $\mathtt{C} = (\lambda a^{\beta}.\,\mathtt{C}_1)\mathtt{L}\,t_2$. The $\equiv_1$-step and the $\rightarrow^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.2.3. $\mathtt{C} = \Box\mathtt{L}_2\,t_2$. The $\equiv_1$-step and the $\rightarrow^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.2.4. $\mathtt{C} = (\lambda a^{\beta}.\,t_1)\mathtt{L}_1[u^{(\beta)}/\mathtt{C}_1]\mathtt{L}_2\,t_2$. The $\equiv_1$-step and the $\rightarrow^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

4.2.5. $\mathtt{C} = (\lambda a^{\beta}.\,t_1)\mathtt{L}\,\mathtt{C}_1$. We resort to the IH.

5. $t = \bullet t_1$. Then by Lem. D.21(1.4.), $s = \bullet s_1$, for some s_1, with $t_1 \equiv_1 s_1$. Moreover, $s \rightarrow^{\alpha}_{\bullet} r$ must follow from $s_1 \rightarrow^{\alpha}_{\bullet} r_1$. We conclude from the IH.

6. $t = \mathtt{req}(t_1)$. Same as the previous case.

7. $t = \mathtt{req}^{\beta}((\bullet t_1)\mathtt{L})$. By Lem. D.21(1.9.), $s = \mathtt{req}^{\beta}(s_1)$ and $(\bullet t_1)\mathtt{L} \equiv_1 s_1$. We now consider all possible ways in which $(\bullet t_1)\mathtt{L} \equiv_1 s_1$.

7.1. $s_1 = (\bullet s_{11})\mathtt{L}$ and $(\bullet t_1)\mathtt{L} \equiv_1 s_1$ follows from $t_1 \equiv_1 s_{11}$. Suppose $s = \mathtt{C}\langle g\rangle \rightarrow^{\alpha}_{\bullet} \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto^{\alpha}_{\bullet} d$. We consider each possible case for $\mathtt{C}$.

7.1.1. $\mathtt{C} = \Box$. Then $\alpha = \beta$ and we have:

$$\begin{array}{ccc}
\mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}) & \equiv_1 & \mathtt{req}^{\beta}((\bullet s_{11})\mathtt{L}) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\
t_1\mathtt{L} & \equiv & s_{11}\mathtt{L}
\end{array}$$

7.1.2. $\mathtt{C} = (\bullet\mathtt{C}_1)\mathtt{L}$. We use the IH.

7.1.3. $\mathtt{C} = \Box\mathtt{L}_2$ and $\mathtt{L} = \mathtt{L}_1[u/(!(\bullet s_2)\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$. The $\equiv_1$-step and the $\rightarrow^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

7.1.4. $\mathtt{C} = (\bullet s_{11})\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1]\mathtt{L}_2$. The $\equiv_1$-step and the $\rightarrow^{\alpha}_{\bullet}$ are disjoint.

$$\begin{array}{ccc}
\mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1\langle g\rangle]\mathtt{L}_2) & \equiv_1 & \mathtt{req}^{\beta}((\bullet s_{11})\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1\langle g\rangle]\mathtt{L}_2) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} \\
\mathtt{req}^{\beta}((\bullet t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1\langle d\rangle]\mathtt{L}_2) & \equiv & \mathtt{req}^{\beta}((\bullet s_{11})\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1\langle d\rangle]\mathtt{L}_2)
\end{array}$$

7.2. $s_1 = (\bullet t_1)\mathtt{L}_1[u^{(\gamma)}/s_{11}]\mathtt{L}_2$ and $\mathtt{L} = \mathtt{L}_1[u^{(\gamma)}/t_2]\mathtt{L}_2$ and $(\bullet t_1)\mathtt{L} \equiv_1 s_1$ follows from $t_2 \equiv_1 s_{11}$. Suppose $s = \mathtt{C}\langle g\rangle \to_{\bullet}^{\alpha} \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto_{\bullet}^{\alpha} d$. We consider each possible case for $\mathtt{C}$.

7.2.1. $\mathtt{C} = \Box$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

7.2.2. $\mathtt{C} = (\bullet \mathtt{C}_1)\mathtt{L}_1[u^{(\gamma)}/s_{11}]\mathtt{L}_2$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

7.2.3. $\mathtt{C} = (\bullet t_1)\mathtt{L}_{11}[v^{(\delta)}/\mathtt{C}_1]\mathtt{L}_{12}[u^{(\gamma)}/s_{11}]\mathtt{L}_2$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

7.2.4. $\mathtt{C} = (\bullet t_1)\mathtt{L}_1[u^{(\gamma)}/\mathtt{C}_1]\mathtt{L}_2$. We resort to the IH.

7.2.5. $\mathtt{C} = (\bullet t_1)\mathtt{L}_1[u^{(\gamma)}/s_{11}]\mathtt{L}_{21}[v^{(\delta)}/\mathtt{C}_1]\mathtt{L}_{22}$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

7.3. $s_1 = (\bullet t_1)\mathtt{L}_1[u^{(\gamma)}/t_{21}][v^{(\delta)}/t_{22}]\mathtt{L}_2$ and $\mathtt{L} = \mathtt{L}_1[u^{(\gamma)}/t_{21}[v^{(\delta)}/t_{22}]]\mathtt{L}_2$ and $v \notin \mathsf{fv}((\bullet t_1)\mathtt{L}_1)$. Suppose $s = \mathtt{C}\langle g\rangle \to_{\bullet}^{\alpha} \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto_{\bullet}^{\alpha} d$. Similar to case 4.1.2..

7.4. $s_1 = (\bullet t_1)\mathtt{L}_1[u^{(\gamma)}/t_{21}[v^{(\delta)}/t_{22}]]\mathtt{L}_2$ and $\mathtt{L} = \mathtt{L}_1[u^{(\gamma)}/t_{21}][v^{(\delta)}/t_{22}]\mathtt{L}_2$ and $v \notin \mathsf{fv}((\bullet t_1)\mathtt{L}_1)$. Suppose $s = \mathtt{C}\langle g\rangle \to_{\bullet}^{\alpha} \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto_{\bullet}^{\alpha} d$. Similar to case 4.1.3..

8. $t = !t_1$. Similar to the case $t = \bullet t_1$.

9. $t = t_1[u/(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2]$ and $u \in \mathsf{fv}^{\mathcal{L}}(t_1)$. By Lem. D.21(1.11.), there are four cases to consider.

9.1. $s = s_1[u/s_2]$ and $t_1 \equiv_1 s_1$ and $(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2 = s_2$. Suppose $s = \mathtt{C}\langle g\rangle \to_{\bullet}^{\alpha} \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto_{\bullet}^{\alpha} d$. We consider each possible case for $\mathtt{C}$.

9.1.1. $\mathtt{C} = \Box$. The step $g \mapsto_{\bullet}^{\alpha} d$ is a $\mapsto_{\bullet\mathsf{ls}}^{\alpha}$-step. Therefore, the step has the form $\mathtt{E}\langle\langle u^{\alpha}\rangle\rangle[u/(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2] \mapsto_{\bullet\mathsf{ls}}^{\alpha} \mathtt{E}\langle\langle(\bullet t_2)\mathtt{L}_1\rangle\rangle[u/(!(\bullet t_2)\mathtt{L}_1)]\mathtt{L}_2$. Moreover, by $t_1 \equiv_1 s_1 = \mathtt{E}\langle\langle u^{\alpha}\rangle\rangle$ and Lem. D.20(1), there exists $\mathtt{D}$ such that $t_1 = \mathtt{D}\langle\langle u^{\alpha}\rangle\rangle$.

$$\begin{array}{ccc}
\mathtt{D}_1\langle\langle u^{\alpha}\rangle\rangle\mathtt{L}_1[u/(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2] & \equiv_1 & \mathtt{E}_1\langle\langle u^{\alpha}\rangle\rangle\mathtt{L}_1[u/(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2] \\
\downarrow \alpha & & \downarrow \alpha \\
\mathtt{D}_1\langle\langle(\bullet t_2)\mathtt{L}_1\rangle\rangle\mathtt{L}_1[u/(!(\bullet t_2)\mathtt{L}_1)]\mathtt{L}_2 & \equiv & \mathtt{E}_1\langle\langle(\bullet t_2)\mathtt{L}_1\rangle\rangle\mathtt{L}_1[u/(!(\bullet t_2)\mathtt{L}_1)]\mathtt{L}_2
\end{array}$$

The bottom equivalence holds from Lem. D.20(2).

9.1.2. $\mathtt{C} = \mathtt{C}_1[u/s_2]$. We use the IH.

9.1.3. $\mathtt{C} = s_1[u/\mathtt{C}_1]$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

9.2. $s = s_1[u/s_2]$ and $t_1 = s_1$ and $(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2 \equiv_1 s_2$. We consider all ways in which $(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2 \equiv_1 s_2$. Similar to case 4.1.3..

9.3. $s = t_1[u/t_{21}][v^{(\beta)}/t_{22}]$ and $(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2 = t_{21}[v^{(\beta)}/t_{22}]$ and $v \notin \mathsf{fv}(t_1)$. Then $s = \mathtt{C}\langle g\rangle \to_{\bullet}^{\alpha} \mathtt{C}\langle d\rangle = r$, follows from $g \mapsto_{\bullet}^{\alpha} d$. We consider each possible case for $\mathtt{C}$

9.3.1. $\mathtt{C} = \mathtt{C}_1[u/t_{21}][v^{(\beta)}/t_{22}]$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

9.3.2. $\mathtt{C} = \Box[v^{(\beta)}/t_{22}]$. Same as case 4.1.3..

9.3.3. $\mathtt{C} = t_1[u/\mathtt{C}_1][v^{(\beta)}/t_{22}]$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

9.3.4. $\mathtt{C} = \Box$. Same as case 4.1.3..

9.3.5. $\mathtt{C} = t_1[u/t_{21}][v^{(\beta)}/\mathtt{C}_1]$. The $\equiv_1$-step and the $\to_{\bullet}^{\alpha}$ are disjoint. We conclude immediately.

9.4. $s = t_{11}[v^{(\beta)}/t_{21}[u/(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2]]$ and $t_1 = t_{11}[v^{(\beta)}/t_{21}]$ and $u \notin \mathsf{fv}(t_{11})$. Suppose $s = \mathsf{C}\langle g\rangle \to^{\alpha}_{\bullet} \mathsf{C}\langle d\rangle = r$, follows from $g \mapsto^{\alpha}_{\bullet} d$. We consider each possible case for C

9.4.1. $\mathsf{C} = \mathsf{C}_1[v^{(\beta)}/t_{21}[u/(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2]]$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

9.4.2. $\mathsf{C} = \Box$. Same as case 4.1.2.4.1.1..

9.4.3. $\mathsf{C} = t_{11}[v^{(\beta)}/\Box]$. Same as case 4.1.2.5..

9.4.4. $\mathsf{C} = t_{11}[v^{(\beta)}/\mathsf{C}_1[u/(!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2]]$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

9.4.5. $\mathsf{C} = t_{11}[v^{(\beta)}/t_{21}[u/\mathsf{C}_1]]$ and $\mathsf{C}_1\langle g\rangle = (!(\bullet t_2)\mathtt{L}_1)\mathtt{L}_2$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

10. $t = t_1[u^{\beta}/(!t_2)\mathtt{L}]$ and $u \notin \mathsf{fv}(t_1)$. By Lem. D.21(1.11.), there are four cases to consider.

10.1. $s = s_1[u^{\beta}/s_2]$ and $t_1 \equiv_1 s_1$ and $(!t_2)\mathtt{L} = s_2$. Suppose $s = \mathsf{C}\langle g\rangle \to^{\alpha}_{\bullet} \mathsf{C}\langle d\rangle = r$, follows from $g \mapsto^{\alpha}_{\bullet} d$. We consider each possible case for C.

10.1.1. $\mathsf{C} = \Box$. Then $\alpha = \beta$ and we reason as follows, making use of Lem. D.17 and reason as follows:

$$\begin{tikzcd} t_1[u^{\alpha}/(!t_2)\mathtt{L}] \arrow[r, equal, "\equiv_1"] \arrow[d, "\alpha"] & s_1[u^{\alpha}/(!t_2)\mathtt{L}] \arrow[d, "\alpha"] \\ t_1\mathtt{L} \arrow[r, "\equiv", no head] & s_1\mathtt{L} \end{tikzcd}$$

10.1.2. $\mathsf{C} = \mathsf{C}_1[u^{\beta}/(!t_2)\mathtt{L}]$. We resort to the IH.

10.1.3. $\mathsf{C} = s_1[u^{\beta}/\mathsf{C}_1]$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

10.2. $s = s_1[u^{\beta}/s_2]$ and $t_1 = s_1$ and $(!t_2)\mathtt{L} \equiv_1 s_2$. We consider all ways in which $(!t_2)\mathtt{L} \equiv_1 s_2$.

10.2.1. $s_2 = (!s_{21})\mathtt{L}$ and $(!t_2)\mathtt{L} \equiv_1 s_2$ follows from $t_2 \equiv_1 s_{21}$. Suppose $s = \mathsf{C}\langle g\rangle \to^{\alpha}_{\bullet} \mathsf{C}\langle d\rangle = r$, follows from $g \mapsto^{\alpha}_{\bullet} d$. We consider each possible case for C.

10.2.1.1. $\mathsf{C} = \Box$. Then $\alpha = \beta$ and we reason as follows, making use of Lem. D.17 and reason as follows:

$$\begin{tikzcd} t_1[u^{\beta}/(!t_2)\mathtt{L}] \arrow[r, equal, "\equiv_1"] \arrow[d, "\alpha"] & s_1[u^{\beta}/(!s_{21})\mathtt{L}] \arrow[d, "\alpha"] \\ t_1\mathtt{L} \arrow[r, equal] & s_1\mathtt{L} \end{tikzcd}$$

10.2.1.2. $\mathsf{C} = \mathsf{C}_1[u^{\beta}/(!s_{21})\mathtt{L}]$. The $\equiv_1$-step and the $\to^{\alpha}_{\bullet}$ are disjoint. We conclude immediately.

10.2.1.3. $\mathsf{C} = s_1[u^{\beta}/(!\mathsf{C}_1)\mathtt{L}]$. We resort to the IH.

10.2.1.4. $\mathsf{C} = s_1[u^{\beta}/\Box\mathtt{L}_2]$. There are two possibilities for the step $g \mapsto^{\alpha}_{\bullet} d$. It can either be a $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$-step or a $\mapsto^{\alpha}_{\bullet\mathsf{gc}}$-step.

10.2.1.4.1. The step $g \mapsto^{\alpha}_{\bullet} d$ is a $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$-step. Then the $\mapsto^{\alpha}_{\bullet\mathsf{ls}}$-step has the form $\mathtt{E}\langle\!\langle v^{\alpha}\rangle\!\rangle[v/(!(\bullet t_3)\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\alpha}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_3)\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_3)\mathtt{K}_1]\mathtt{K}_2$ and $\mathtt{E}\langle\!\langle v^{\alpha}\rangle\!\rangle = (!t_2)\mathtt{L}_1$. We consider each possible form for $\mathtt{E}$.

10.2.1.4.1.1. $\mathtt{E} = (!\mathtt{E}_1)\mathtt{L}_1$. The $\equiv_1$-step and the $\rightarrow_\bullet^\alpha$ are disjoint. We conclude immediately.

$$\begin{tikzcd}
t_1[u^\beta/(!\mathtt{E}_1\langle\!\langle v^\alpha\rangle\!\rangle)\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2] \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & s_1[u^\beta/(!\mathtt{E}_1\langle\!\langle v^\alpha\rangle\!\rangle)\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2] \arrow[d, "\alpha"] \\
t_1[u^\beta/(!\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2] \arrow[r, phantom, "\equiv"] & s_1[u^\beta/(!\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2]
\end{tikzcd}$$

10.2.1.4.1.2. $\mathtt{E} = (!t_2)\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}$ and $\mathtt{L} = \mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{11}[u/(!(\bullet t_{11})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$. The $\equiv_1$-step and the $\rightarrow_\bullet^\alpha$ are disjoint. We conclude immediately.

10.2.1.4.2. The step $g \mapsto_\bullet^\alpha d$ is a $\mapsto_{\bullet\mathsf{gc}}^\alpha$-step. Then it has the form $(!t_{12})\mathtt{L}_1[v^\alpha/(!t_3)\mathtt{K}] \mapsto_{\bullet\mathsf{gc}}^\alpha (!t_{12})\mathtt{L}_1\mathtt{K}$ and $u \notin \mathsf{fv}((!t_{12})\mathtt{L}_1)$. We make use of Lem. D.17 and reason as follows:

$$\begin{tikzcd}
t_1[u^\beta/(!t_{12})\mathtt{L}_1[v^\alpha/(!t_3)\mathtt{K}]\mathtt{L}_2] \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & s_1[u^\beta/(!t_{12})\mathtt{L}_1[v^\alpha/(!t_3)\mathtt{K}]\mathtt{L}_2] \arrow[d, "\alpha"] \\
t_1[u^\beta/(!t_{12})\mathtt{L}_1\mathtt{K}\mathtt{L}_2] \arrow[r, phantom, "\equiv"] & s_1[u^\beta/(!t_{12})\mathtt{L}_1\mathtt{K}\mathtt{L}_2]
\end{tikzcd}$$

10.2.1.5. $s_2 = (!t_2)\mathtt{L}_{11}[u^{(\gamma)}/s][v^{(\delta)}/r]\mathtt{L}_{12}$ and $(!t_2)\mathtt{L} = (!t_2)\mathtt{L}_{11}[u^{(\gamma)}/s[v^{(\delta)}/r]]\mathtt{L}_{12}$. Similar to case 4.1.2.

10.2.1.6. $s_2 = (!t_2)\mathtt{L}_{11}[u^{(\gamma)}/s[v^{(\delta)}/r]]\mathtt{L}_{12}$ and $(!t_2)\mathtt{L} = (!t_2)\mathtt{L}_{11}[u^{(\gamma)}/s][v^{(\delta)}/r]\mathtt{L}_{12}$. Similar to case 4.1.3.

10.2.2. $s_2 = (!t_2)\mathtt{L}_{11}[v^{(\gamma)}/s_{21}]\mathtt{L}_{12}$ and $\mathtt{L} = \mathtt{L}_{11}[v^{(\gamma)}/t_{21}]\mathtt{L}_{12}$ and $(!t_2)\mathtt{L} \equiv_1 s_2$ follows from $t_{21} \equiv_1 s_{21}$. Suppose $s = \mathsf{C}\langle g\rangle \rightarrow_\bullet^\alpha \mathsf{C}\langle d\rangle = r$, follows from $g \mapsto_\bullet^\alpha d$. We consider each possible case for C.

10.2.2.1. $\mathsf{C} = \Box$. Then $\alpha = \beta$ and we reason as follows, making use of Lem. D.17 and reason as follows:

$$\begin{tikzcd}
t_1[u^\beta/(!t_2)\mathtt{L}_{11}[v^{(\gamma)}/s_{21}]\mathtt{L}_{12}] \arrow[r, phantom, "\equiv_1"] \arrow[d, "\alpha"] & s_1[u^\beta/(!t_2)\mathtt{L}_{11}[v^{(\gamma)}/t_{21}]\mathtt{L}_{12}] \arrow[d, "\alpha"] \\
t_1\mathtt{L}_{11}[v^{(\gamma)}/s_{21}]\mathtt{L}_{12} \arrow[r, phantom, "\equiv"] & s_1\mathtt{L}_{11}[v^{(\gamma)}/t_{21}]\mathtt{L}_{12}
\end{tikzcd}$$

10.2.2.2. $\mathsf{C} = \mathsf{C}_1[u^\beta/(!t_2)\mathtt{L}_{11}[v^{(\gamma)}/t_{21}]\mathtt{L}_{12}]$. The $\equiv_1$-step and the $\rightarrow_\bullet^\alpha$ are disjoint. We conclude immediately.

10.2.2.3. $\mathsf{C} = s_1[u^\beta/(!\mathsf{C}_1)\mathtt{L}_{11}[v^{(\gamma)}/t_{21}]\mathtt{L}_{12}]$. The $\equiv_1$-step and the $\rightarrow_\bullet^\alpha$ are disjoint. We conclude immediately.

10.2.2.4. $\mathsf{C} = s_1[u^\beta/\Box\mathtt{L}_{12}]$. There are two possibilities for the step $g \mapsto_\bullet^\alpha d$. It can either be a $\mapsto_{\bullet\mathsf{ls}}^\alpha$-step or a $\mapsto_{\bullet\mathsf{gc}}^\alpha$-step. Similar to case 10.2.1.4.

10.2.2.5. $\mathsf{C} = s_1[u^\beta/(!t_2)\mathtt{L}_{111}[w^{(\delta)}/\mathsf{C}_1]\mathtt{L}_{112}[v^{(\gamma)}/t_{21}]\mathtt{L}_{12}]$. The $\equiv_1$-step and the $\rightarrow_\bullet^\alpha$ are disjoint. We conclude immediately.

10.2.2.6. $\mathsf{C} = s_1[u^\beta/(!t_2)\mathtt{L}_{11}[v^{(\gamma)}/\mathsf{C}_1]\mathtt{L}_{12}]$. We resort to the IH.

10.2.2.7. $\mathsf{C} = s_1[u^\beta/(!t_2)\mathsf{L}_{11}[v^{(\gamma)}/t_{21}]\mathsf{L}_{121}[w^{(\delta)}/\mathsf{C}_1]\mathsf{L}_{122}]$. The $\equiv_1$-step and the $\rightarrow^\alpha_\bullet$ are disjoint. We conclude immediately.

10.2.3. $\mathsf{L} = \mathsf{L}_{11}[w^{(\delta)}/t_3[v^{(\gamma)}/t_{22}]]\mathsf{L}_{12}$ and $s = t_1[u^\beta/(!t_2)\mathsf{L}_{11}[w^{(\delta)}/t_3][v^{(\gamma)}/t_{22}]\mathsf{L}_{12}]$ and $v \notin \mathsf{fv}((!t_2)\mathsf{L}_{11})$. Similar to case 4.1.2.

10.2.4. $\mathsf{L} = \mathsf{L}_{11}[w^{(\delta)}/t_3][v^{(\gamma)}/t_{22}]\mathsf{L}_{12}$ and $s = t_1[u^\beta/(!t_2)\mathsf{L}_{11}[w^{(\delta)}/t_3[v^{(\gamma)}/t_{22}]]\mathsf{L}_{12}]$ and $v \notin \mathsf{fv}((!t_2)\mathsf{L}_{11})$. Similar to case 4.1.3.

10.3. $t = t_1[u/t_2]$ and $u \notin \mathsf{fv}^{\mathcal{L}}(t_1)$. By Lem. D.21(1.11.), there are four cases to consider.

10.3.1. $s = s_1[u/s_2]$ and $t_1 \equiv_1 s_1$ and $t_2 = s_2$. Suppose $s = \mathsf{C}\langle g\rangle \rightarrow^\alpha_\bullet \mathsf{C}\langle d\rangle = r$, follows from $g \mapsto^\alpha_\bullet d$. We consider each possible case for C.

10.3.1.1. $\mathsf{C} = \Box$. Not possible.

10.3.1.2. $\mathsf{C} = \mathsf{C}_1[u/s_2]$. We resort to the IH.

10.3.1.3. $\mathsf{C} = s_1[u/\mathsf{C}_2]$. The $\equiv_1$-step and the $\rightarrow^\alpha_\bullet$ are disjoint. We conclude immediately.

10.3.2. $s = s_1[u/s_2]$ and $t_1 = s_1$ and $t_2 \equiv_1 s_2$. Suppose $s = \mathsf{C}\langle g\rangle \rightarrow^\alpha_\bullet \mathsf{C}\langle d\rangle = r$, follows from $g \mapsto^\alpha_\bullet d$. We consider each possible case for C.

10.3.2.1. $\mathsf{C} = \Box$. Not possible.

10.3.2.2. $\mathsf{C} = \mathsf{C}_1[u/s_2]$. The $\equiv_1$-step and the $\rightarrow^\alpha_\bullet$ are disjoint. We conclude immediately.

10.3.2.3. $\mathsf{C} = s_1[u/\mathsf{C}_2]$. We resort to the IH.

10.3.3. $t_2 = t_{21}[v^{(\beta)}/t_{22}]$ and $s = t_1[u/t_{21}][v^{(\beta)}/t_{22}]$ and $v \notin \mathsf{fv}(t_1)$. Similar to case 4.1.2.4..

10.3.4. $t_1 = t_{11}[v^{(\beta)}/t_{12}]$ and $s = t_{11}[v^{(\beta)}/t_{21}[u/t_2]]$ and $u \notin \mathsf{fv}(t_{11})$. Similar to case 4.1.2.4..

Proposition D.3 (Flattening is a strong $\rightarrow_\bullet$-bisimulation). $\equiv\rightarrow^\alpha_\bullet \subseteq \rightarrow^\alpha_\bullet\equiv$

Proof. Suppose $t \overset{\pi}{\equiv} s$. By Rem. D.1 there exists $n > 0$ and $r_1, \ldots r_n$ and $\pi_1, \ldots, \pi_{n-1}$ such that $r_1 = t$ and $r_n = s$ and $r_1 \overset{\pi_1}{\equiv}_1 r_2, r_2 \overset{\pi_2}{\equiv}_1 r_3, \ldots, r_{n-1} \overset{\pi_{n-1}}{\equiv}_1 r_n$, where $n-1$ is the number of times $\equiv$F was used in π. We therefore prove $(\equiv_1)^{n-1} \rightarrow^\alpha_\bullet \subseteq \rightarrow^\alpha_\bullet\equiv$, by induction on n. If $n = 1$, then $t = s$ and the result hold immediately. If $n = 2$, then the result follows from Lem. D.22. Suppose $n > 2$. Then we rely on Lem. D.22 and the IH:

$$\begin{tikzcd}
r_1 \arrow[r, equal, "n-2", "1"'] \arrow[d, "\bullet"', "\alpha"] \arrow[dr, phantom, "IH"] & r_{n-1} \arrow[r, equal, "\pi_{n-1}", "1"'] \arrow[d, "\bullet"', "\alpha"] \arrow[dr, phantom, "Lem.\ D.22"] & r_n \arrow[d, "\bullet"', "\alpha"] \\
r'_1 \arrow[r, equal] & r'_{n-1} \arrow[r, equal] & r'_n
\end{tikzcd}$$

Lemma D.23. *Let $t, s, r \in \mathsf{T}^{\mathcal{WL}}_\bullet$ be well-labeled terms. Suppose $t \rightarrow^\alpha s$. Then*

1. $t\{a := r\} \rightarrow^\alpha s\{a := r\}$
2. $r\{a := t\} \twoheadrightarrow^\alpha r\{a := s\}$

Proof. The first item is by induction on t and the second by induction on r.

1. $t = a, t = u^{(\alpha)}$. Immediate.
2. $t = \lambda b.\, t_1$. Then the step $t \to^\alpha s$ is in t_1 and we conclude from the IH.
3. $t = t_1\, t_2$. Then the step $t \to^\alpha s$ must be in t_1 or t_2, we conclude from the IH.
4. $t = (\lambda b^\beta.\, t_1)\mathsf{L}\, t_2$. If the step $t \to^\alpha s$ is in t_1 or t_2 or L, we conclude from the IH. Suppose it is at the root. Then $\beta = \alpha$ and $t = (\lambda b^\alpha.\, t_1)\mathsf{L}\, t_2 \mapsto^\alpha_{\bullet\mathsf{db}} t_1\{b := t_2\}\mathsf{L} = s$. We conclude from
$$\begin{array}{rl} & ((\lambda b^\alpha.\, t_1)\mathsf{L}\, t_2)\{a := r\} \\ = & ((\lambda b^\alpha.\, t_1\{a := r\})\mathsf{L}\{a := r\}\, t_2\{a := r\}) \\ \mapsto^\alpha_{\bullet\mathsf{db}} & t_1\{a := r\}\{b := t_2\{a := r\}\}\mathsf{L}\{a := r\} \\ = & (t_1\{b := t_2\}\mathsf{L})\{a := r\} \end{array}$$
We make use of the substitution lemma stating that $t_1\{a := r\}\{b := t_2\{a := r\}\} = t_1\{b := t_2\}\{a := r\}$, if $b \notin \mathsf{fv}(r)$.
5. $t = \bullet t_1$. Then the step $t \to^\alpha s$ must be in t_1, we conclude from the IH.
6. $t = \mathtt{req}(t_1)$. Then the step $t \to^\alpha s$ must be in t_1, we conclude from the IH.
7. $t = \mathtt{req}^\beta((\bullet t_1)\mathsf{L})$. If the step $t \to^\alpha s$ is in t_1 or L, we conclude from the IH. Suppose it is at the root. Then $\beta = \alpha$ and $t = \mathtt{req}^\alpha((\bullet t_1)\mathsf{L}) \mapsto^\alpha_{\bullet\mathsf{req}} t_1 lsctx = s$. We conclude from
$$\begin{array}{rl} & (\mathtt{req}^\alpha((\bullet t_1)\mathsf{L}))\{a := r\} \\ = & \mathtt{req}^\alpha((\bullet t_1\{a := r\})\mathsf{L}\{a := r\}) \\ \mapsto^\alpha_{\bullet\mathsf{req}} & t_1\{a := r\}\mathsf{L}\{a := r\} \\ = & (t_1\mathsf{L})\{a := r\} \end{array}$$
8. $t = !t_1$. Then the step $t \to^\alpha s$ must be in t_1, we conclude from the IH.
9. $t = t_1[u/(!(\bullet t_2)\mathsf{L}_1)\mathsf{L}_2]$. If the step $t \to^\alpha s$ is in t_1 or t_2 or L_1 or L_2, we conclude from the IH. Suppose it is at the root. Then $\alpha = \beta$ and $t = \mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle[u/(!(\bullet t_2)\mathsf{L}_1)\mathsf{L}_2] \mapsto^\alpha_{\bullet\mathsf{ls}} \mathsf{C}\langle\!\langle(\bullet t_2)\mathsf{L}_1\rangle\!\rangle[u/!(\bullet t_2)\mathsf{L}_1]\mathsf{L}_2 = s$, where $u \notin \mathsf{fv}(t_2)$ and $\mathsf{fv}(\mathsf{C}) \cap \mathsf{dom}(\mathsf{L}_1\mathsf{L}_2) = \emptyset$. We conclude from
$$\begin{array}{rl} & (\mathsf{C}\langle\!\langle u^\alpha \rangle\!\rangle[u/(!(\bullet t_2)\mathsf{L}_1)\mathsf{L}_2])\{a := r\} \\ = & \mathsf{C}\{a := r\}\langle\!\langle u^\alpha \rangle\!\rangle[u/(!(\bullet t_2\{a := r\})\mathsf{L}_1\{a := r\})\mathsf{L}_2\{a := r\}] \\ \mapsto^\alpha_{\bullet\mathsf{ls}} & \mathsf{C}\{a := r\}\langle\!\langle(\bullet t_2\{a := r\})\mathsf{L}_1\{a := r\}\rangle\!\rangle[u/!(\bullet t\{a := r\})\mathsf{L}_1\{a := r\}]\mathsf{L}_2\{a := r\} \\ = & (\mathsf{C}\langle\!\langle(\bullet t_2)\mathsf{L}_1\rangle\!\rangle[u/!(\bullet t_2)\mathsf{L}_1]\mathsf{L}_2)\{a := r\} \end{array}$$
10. $t = t_1[u^\beta/(!t_2)\mathsf{L}]$. If the step $t \to^\alpha s$ is in t_1 or t_2 or L, we conclude from the IH. Suppose it is at the root. Then $\alpha = \beta$ and $t = t_1[u^\alpha/(!t_2)\mathsf{L}] \mapsto^\alpha_{\bullet\mathsf{gc}} t_1\mathsf{L} = s$, where $u \notin \mathsf{fv}(t_1)$. We conclude from
$$\begin{array}{rl} & (t_1[u^\alpha/(!t_2)\mathsf{L}])\{a := r\} \\ = & t_1\{a := r\}[u^\alpha/(!t_2\{a := r\})\mathsf{L}\{a := r\}] \\ \mapsto^\alpha_{\bullet\mathsf{gc}} & t_1\{a := r\}\mathsf{L}\{a := r\} \\ = & (t_1\mathsf{L})\{a := r\} \end{array}$$
Note that $u \notin \mathsf{fv}(r)$.
11. $t = t_1[u/t_2]$ with $t_1 \in \mathsf{T}^{\mathcal{WL}}_\bullet$, $t_2 \in \mathsf{T}^{\mathcal{WL}}_\bullet$ and $u \notin \mathsf{fv}^{\mathcal{L}}(t_1)$. Then the step $t \to^\alpha s$ is in t_1 or t_2 and we conclude from the IH.

We now address the second item.

1. $r = b$. If $a \neq b$, then the result is immediate. Otherwise, it follows from the hypothesis $t \to^{\alpha} s$.
2. $r = u^{(\alpha)}$. Immediate.
3. $r = \lambda b.\, r_1$. The result follows from the IH.
4. $r = r_1\, r_2$. The result follows from the IH, applied twice. First to obtain $r_1\{a := t\} \twoheadrightarrow^{\alpha} r_1\{a := s\}$, then to obtain $r_2\{a := t\} \twoheadrightarrow^{\alpha} r_2\{a := s\}$. From which we conclude $r_1\{a := t\}\, r_2\{a := t\} \twoheadrightarrow^{\alpha} r_1\{a := t\}\, r_2\{a := s\}$.
5. $r = (\lambda b^{\beta}.\, r_1)\mathtt{L}\, r_2$. The result follows from the IH, applied multiple times. First to obtain $r_1\{a := t\} \twoheadrightarrow^{\alpha} r_1\{a := s\}$, then to obtain $r_2\{a := t\} \twoheadrightarrow^{\alpha} r_2\{a := s\}$, and finally to obtain $r_{3i}\{a := t\} \twoheadrightarrow^{\alpha} r_{3i}\{a := s\}$, for each r_{3i} in $\mathtt{L} = [u_1/r_{31}] \ldots [u_n/r_{3n}]$. This suffices to conclude that $r\{a := t\} \twoheadrightarrow^{\alpha} r\{a := s\}$.
6. $r = \bullet r_1$. The result follows from the IH.
7. $r = \mathtt{req}(r_1)$. The result follows from the IH.
8. $r = \mathtt{req}^{\beta}((\bullet r_1)\mathtt{L})$. The result follows from the IH, applied multiple times.
9. $r = !r_1$. The result follows from the IH.
10. $r = r_1[u/(!(\bullet r_2)\mathtt{L}_1)\mathtt{L}_2]$ and $u \in \mathsf{fv}^{\mathcal{L}}(r_1)$. The result follows from the IH, applied multiple times.
11. $r = r_1[u^{(\beta)}/(!r_2)\mathtt{L}]$ and $u \notin \mathsf{fv}(r_1)$. The result follows from the IH, applied multiple times.
12. $r = r_1[u/r_2]$ with $r_1 \in \mathsf{T}_{\bullet}^{\mathcal{WL}}$, $r_2 \in \mathsf{T}_{\bullet}^{\mathcal{WL}}$ and $u \notin \mathsf{fv}^{\mathcal{L}}(r_1)$. The result follows from the IH, applied multiple times.

Definition D.8 (Multi-hole Labeled contexts).

$$
\begin{aligned}
\mathtt{F} ::= \Box \mid \lambda a.\, \mathtt{F} \mid \lambda a^{\alpha}.\, \mathtt{F} \mid \mathtt{F}\, t \mid t\, \mathtt{F} \mid \mathtt{F}\, \mathtt{F} \mid \bullet \mathtt{F} \mid \mathtt{req}(\mathtt{F}) \mid \mathtt{req}^{\alpha}(\mathtt{F}) \mid !\mathtt{F} \\
\mid \mathtt{F}[u/t] \mid \mathtt{F}[u^{\alpha}/t] \mid \mathtt{F}[u^{\alpha}/\mathtt{F}] \mid t[u/\mathtt{F}] \mid t[u^{\alpha}/\mathtt{F}] \mid \mathtt{F}[u^{\alpha}/\mathtt{F}]
\end{aligned}
$$

Lemma D.24. *Let $\mathtt{C}, \mathtt{D} \in \mathsf{Ctxs}_{\bullet}^{\mathcal{L}}$ be labeled contexts. If $\mathtt{C}\langle\!\langle u^{(\alpha)} \rangle\!\rangle = \mathtt{D}\langle\!\langle v^{(\beta)} \rangle\!\rangle$, $\mathtt{C} \neq \mathtt{D}$, and $u^{(\alpha)} \neq v^{(\beta)}$, then there exists $\mathtt{F}$ such that $\mathtt{C}\langle\!\langle u^{(\alpha)} \rangle\!\rangle = \mathtt{D}\langle\!\langle v^{(\beta)} \rangle\!\rangle = \mathtt{F}\langle\!\langle u^{(\alpha)}, v^{(\beta)} \rangle\!\rangle$.*

Proof. By induction on $\mathtt{C}$.

1. $\mathtt{C} = \Box$. Immediate from hypothesis.
2. $\mathtt{C} = \lambda a^{(\gamma)}.\, \mathtt{C}_1$. Then $\mathtt{D} = \lambda a^{(\gamma)}.\, \mathtt{D}_1$ and we conclude from the IH.
3. $\mathtt{C} = \mathtt{C}_1\, t$. Then either $\mathtt{D} = \mathtt{D}_1\, t$ or $\mathtt{D} = \mathtt{C}_1\langle\!\langle u^{(\alpha)} \rangle\!\rangle\, \mathtt{D}_1$ with $\mathtt{D}_1\langle\!\langle v^{(\beta)} \rangle\!\rangle = t$. In the former we resort to the IH; in the latter we set $\mathtt{F}$ to be $\mathtt{C}_1\, \mathtt{D}_1$.
4. $\mathtt{C} = t\, \mathtt{C}_1$. Similar to previous case.
5. $\mathtt{C} = \bullet \mathtt{C}_1$. Then $\mathtt{D} = \bullet \mathtt{D}_1$ and we conclude from the IH.
6. $\mathtt{C} = \mathtt{req}^{(\gamma)}(\mathtt{C}_1)$. Then $\mathtt{D} = \mathtt{req}^{(\gamma)}(\mathtt{D}_1)$ and we conclude from the IH.
7. $\mathtt{C} = !\mathtt{C}_1$. Similar to previous case.
8. $\mathtt{C} = \mathtt{C}_1[w^{(\gamma)}/t]$. Similar to the case for application.
9. $\mathtt{C} = t[w^{(\gamma)}/\mathtt{C}_1]$. Similar to the case for application.

We next prove Semantic Orthogonality. Note that p_1 has no α-labels since the only α-label was reduced in the step $t \to_{\bullet}^{\alpha} s$. Similarly, p_2 has no β-labels since the only β-label was reduced in the step $t \to_{\bullet}^{\beta} r$. Moreover, since $p_1 \equiv p_2$ implies they have the same labels (*cf.* Lem. D.25), p_1 has no β-labels. Thus $s \twoheadrightarrow^{\beta} p_1$ is a complete β-development. Similarly, $r \twoheadrightarrow^{\beta} p_2$ is a complete α-development.

Proposition D.4 (Semantic Orthogonality). *Let t be a well-labeled term with a unique occurrence of α and a unique occurrence of β. Then,*

$$\begin{tikzcd}
 & s \arrow[r, dotted, two heads, "\beta"] & p_1 \arrow[d, phantom, "\equiv"] \\
t \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \\
 & r \arrow[r, dotted, two heads, "\alpha"'] & p_2
\end{tikzcd}$$

Proof. Suppose $t = \mathtt{C}\langle t_1\rangle \to^{\alpha} \mathtt{C}\langle s_1\rangle = s$, referred to as the α-step, follows from $t_1 \mapsto^{\alpha} s_1$ and $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$, referred to as the β-step, follows from $t_2 \mapsto^{\beta} r_1$. We proceed by induction on $\mathtt{C}$ and, for each case, consider all possible forms for $\mathtt{D}$.

1. $\mathtt{C} = \Box$. Then $t = t_1$ and $s = s_1$.

   1.1. The $t \to^{\alpha} s$ step follows from $t = (\lambda a^{\alpha}.\, t_{11})\mathtt{L}\, t_{12} \mapsto^{\alpha}_{\bullet\mathsf{db}} t_{11}\{a := t_{12}\}\mathtt{L}$, with $\mathsf{fv}(t_{12}) \cap \mathsf{dom}(\mathtt{L}) = \emptyset$. We next consider all possible forms for $\mathtt{D}$.

   1.1.1. $\mathtt{D} = \Box$. Then $\alpha = \beta$ and we conclude immediately.

   1.1.2. $\mathtt{D} = (\lambda a^{\alpha}.\, \mathtt{D}_1)\mathtt{L}\, t_{12}$. We define $\mathtt{D}_1^{\star}$ to be $\mathtt{D}_1\{a := t_{12}\}$ and similarly for $t_2^{\star}$. Note that $\mathtt{D}_1\langle t_2\rangle^{\star} = \mathtt{D}_1^{\star}\langle t_2^{\star}\rangle$.

$$\begin{tikzcd}
 & \mathtt{D}_1^{\star}\langle t_2^{\star}\rangle\mathtt{L} \arrow[dr, dotted, "\beta\ (\textit{Lem. } D.23(1))"] & \\
(\lambda a^{\alpha}.\, \mathtt{D}_1\langle t_2\rangle)\mathtt{L}\, t_{12} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1^{\star}\langle r_1^{\star}\rangle\mathtt{L} \\
 & (\lambda a^{\alpha}.\, \mathtt{D}_1\langle r_1\rangle)\mathtt{L}\, t_{12} \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

   1.1.3. $\mathtt{D} = \Box\mathtt{L}_2\, t_{12}$.

   1.1.3.1. The $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ step follows from $t_2 = \mathtt{E}\langle\!\langle u^{\beta}\rangle\!\rangle[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. We consider each possible form for $\mathtt{E}$.

   1.1.3.1.1. $\mathtt{E} = (\lambda a^{\alpha}.\, \mathtt{E}_1)\mathtt{L}_1$ and $\mathtt{L} = \mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$ and $a \notin \mathsf{fv}((!(\bullet t_{21})\mathtt{K}_1))$.

$$\begin{tikzcd}
 & \mathtt{E}_1^{\star}\langle\!\langle u^{\beta}\rangle\!\rangle\mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
(\lambda a^{\alpha}.\, \mathtt{E}_1\langle\!\langle u^{\beta}\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\, t_{12} \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{E}_1^{\star}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle\mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2 \\
 & (\lambda a^{\alpha}.\, \mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2\mathtt{L}_2\, t_{12} \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

   1.1.3.1.2. $\mathtt{E} = (\lambda a^{\alpha}.\, t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}$ and $\mathtt{L} = \mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$.

$$\begin{tikzcd}
 & t^\star_{11}\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle u^\beta\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
(\lambda a^\alpha.\, t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle u^\beta\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2\; t_{12} \arrow[ur, "\alpha"] \arrow[dr, "\beta"] & & t^\star_{11}\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2\mathtt{L}_2 \\
 & (\lambda a^\alpha.\, t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2\mathtt{L}_2\; t_{12} \arrow[ur, dotted, "\alpha"] &
\end{tikzcd}$$

1.1.3.2. The $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ step follows from $t_2 = (\lambda a^\alpha.\, t_{11})\mathtt{L}_1[u^\beta/(!t_{21})\mathtt{K}] \mapsto^{\beta}_{\bullet\mathsf{gc}} (\lambda a^\alpha.\, t_{11})\mathtt{L}_1\mathtt{K} = r_1$ and $\mathtt{L} = \mathtt{L}_1[u^\beta/(!t_{21})\mathtt{K}]\mathtt{L}_2$. Note that $u \notin \mathsf{fv}(t_{12})$.

$$\begin{tikzcd}
 & t^\star_{11}\mathtt{L}_1[u^\beta/(!t_{21})\mathtt{K}]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
(\lambda a^\alpha.\, t_{11})\mathtt{L}_1[u^\beta/(!t_{21})\mathtt{K}]\mathtt{L}_2\; t_{12} \arrow[ur, "\alpha"] \arrow[dr, "\beta"] & & t^\star_{11}\mathtt{L}_1\mathtt{K}\mathtt{L}_2 \\
 & (\lambda a^\alpha.\, t_{11})\mathtt{L}_1\mathtt{K}\mathtt{L}_2\; t_{12} \arrow[ur, dotted, "\alpha"] &
\end{tikzcd}$$

1.1.4. $\mathtt{D} = (\lambda a^\alpha.\, t_{11})\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_2\; t_{12}$

$$\begin{tikzcd}
 & t^\star_{11}\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
(\lambda a^\alpha.\, t_{11})\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_2\; t_{12} \arrow[ur, "\alpha"] \arrow[dr, "\beta"] & & t^\star_{11}\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_2 \\
 & (\lambda a^\alpha.\, t_{11})\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_2\; t_{12} \arrow[ur, dotted, "\alpha"] &
\end{tikzcd}$$

1.1.5. $\mathtt{D} = (\lambda a^\alpha.\, t_{11})\mathtt{L}\,\mathtt{D}_1$ and $\mathtt{D}_1\langle t_2\rangle = t_{12}$.

$$\begin{tikzcd}
 & t_{11}\{a := \mathtt{D}_1\langle t_2\rangle\}\mathtt{L} \arrow[dr, dotted, two heads, "\beta\ (\mathit{Lem.}\ D.23)(2)"] & \\
(\lambda a^{\alpha}.\, t_{11})\mathtt{L}\,\mathtt{D}_1\langle t_2\rangle \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & t_{11}\{a := \mathtt{D}_1\langle r_1\rangle\}\mathtt{L} \\
 & (\lambda a^{\alpha}.\, t_{11})\mathtt{L}\,\mathtt{D}_1\langle r_1\rangle \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.2. The $t \to^{\alpha} s$ step follows from $t = \mathtt{req}^{\alpha}((\bullet t_{11})\mathtt{L}) \mapsto^{\alpha}_{\bullet\mathsf{req}} t_{11}\mathtt{L}$

1.2.1. $\mathtt{D} = \square$. Then $\alpha = \beta$ and we conclude immediately.

1.2.2. $\mathtt{D} = \mathtt{req}^{\alpha}((\bullet\mathtt{D}_1)\mathtt{L})$ and $\mathtt{D}_1\langle t_2\rangle = t_{11}$.

$$\begin{tikzcd}
 & \mathtt{D}_1\langle t_2\rangle\mathtt{L} \arrow[dr, dotted, "\beta"] & \\
\mathtt{req}^{\alpha}((\bullet\mathtt{D}_1\langle t_2\rangle)\mathtt{L}) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle r_1\rangle\mathtt{L} \\
 & \mathtt{req}^{\alpha}((\bullet\mathtt{D}_1\langle r_1\rangle)\mathtt{L}) \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.2.3. $\mathtt{D} = \mathtt{req}^{\alpha}(\square\mathtt{L}_2)$.

1.2.3.1. The $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ step follows from $t_2 = \mathtt{E}\langle\!\langle u^{\beta}\rangle\!\rangle[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}}$ $\mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. We consider each possible form for $\mathtt{E}$.

1.2.3.1.1. $\mathtt{E} = (\bullet\mathtt{E}_1)\mathtt{L}_1$ and $\mathtt{L} = \mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$ and $a \notin \mathsf{fv}((!(\bullet t_{21})\mathtt{K}_1))$.

$$\begin{tikzcd}
 & \mathtt{E}_1\langle\!\langle u^{\beta}\rangle\!\rangle\mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{req}^{\alpha}((\bullet\mathtt{E}_1\langle\!\langle u^{\beta}\rangle\!\rangle)\mathtt{L}_1[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle\mathtt{L}_1[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2\mathtt{L}_2 \\
 & \mathtt{req}^{\alpha}((\bullet\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2\mathtt{L}_2) \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.2.3.1.2. $\mathtt{E} = (\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}$ and $\mathtt{L} = \mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$.

$$\begin{tikzcd}
& t_{11}^{\star}\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{req}^{\alpha}((\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & t_{11}^{\star}\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2\mathtt{L}_2 \\
& (\lambda a^{\alpha}.\, t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2\mathtt{L}_2\, t_{12} \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.2.3.2. The $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ step follows from $t_2 = (\bullet t_{11})\mathtt{L}_1[u^{\beta}/(!t_{21})\mathtt{K}] \mapsto^{\beta}_{\bullet\mathsf{gc}} (\bullet t_{11})\mathtt{L}_1\mathtt{K} = r_1$ and $\mathtt{L} = \mathtt{L}_1[u^{\beta}/(!t_{21})\mathtt{K}]\mathtt{L}_2$

$$\begin{tikzcd}
& t_{11}\mathtt{L}_1[u^{\beta}/(!t_{21})\mathtt{K}]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{req}^{\alpha}((\bullet t_{11})\mathtt{L}_1[u^{\beta}/(!t_{21})\mathtt{K}]\mathtt{L}_2) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & t_{11}\mathtt{L}_1\mathtt{K}\mathtt{L}_2 \\
& \mathtt{req}^{\alpha}((\bullet t_{11})\mathtt{L}_1\mathtt{K}\mathtt{L}_2) \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.2.4. $\mathtt{D} = \mathtt{req}^{\alpha}((\bullet t_{11})\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_2)$.

$$\begin{tikzcd}
& t_{11}\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{req}^{\alpha}((\bullet t_{11})\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_2) \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & t_{11}\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_2\mathtt{L}_2 \\
& \mathtt{req}^{\alpha}((\bullet t_{11})\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_2) \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3. The $t \to^{\alpha} s$ step follows from $t = \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \mapsto^{\alpha}_{\bullet\mathsf{ls}} \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle[u/!(\bullet t)\mathtt{L}_1]\mathtt{L}_2$ with $u \notin \mathsf{fv}(t_{11})$ and $\mathsf{fv}(\mathtt{C}) \cap \mathsf{dom}(\mathtt{L}_1\mathtt{L}_2) = \varnothing$.

1.3.1. $\mathtt{D} = \Box$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = \mathtt{E}\langle\!\langle u^{\beta}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle[u/!(\bullet t_{11})\mathtt{L}_1]\mathtt{L}_2 = r_1$. We consider each possible form for $\mathtt{E}$.

1.3.1.1. $\mathtt{E} = \mathtt{C}_1$. Then $\alpha = \beta$ and the result is immediate.

1.3.1.2. $\mathtt{E} \neq \mathtt{C}_1$. Then by Lem. D.24 there exists a multi-hole context $\mathtt{F}$ such that $\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle = \mathtt{E}\langle\!\langle u^{\beta}\rangle\!\rangle = \mathtt{F}\langle\!\langle u^{\alpha}, u^{\beta}\rangle\!\rangle$.

$$
\begin{tikzcd}
& \mathtt{F}\langle\!\langle(\bullet t_{11})\mathtt{L}_1, u^\beta\rangle\!\rangle[v/!(\bullet t_{11})\mathtt{L}_1]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{F}\langle\!\langle u^\alpha, u^\beta\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{F}\langle\!\langle(\bullet t_{11})\mathtt{L}_1, (\bullet t_{11})\mathtt{L}_1\rangle\!\rangle[v/!(\bullet t_{11})\mathtt{L}_1]\mathtt{L}_2 \\
& \mathtt{F}\langle\!\langle u^\alpha, (\bullet t_{11})\mathtt{L}_1\rangle\!\rangle[v/!(\bullet t_{11})\mathtt{L}_1]\mathtt{L}_2 \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}
$$

1.3.2. $\mathtt{D} = \mathtt{D}_1[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2]$ and $\mathtt{D}_1\langle t_2\rangle = \mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle$. We consider two further cases.

1.3.2.1. $u^\alpha \in \mathtt{D}_1$. Consider the context $\mathtt{D}_{11}$ obtained from placing a hole in the unique occurrence of u^α in $\mathtt{D}_1\langle v\rangle$, for some fresh variable v. By Lem. D.24 there exists $\mathtt{F}$ such that $\mathtt{C}_1\langle\!\langle u^\alpha\rangle\!\rangle = \mathtt{D}_{11}\langle\!\langle v\rangle\!\rangle = \mathtt{F}\langle\!\langle u^\alpha, v\rangle\!\rangle$. Moreover, $\mathtt{F}\langle u^{(\alpha)}, t_2\rangle = \mathtt{D}_1\langle t_2\rangle$.

$$
\begin{tikzcd}
& \mathtt{F}\langle(\bullet t_{11})\mathtt{L}_1, t_2\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{F}\langle u^\alpha, t_2\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{F}\langle(\bullet t_{11})\mathtt{L}_1, r_1\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \\
& \mathtt{F}\langle u^\alpha, r_1\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}
$$

1.3.2.2. $u^\alpha \in t_2$. Then $\mathtt{C}_1 = \mathtt{D}_1\langle\mathtt{D}_2\rangle$ for some $\mathtt{D}_2$ such that $\mathtt{D}_2\langle\!\langle u^\alpha\rangle\!\rangle = t_2$. We consider each possible step for $t_2 \mapsto^\beta r_1$.

1.3.2.2.1. The $t_2 \mapsto^\beta r_1$ step follows from $t_2 = (\lambda b^\beta. t_{21})\mathtt{K}\, t_{22} \mapsto^\beta_{\bullet\mathsf{db}} t_{21}\{b := t_{22}\}\mathtt{K} = r_1$, with $\mathsf{fv}(t_{22}) \cap \mathsf{dom}(\mathtt{L}) = \varnothing$. We next consider each possible location of u^α.

1.3.2.2.1.1. $u^\alpha \in t_{21}$. Then $\mathtt{D}_2 = (\lambda b^\beta. \mathtt{D}_{21})\mathtt{K}\, t_{22}$.

$$\begin{tikzcd}[column sep=tiny]
& \mathtt{D}_1\langle(\lambda b^{\beta}.\,\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle)\mathtt{K}\,t_{22}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle(\lambda b^{\beta}.\,\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{K}\,t_{22}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle\mathtt{D}^{\star}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle\mathtt{K}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \\
& \mathtt{D}_1\langle\mathtt{D}^{\star}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle\mathtt{K}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.1.2. $u^{\alpha} \in \mathtt{K}$. Then $\mathtt{D}_2 = (\lambda b^{\beta}.\,t_{21})\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}]\mathtt{K}_2\,t_{22}$.

$$\begin{tikzcd}[column sep=tiny]
& \mathtt{D}_1\langle(\lambda b^{\beta}.\,t_{21})\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle]\mathtt{K}_2\,t_{22}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle(\lambda b^{\beta}.\,t_{21})\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_2\,t_{22}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle t^{\star}_{21}\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_2\rangle[u/(!(\bullet t_{11})\mathtt{L}_1 \\
& \mathtt{D}_1\langle t^{\star}_{21}\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_2\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.1.3. $u^{\alpha} \in t_{22}$. Then $\mathtt{D}_2 = (\lambda b^{\beta}.\,t_{21})\mathtt{K}\,\mathtt{D}_{21}$.

$$\begin{tikzcd}[column sep=tiny]
& \mathtt{D}_1\langle(\lambda b^{\beta}.\,t_{21})\mathtt{K}\,\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle(\lambda b^{\beta}.\,t_{21})\mathtt{K}\,\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle t_{21}\{b := \mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle\}\mathtt{K}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \\
& \mathtt{D}_1\langle t_{21}\{b := \mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle\}\mathtt{K}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, two heads, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.2. The $t_2 \mapsto^{\beta} r_1$ step follows from $t_2 = \mathtt{req}^{\beta}((\bullet t_{21})\mathtt{K}) \mapsto^{\beta}_{\bullet\mathsf{req}} t_{21}\mathtt{K} = r_1$. We next consider each possible location of u^{α}.

1.3.2.2.2.1. $u^{\alpha} \in t_{21}$. Then $\mathtt{D}_2 = \mathtt{req}^{\beta}((\bullet\mathtt{D}_{21})\mathtt{K})$.

$$\begin{tikzcd}[column sep=tiny]
& \mathtt{D}_1\langle\mathtt{req}^{\beta}((\bullet\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle)\mathtt{K})\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle\mathtt{req}^{\beta}((\bullet\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{K})\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle\rangle\mathtt{K}[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \\
& \mathtt{D}_1\langle\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle\rangle\mathtt{K}[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.2.2. $u^{\alpha} \in \mathtt{K}$. Then $\mathtt{D}_2 = \mathtt{req}^{\beta}((\bullet t_{21})\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}]\mathtt{K}_2)$.

$$\begin{tikzcd}[column sep=tiny]
& \mathtt{D}_1\langle\mathtt{req}^{\beta}((\bullet t_{21})\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle]\mathtt{K}_2)\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle\mathtt{req}^{\beta}((\bullet t_{21})\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_2)\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle t_{21}\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle]\mathtt{K}_2\rangle[u/(!(\bullet \\
& \mathtt{D}_1\langle t_{21}\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_2\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.3. The $t_2 \mapsto^{\beta} r_1$ step follows from $t_2 = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}}$ $\mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. Recall that $\mathtt{D}_2\langle\!\langle u^{\alpha}\rangle\!\rangle = t_2 = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]$. We consider each case for $u^{\alpha} \in t_2$.

1.3.2.2.3.1. $u^{\alpha} \in \mathtt{E}$. Then there exists $\mathtt{E}_1$ such that $\mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle$. Note that $\mathtt{E}_1 \neq \mathtt{E}$. By Lem. D.24, there exists $\mathtt{F}$ such that $\mathtt{E}_1\langle\!\langle u^{\alpha}\rangle\!\rangle = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle = \mathtt{F}\langle\!\langle u^{\alpha}, v^{\beta}\rangle\!\rangle$.

$$\begin{tikzcd}[column sep=tiny]
& \mathtt{D}_1\langle\mathtt{F}\langle\!\langle(\bullet t_{11})\mathtt{L}_1, v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle\mathtt{F}\langle\!\langle u^{\alpha}, (\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle\mathtt{F}\langle\!\langle(\bullet t_{11})\mathtt{L}_1, v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\rangle[u/(!(\bullet t \\
& \mathtt{D}_1\langle\mathtt{F}\langle\!\langle u^{\alpha}, (\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.3.2. $u^{\alpha} \in t_{21}$. Then $\mathtt{D}_2\langle\!\langle u^{\alpha}\rangle\!\rangle = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{K}_1)\mathtt{K}_2]$.

$$\begin{tikzcd}[column sep=tiny]
 & \mathtt{D}_1\langle\mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle)\mathtt{K}_1)\mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle\mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{K}_1)\mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle\mathtt{E}\langle\!\langle(\bullet\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle)\mathtt{K}_1\rangle\!\rangle[v/(!(\bullet\mathtt{D}_{21}\langle\!\langle(\\
 & \mathtt{D}_1\langle\mathtt{E}\langle\!\langle(\bullet\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{K}_1\rangle\!\rangle[v/(!(\bullet\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle)\mathtt{K}_1)]\mathtt{K}_2\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, two heads, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.3.3. $u^{\alpha} \in \mathtt{K}_1$. Then $\mathtt{D}_2\langle\!\langle u^{\alpha}\rangle\!\rangle = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_{11}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_{12})\mathtt{K}_2]$.

$$\begin{tikzcd}[column sep=tiny]
 & \mathtt{D}_1\langle\mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_{11}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle]\mathtt{K}_{12})\mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle\mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_{11}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_{12})\mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle\mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_{11}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_{12}\rangle \\
 & \mathtt{D}_1\langle\mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_{11}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_{12}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_{11}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_{12})\mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.3.4. $u^{\alpha} \in \mathtt{K}_2$. Then $\mathtt{D}_2\langle\!\langle u^{\alpha}\rangle\!\rangle = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_{21}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_{22}]$.

$$\begin{tikzcd}[column sep=tiny]
 & \mathtt{D}_1\langle\mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_{21}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle]\mathtt{K}_{22}]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle\mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_{21}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_{22}]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle\mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_{21}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle] \\
 & \mathtt{D}_1\langle\mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_{21}[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle]\mathtt{K}_{22}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.4. The $t_2 \mapsto^{\beta} r_1$ step follows from $t_2 = t_{21}[v^{\beta}/(!t_{22})\mathtt{K}] \mapsto^{\beta}_{\bullet\mathtt{gc}} t_{21}\mathtt{K} = r_1$. Then $\mathtt{D}_2\langle\!\langle u^{\alpha}\rangle\!\rangle = t_{21}[v^{\beta}/(!t_{22})\mathtt{K}] \mapsto^{\beta}_{\bullet\mathtt{gc}} t_{21}\mathtt{K}$. We consider all possible locations for u^{α}.

1.3.2.2.4.1. $u^{\alpha} \in t_{21}$. Then $\mathtt{D}_2\langle\!\langle u^{\alpha}\rangle\!\rangle = \mathtt{D}_{21}\langle\!\langle u^{\alpha}\rangle\!\rangle[v^{\beta}/(!t_{22})\mathtt{K}]$, for some $\mathtt{D}_{21}$, and $\mathtt{D}_2\langle\!\langle u^{\alpha}\rangle\!\rangle = t_{21}$.

$$\begin{tikzcd}[column sep=tiny]
 & \mathtt{D}_1\langle \mathtt{D}_{21}\langle\!\langle u^\alpha \rangle\!\rangle \mathtt{K}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle \mathtt{D}_{21}\langle\!\langle u^\alpha \rangle\!\rangle[v^\beta/(!t_{22})\mathtt{K}]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle \mathtt{D}_{21}\langle\!\langle (\bullet t_{11})\mathtt{L}_1 \rangle\!\rangle \mathtt{K}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \\
 & \mathtt{D}_1\langle \mathtt{D}_{21}\langle\!\langle (\bullet t_{11})\mathtt{L}_1 \rangle\!\rangle[v^\beta/(!t_{22})\mathtt{K}]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.4.2. $u^\alpha \in t_{22}$. Then $\mathtt{D}_2\langle\!\langle u^\alpha \rangle\!\rangle = t_{21}[v^\beta/(!\mathtt{D}_{21}\langle\!\langle u^\alpha \rangle\!\rangle)\mathtt{K}]$, for some $\mathtt{D}_{21}$, and $\mathtt{D}_2\langle\!\langle u^\alpha \rangle\!\rangle = t_{22}$.

$$\begin{tikzcd}[column sep=tiny]
 & \mathtt{D}_1\langle t_{21}[v^\beta/(!\mathtt{D}_{21}\langle\!\langle (\bullet t_{11})\mathtt{L}_1 \rangle\!\rangle)\mathtt{K}]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle t_{21}[v^\beta/(!\mathtt{D}_{21}\langle\!\langle u^\alpha \rangle\!\rangle)\mathtt{K}]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle \mathtt{D}_{21}\langle\!\langle (\bullet t_{11})\mathtt{L}_1 \rangle\!\rangle \mathtt{K}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \\
 & \mathtt{D}_1\langle t_{21}\mathtt{K}\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, equal, "\alpha"'] &
\end{tikzcd}$$

1.3.2.2.4.3. $u^\alpha \in \mathtt{K}$. Then $\mathtt{D}_2\langle\!\langle u^\alpha \rangle\!\rangle = t_{21}[v^\beta/(!t_{22})\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^\alpha \rangle\!\rangle]\mathtt{K}_2]$, for some $\mathtt{D}_{21}$, and $\mathtt{K} = \mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^\alpha \rangle\!\rangle]\mathtt{K}_2$.

$$\begin{tikzcd}[column sep=tiny]
 & \mathtt{D}_1\langle t_{21}[v^\beta/(!t_{22})\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle (\bullet t_{11})\mathtt{L}_1 \rangle\!\rangle]\mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle t_{21}[v^\beta/(!t_{22})\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^\alpha \rangle\!\rangle]\mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle t_{21}\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle (\bullet t_{11})\mathtt{L}_1 \rangle\!\rangle \mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_2 \\
 & \mathtt{D}_1\langle t_{21}\mathtt{K}_1[w^{(\gamma)}/\mathtt{D}_{21}\langle\!\langle u^\alpha \rangle\!\rangle \mathtt{K}_2]\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.3. $\mathtt{D} = \mathtt{C}_1\langle\!\langle u^\alpha \rangle\!\rangle[u/(!(\bullet \mathtt{D}_1)\mathtt{L}_1)\mathtt{L}_2]$ and $\mathtt{D}_1\langle t_2 \rangle = t_{11}$.

$$
\begin{tikzcd}
 & \mathtt{C}_1\langle\!\langle(\bullet\mathtt{D}_1\langle t_2\rangle)\mathtt{L}_1\rangle\!\rangle[u/(!(\bullet\mathtt{D}_1\langle t_2\rangle)\mathtt{L}_1)]\mathtt{L}_2 \arrow[dr, dotted, ->>, "\beta"] & \\
\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet\mathtt{D}_1\langle t_2\rangle)\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{C}_1\langle\!\langle(\bullet\mathtt{D}_1\langle r_1\rangle)\mathtt{L}_1\rangle\!\rangle[u/(!(\bullet\mathtt{D}_1\langle r_1\rangle)\mathtt{L}_1)]\mathtt{L}_2 \\
 & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet\mathtt{D}_1\langle r_1\rangle)\mathtt{L}_1)\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}
$$

1.3.4. $\mathtt{D} = \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_{12})\mathtt{L}_2]$ and $\mathtt{L}_1 = \mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_{12}$.

$$
\begin{tikzcd}
 & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_{12}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_{12})]\mathtt{L}_2 \arrow[dr, dotted, ->>, "\beta"] & \\
\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_{12})\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_{12}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_{12})]\mathtt{L}_2 \\
 & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_{12})\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}
$$

1.3.5. $\mathtt{D} = \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/\Box\mathtt{L}_{12}\mathtt{L}_2]$. The context $\mathtt{L}_1 = \mathtt{L}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{12}$ and the step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. We consider each possible form for $\mathtt{E}$.

1.3.5.1. $\mathtt{E} = !(\bullet\mathtt{E}_1)\mathtt{L}_{11}$.

$$
\begin{tikzcd}
 & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle)\mathtt{L}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{12})\mathtt{L}_2] \arrow[dr, dotted, ->>, "\beta"] & \\
\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle)\mathtt{L}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{12})\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_{12}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_{12})]\mathtt{L}_2 \\
 & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle)\mathtt{L}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{12})\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}
$$

1.3.5.2. $\mathtt{E} = !(\bullet t_{11})\mathtt{L}_{111}[w^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{112}$ where $\mathtt{L}_{11} = \mathtt{L}_{111}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle]\mathtt{L}_{112}$. We write $\mathtt{L}^{\star}_{11}$ for $\mathtt{L}_{111}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{112}$

$$\begin{tikzcd}
 & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{12}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{12})]\mathtt{L}_2 \arrow[dr, dotted, two heads, "\beta"] & \\
\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{12})\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}^{\star}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_{12})]\mathtt{L}_2 \\
 & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}^{\star}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_{12})\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.6. $\mathtt{D} = \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_{21}[v^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_{22}]$

$$\begin{tikzcd}
 & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_{21}[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_{22} \arrow[dr, dotted, "\beta"] & \\
\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_{21}[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_{22}] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_{21}[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_{22} \\
 & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_{21}[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_{22}] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.7. $\mathtt{D} = \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/\square\mathtt{L}_{22}]$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. We consider each possible form for $\mathtt{E}$.

1.3.7.1. $\mathtt{E} = (!(\bullet\mathtt{E}_1)\mathtt{L}_1)\mathtt{L}_{21}$

$$\begin{tikzcd}
 & \mathtt{C}_1\langle\!\langle(\bullet\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle)\mathtt{L}_1\rangle\!\rangle[u/(!(\bullet\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle)\mathtt{L}_1)]\mathtt{L}_{21}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{22} \arrow[dr, dotted, two heads, "\beta"] & \\
\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle)\mathtt{L}_1)\mathtt{L}_{21}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{22}] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{C}_1\langle\!\langle(\bullet\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1\rangle\!\rangle[u/(!(\bullet\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1)]\mathtt{L}_{21}[v/(!(\bullet t_2 \\
 & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1)\mathtt{L}_{21}[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_{22}] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.7.2. $\mathtt{E} = (!(\bullet t_{11})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12})\mathtt{L}_{21}$

$$\begin{tikzcd}[column sep=tiny]
 & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle]\mathtt{L}_{12}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle]\mathtt{L}_{12})]\mathtt{L}_{21}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{22} \arrow[dr, dotted, two heads, "\beta"] & \\
\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle]\mathtt{L}_{12})\mathtt{L}_{21}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{22}] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\\
 & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12})\mathtt{L}_{21}[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_{22}] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.3.7.3. $\mathtt{E} = (!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_{211}[w^{\gamma}/\mathtt{E}_1]\mathtt{L}_{212}$. The α and β-steps are disjoint.

$$\begin{tikzcd}[column sep=tiny]
 & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_{211}[w^{\gamma}/\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle]\mathtt{L}_{212}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{22} \arrow[dr, dotted, "\beta"] & \\
\mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_{211}[w^{\gamma}/\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle]\mathtt{L}_{212}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{22}] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{C}_1\langle\!\langle(\bullet t_{11})\mathtt{L}_1\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)]\mathtt{L}_{211}[w^{\gamma}/\mathtt{E}_1 \\
 & \mathtt{C}_1\langle\!\langle u^{\alpha}\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{L}_1)\mathtt{L}_{211}[w^{\gamma}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{212}[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_{22}] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.4. The $t \to^{\alpha} s$ step follows from $t = t_{11}[u^{\alpha}/(!t_{12})\mathtt{L}] \mapsto^{\alpha}_{\bullet\mathsf{gc}} t_{11}\mathtt{L}$, with $u \notin \mathsf{fv}(t_{11})$.
We consider each possible form for $\mathtt{D}$:

1.4.1. $\mathtt{D} = \Box$. Then $\alpha = \beta$ and the result holds immediately.

1.4.2. $\mathtt{D} = \mathtt{D}_1[u^{\alpha}/(!t_{12})\mathtt{L}]$

$$\begin{tikzcd}
 & \mathtt{D}_1\langle t_2\rangle\mathtt{L} \arrow[dr, dotted, "\beta"] & \\
\mathtt{D}_1\langle t_2\rangle[u^{\alpha}/(!t_{12})\mathtt{L}] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \mathtt{D}_1\langle r_1\rangle\mathtt{L} \\
 & \mathtt{D}_1\langle r_1\rangle[u^{\alpha}/(!t_{12})\mathtt{L}] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.4.3. $\mathtt{D} = t_{11}[u^{\alpha}/(!\mathtt{D}_1)\mathtt{L}]$

$$\begin{tikzcd}
 & t_{11}\mathtt{L} \arrow[dr, equal, "\beta"] & \\
t_{11}[u^\alpha/(!\mathtt{D}_1\langle t_2\rangle)\mathtt{L}] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & t_{11}\mathtt{L} \\
 & t_{11}[u^\alpha/(!\mathtt{D}_1\langle r_1\rangle)\mathtt{L}] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.4.4. $\mathtt{D} = t_{11}[u^\alpha/(!t_{12})\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_2]$

$$\begin{tikzcd}
 & t_{11}\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
t_{11}[u^\alpha/(!t_{12})\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle t_2\rangle]\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & t_{11}\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_2 \\
 & t_{11}[u^\alpha/(!t_{12})\mathtt{L}_1[v^{(\gamma)}/\mathtt{D}_1\langle r_1\rangle]\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.4.5. $\mathtt{D} = t_{11}[u^\alpha/\Box\mathtt{L}_2]$ and $\mathtt{L} = (!t_{12})\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$. The step $t = \mathtt{D}\langle t_2\rangle \to^\beta \mathtt{D}\langle r_1\rangle = r$ is $t_2 = \mathtt{E}\langle\!\langle v^\beta\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^\beta_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. We consider each possible form for $\mathtt{E}$.

1.4.5.1. $\mathtt{E} = (!\mathtt{E}_1)\mathtt{L}_1$

$$\begin{tikzcd}
 & t_{11}\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \arrow[dr, equal, "\beta"] & \\
t_{11}[u^\alpha/(!\mathtt{E}_1\langle\!\langle v^\beta\rangle\!\rangle)\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & \begin{array}{c} t_{11}\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \\ \equiv \\ t_{11}\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2 \end{array} \\
 & t_{11}[u^\alpha/(!\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle)\mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

1.4.5.2. $\mathtt{E} = (!t_{12})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1]\mathtt{L}_{12}$

$$
\begin{tikzcd}[column sep=small]
 & t_{11}\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle]\mathtt{L}_{12}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
t_{11}[u^{\alpha}/(!t_{12})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle v^{\beta}\rangle\!\rangle]\mathtt{L}_{12}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & t_{11}\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K} \\
 & t_{11}[u^{\alpha}/(!t_{12})\mathtt{L}_{11}[w^{(\gamma)}/\mathtt{E}_1\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle]\mathtt{L}_{12}[v/(!(\bullet t_{21})\mathtt{K}_1)]\mathtt{K}_2\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}
$$

1.4.6. $\mathtt{D} = t_{11}[u^{\alpha}/\Box\mathtt{L}_2]$ and $\mathtt{L} = \mathtt{L}_1[v^{\beta}/\mathtt{D}_1]\mathtt{L}_2$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = (!t_{12})\mathtt{L}_1[v^{\beta}/(!t_{21})\mathtt{K}] \mapsto^{\beta}_{\bullet\mathsf{gc}} (!t_{12})\mathtt{L}_1\mathtt{K} = r_1$.

$$
\begin{tikzcd}
 & t_{11}\mathtt{L}_1[v^{\beta}/(!t_{21})\mathtt{K}]\mathtt{L}_2 \arrow[dr, dotted, "\beta"] & \\
t_{11}[u^{\alpha}/(!t_{12})\mathtt{L}_1[v^{\beta}/(!t_{21})\mathtt{K}]\mathtt{L}_2] \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & & t_{11}\mathtt{L}_1\mathtt{K}\mathtt{L}_2 \\
 & t_{11}[u^{\alpha}/(!t_{12})\mathtt{L}_1\mathtt{K}\mathtt{L}_2] \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}
$$

2. $\mathtt{C} = \lambda a.\mathtt{C}_1$. Then both the α and β-steps are inside $\mathtt{C}_1\langle t_1\rangle$ and we conclude from the IH.
3. $\mathtt{C} = (\lambda a^{\gamma}.\mathtt{C}_1)\mathtt{L}\,t_{12}$. We next consider all possible forms for $\mathtt{D}$.
   3.1. $\mathtt{D} = \Box$. We proceed as in case 1.1..
   3.2. $\mathtt{D} = (\lambda a^{\gamma}.\mathtt{D}_1)\mathtt{L}\,t_{12}$. Then we conclude from the IH.
   3.3. $\mathtt{D} = (\lambda a^{\gamma}.\mathtt{C}_1\langle t_1\rangle)\mathtt{L}_1[u^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_2\,t_{12}$. Then we conclude immediately since the α and β-steps are disjoint.
   3.4. $\mathtt{D} = \Box\,\mathtt{L}_2\,t_{12}$ and $\mathtt{L} = \mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. We proceed as in case 1.3.2.2..
   3.5. $\mathtt{D} = \Box\,\mathtt{L}_2\,t_{12}$ and $\mathtt{L} = \mathtt{L}_1[v/(!t_{21})\mathtt{K}]\mathtt{L}_2$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = (\lambda a^{\gamma}.\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle)\mathtt{L}_1[u^{\beta}/(!t_{21})\mathtt{K}] \mapsto^{\beta}_{\bullet\mathsf{gc}} (\lambda a^{\gamma}.\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle)\mathtt{L}_1\mathtt{K} = r_1$. Then we conclude immediately since the α and β-steps are disjoint.
   3.6. $\mathtt{D} = (\lambda a^{\gamma}.\mathtt{C}_1\langle t_1\rangle)\mathtt{L}\,\mathtt{D}_1$. Then we conclude immediately since the α and β-steps are disjoint.
4. $\mathtt{C} = (\lambda a^{\gamma}.t_{11})\mathtt{L}_1[v^{(\delta)}/\mathtt{C}_1]\mathtt{L}_2\,t_{12}$. We next consider all possible forms for $\mathtt{D}$.
   4.1. $\mathtt{D} = \Box$. We proceed as in case 1.1..
   4.2. $\mathtt{D} = (\lambda a^{\gamma}.\mathtt{D}_1)\mathtt{L}_1[v^{(\delta)}/\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle]\mathtt{L}_2\,t_{12}$. Then we conclude immediately since the α and β-steps are disjoint.
   4.3. $\mathtt{D} = (\lambda a^{\gamma}.t_{11})\mathtt{L}_{11}[u^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_{12}[v^{(\delta)}/\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle]\mathtt{L}_2\,t_{12}$. Then we conclude immediately since the α and β-steps are disjoint.

4.4. $\mathtt{D} = (\lambda a^{\gamma}.\, t_{11})\mathtt{L}_1[v^{(\delta)}/\mathtt{D}_1]\mathtt{L}_2\, t_{12}$. Then we conclude from the IH.

4.5. $\mathtt{D} = (\lambda a^{\gamma}.\, t_{11})\mathtt{L}_1[v^{(\delta)}/\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle]\mathtt{L}_{21}[u^{(\gamma)}/\mathtt{D}_1]\mathtt{L}_{22}\, t_{12}$. Then we conclude immediately since the α and β-steps are disjoint.

4.6. $\mathtt{D} = \Box\,\mathtt{L}_{12}[v^{(\delta)}/\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle]\mathtt{L}_2\, t_{12}$ and $\mathtt{L}_1 = \mathtt{L}_{11}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{12}$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. We consider each possible location for v^{β}.

4.6.1. $v^{\beta} \in t_1$. We proceed as in case 1.3.2.2..

4.6.2. $v^{\beta} \in \mathtt{L}_{11}$. Then we conclude immediately since the α and β-steps are disjoint.

4.7. $\mathtt{D} = \Box\,\mathtt{L}_{12}[v^{(\delta)}/\mathtt{C}_1\langle\!\langle t_1\rangle\!\rangle]\mathtt{L}_2\, t_{12}$ and $\mathtt{L}_1 = \mathtt{L}_{11}[u^{\beta}/(!t_{21})\mathtt{K}]\mathtt{L}_{12}$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = (\lambda a^{\alpha}.\, t_{11})\mathtt{L}_{11}[u^{\beta}/(!t_{21})\mathtt{K}] \mapsto^{\beta}_{\bullet\mathsf{gc}} (\lambda a^{\alpha}.\, t_{11})\mathtt{L}_{11}\mathtt{K} = r_1$. Then we conclude immediately since the α and β-steps are disjoint.

4.8. $\mathtt{D} = \Box\,\mathtt{L}_{22}\, t_{12}$ and $\mathtt{L}_2 = \mathtt{L}_{21}[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_{22}$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. We consider each possible location for v^{β}.

4.8.1. $v^{\beta} \in t_{11}$. Then we conclude immediately since the α and β-steps are disjoint.

4.8.2. $v^{\beta} \in \mathtt{L}_1$. Then we conclude immediately since the α and β-steps are disjoint.

4.8.3. $v^{\beta} \in \mathtt{C}_1$. Then we conclude immediately since the α and β-steps are disjoint.

4.8.4. $v^{\beta} \in t_1$. We proceed as in case 1.3.2.2..

4.8.5. $v^{\beta} \in \mathtt{L}_{21}$. Then we conclude immediately since the α and β-steps are disjoint.

4.9. $\mathtt{D} = (\lambda a^{\gamma}.\, t_{11})\mathtt{L}_1[v^{(\delta)}/\mathtt{C}_1\langle t_1\rangle]\mathtt{L}_2\,\mathtt{D}_1$. Then we conclude immediately since the α and β-steps are disjoint.

5. $\mathtt{C} = (\lambda a^{\gamma}.\, t_{11})\mathtt{L}\,\mathtt{C}_1$. We next consider all possible forms for $\mathtt{D}$.

5.1. $\mathtt{D} = \Box$. We proceed as in case 1.1..

5.2. $\mathtt{D} = (\lambda a^{\gamma}.\mathtt{D}_1)\mathtt{L}\,\mathtt{C}_1\langle t_1\rangle$. Then we conclude immediately since the α and β-steps are disjoint.

5.3. $\mathtt{D} = (\lambda a^{\gamma}.\, t_{11})\mathtt{L}_1[u^{(\gamma)}/\mathtt{D}_1]\,\mathtt{C}_1\langle t_1\rangle$. Then we conclude immediately since the α and β-steps are disjoint.

5.4. $\mathtt{D} = \Box\,\mathtt{L}_2\, t_{12}$ and $\mathtt{L} = \mathtt{L}_1[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2]\mathtt{L}_2$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = \mathtt{E}\langle\!\langle v^{\beta}\rangle\!\rangle[v/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \mapsto^{\beta}_{\bullet\mathsf{ls}} \mathtt{E}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[v/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 = r_1$. Then we conclude immediately since the α and β-steps are disjoint.

5.5. $\mathtt{D} = \Box\,\mathtt{L}_2\, t_{12}$ and $\mathtt{L} = \mathtt{L}_1[u^{\beta}/(!t_{21})\mathtt{K}]\mathtt{L}_2$. The step $t = \mathtt{D}\langle t_2\rangle \to^{\beta} \mathtt{D}\langle r_1\rangle = r$ is $t_2 = (\lambda a^{\alpha}.\, t_{11})\mathtt{L}_{11}[u^{\beta}/(!t_{21})\mathtt{K}] \mapsto^{\beta}_{\bullet\mathsf{gc}} (\lambda a^{\alpha}.\, t_{11})\mathtt{L}_1\mathtt{K} = r_1$. Then we conclude immediately since the α and β-steps are disjoint.

5.6. $\mathtt{D} = (\lambda a^{\gamma}.\, t_{11})\mathtt{L}\,\mathtt{D}_1$. We conclude from the IH.

6. $\mathtt{C} = \mathtt{C}_1\, t_{11}$. We next consider all possible forms for $\mathtt{D}$.

6.1. $\mathtt{D} = \Box$. We proceed as in case 1.1..

6.2. $\mathtt{D} = \mathtt{D}_1\, t_{11}$. We conclude from the IH.

6.3. $\mathtt{D} = \mathtt{C}_1\langle t_1\rangle\,\mathtt{D}_1$. Then we conclude immediately since the α and β-steps are disjoint.

7. $\mathtt{C} = t_{11}\,\mathtt{C}_1$. Same as previous case.

8. $\mathtt{C} = \bullet\mathtt{C}_1$. We conclude from the IH since $\mathtt{D}$ must be of the form $\bullet\mathtt{D}_1$.
9. $\mathtt{C} = \mathtt{req}^{(\gamma)}(\mathtt{C}_1)$. We next consider all possible forms for $\mathtt{D}$.
   9.1. $\mathtt{D} = \Box$. We proceed as in case 1.2..
   9.2. $\mathtt{D} = \mathtt{req}^{(\gamma)}(\mathtt{D}_1)$. We conclude from the IH.
10. $\mathtt{C} = !\mathtt{C}_1$. Then both the α and β-steps are inside $\mathtt{C}_1\langle t_1\rangle$ and we conclude from the IH.
11. $\mathtt{C} = \mathtt{C}_1[v^{(\gamma)}/t_{11}]$. We next consider all possible forms for $\mathtt{D}$.
   11.1. $\mathtt{D} = \Box$. We proceed as in case 1.3..
   11.2. $\mathtt{D} = \mathtt{D}_1[v^{(\gamma)}/t_{11}]$. We conclude from the IH.
   11.3. $\mathtt{D} = \mathtt{C}_1\langle t_1\rangle[v^{(\gamma)}/\mathtt{D}_1]$. Then we conclude immediately since the α and β-steps are disjoint.
12. $\mathtt{C} = s[v^{(\gamma)}/\mathtt{C}_1]$. Same as the previous case.

D.1 The $\lambda^{!\bullet}$-calculus as an ARS

Residuals after flattening

Remark D.3. If $t \overset{\pi}{\equiv} s$, $\mathfrak{r} \in \mathsf{steps}(t)$, then $\mathsf{lift}(t, \mathfrak{r}, \alpha) \overset{\pi}{\equiv} s'$, for some s' variant of s.

Definition D.9 (Residuals after flattening). *Given $t \overset{\pi}{\equiv} s$, $\mathfrak{r} \in \mathsf{steps}(t)$ and $\alpha \notin \mathsf{lab}(t)$, consider $\mathsf{lift}(t, \mathfrak{r}, \alpha) \overset{\pi}{\equiv} s'$, where s' is some variant of s. The set of* residuals of $\mathfrak{r}$ after *$t \overset{\pi}{\equiv} s$, is defined as*

$$\mathfrak{r}[\![t \overset{\pi}{\equiv} s]\!] := \{\mathsf{steps}_\alpha(s') \mid \mathsf{lift}(t, \mathfrak{r}, \alpha) \overset{\pi}{\equiv} s'\}$$

We write $\mathfrak{r}[\![t \overset{\pi}{\equiv} s]\!]\mathfrak{r}'$ if $\mathfrak{r}' \in \mathfrak{r}[\![t \overset{\pi}{\equiv} s]\!]$.

We must verify that $\mathfrak{r}[\![t \overset{\pi}{\equiv} s]\!]$ in Def. D.9, does not rely on π. In other words, that $[\![t \overset{\pi}{\equiv} s]\!] = [\![t \overset{\pi'}{\equiv} s]\!]$, for any pair of $t \overset{\pi}{\equiv} s$ and $t \overset{\pi'}{\equiv} s$. This requires making sure that if there are s' and s'' such that $\mathsf{lift}(t, \mathfrak{r}, \alpha) \overset{\pi}{\equiv} s'$ and $\mathsf{lift}(t, \mathfrak{r}, \alpha) \overset{\pi'}{\equiv} s''$, then $s' = s''$. By transitivity of $\equiv$, it suffices to check that $t \overset{\pi}{\equiv} s$ and $t^\circ = s^\circ$ implies $t = s$ (*cf.* Lem. D.27). First we introduce some auxiliary notions and results.

Definition D.10 (Well-named labeled term). *A labeled term $t \in \mathsf{T}^{\mathcal{L}}_{\bullet}$ is* well-named *if 1) all its bound variables are pairwise distinct and 2) all its labels are pairwise distinct.*

Definition D.11 (Label ordering). *Let $t \in \mathsf{T}^{\mathcal{L}}_{\bullet}$ be a well-named term. The* label ordering, *$\prec_t \subseteq \mathsf{lab}(t) \times \mathsf{lab}(t)$, is the total order on its labels defined as the left-to-right order when reading t as a string.*

Lemma D.25. *Let t be well-named and $\pi \vdash t \equiv s$. Then:*

1. *s is well-named;*
2. *$\mathsf{lab}(t) = \mathsf{lab}(s)$; and*
3. *$\prec_t = \prec_s$*

Proof. By induction on π.

Definition D.12 (Equally labeled terms). *Let $t, s \in \mathsf{T}^{\mathcal{L}}_{\bullet}$ be variants (*i.e. $t^\circ = s^\circ$*). We say that t and s are* equally labeled *if they have labels on exactly the same symbols in t°, although these labels might not be identical.*

For example, $(\lambda a^\alpha.\, a)[u^\beta/v]$ and $(\lambda a^\beta.\, a)[u^\alpha/v]$ are equally labeled.

Lemma D.26. *Let t, s be well-named and equally labeled. If*

1. $\mathsf{lab}(t) = \mathsf{lab}(s)$*; and*
2. $\prec_t = \prec_s$

then $t = s$

Proof. By induction on t.

1. $t = a$ or $t = u^{(\alpha)}$. The result is immediate.
2. $t = \lambda a.\, t_1$. From $t^\circ = s^\circ$, it must be the case that $s = \lambda a.\, s_1$ and ${t_1}^\circ = {s_1}^\circ$. Moreover, t_1 and s_1 must be well-named and equally labeled. We thus resort to the IH and conclude.
3. $t = \lambda a^\alpha.\, t_1$. Since t and s are equally labeled, it must be the case that $s = \lambda a^\alpha.\, s_1$ and ${t_1}^\circ = {s_1}^\circ$. Moreover, it also follows that t_1 and s_1 must be well-named and equally labeled too. We thus resort to the IH and conclude.
4. $t = t_1\, t_2$. From $t^\circ = s^\circ$, it must be the case that $s = s_1\, s_2$ and ${t_1}^\circ = {s_1}^\circ$ and ${t_2}^\circ = {s_2}^\circ$.
   4.1. $\mathsf{lab}(t) = \mathsf{lab}(s)$. First we verify that $\mathsf{lab}(t_1) = \mathsf{lab}(s_1)$. Suppose that, on the contrary, there is a $\alpha \in \mathsf{lab}(t_1) \setminus \mathsf{lab}(s_1)$. Then $\alpha \in \mathsf{lab}(s_2)$. Since $\mathsf{lab}(t) = \mathsf{lab}(s)$ and t and s are equally labeled, there is a $\beta \in \mathsf{lab}(s_1) \setminus \mathsf{lab}(t_1)$. Then $\beta \in t_2$. Then we have $\alpha \prec_t \beta$ and $\beta \prec_s \alpha$, contradicting $\prec_t = \prec_s$.
   If we assume $\alpha \in \mathsf{lab}(s_1) \setminus \mathsf{lab}(t_1)$, rather than $\alpha \in \mathsf{lab}(t_1) \setminus \mathsf{lab}(s_1)$, we reach a contradiction in a similar way. Therefore, $\mathsf{lab}(t_1) = \mathsf{lab}(s_1)$. A similar argument shows that $\mathsf{lab}(t_2) = \mathsf{lab}(s_2)$.
   4.2. $\prec_t = \prec_s$. Note that $\prec_{t_1} = \prec_{s_1}$ and $\prec_{t_2} = \prec_{s_2}$, follow immediately from $\prec_t = \prec_s$.
   We thus conclude from the IH applied twice.
5. $t = \bullet t_1$. From $t^\circ = s^\circ$, it must be the case that $s = \bullet s_1$ and ${t_1}^\circ = {s_1}^\circ$. Moreover, $\mathsf{lab}(t_1) = \mathsf{lab}(s_1)$ follows from $\mathsf{lab}(t) = \mathsf{lab}(s)$. Likewise, $\prec_{t_1} = \prec_{s_1}$ follows from $\prec_t = \prec_s$. We thus conclude from the IH.
6. $t = \mathtt{req}(t_1)$. Same as above.
7. $t = \mathtt{req}^\alpha(t_1)$. Same as above.
8. $t = !t_1$. Same as above.
9. $t = t_1[u/t_2]$. This is similar to the case for application. From $t^\circ = s^\circ$, it must be the case that $s = s_1[u/s_2]$ and ${t_1}^\circ = {s_1}^\circ$ and ${t_2}^\circ = {s_2}^\circ$.
   9.1. $\mathsf{lab}(t) = \mathsf{lab}(s)$. First we verify that $\mathsf{lab}(t_1) = \mathsf{lab}(s_1)$. Suppose that, on the contrary, there is a $\alpha \in \mathsf{lab}(t_1) \setminus \mathsf{lab}(s_1)$. Then $\alpha \in \mathsf{lab}(s_2)$. Since $\mathsf{lab}(t) = \mathsf{lab}(s)$ and t and s are equally labeled, there is a $\beta \in \mathsf{lab}(s_1) \setminus \mathsf{lab}(t_1)$. Then $\beta \in t_2$. Then we have $\alpha \prec_t \beta$ and $\beta \prec_s \alpha$, contradicting $\prec_t = \prec_s$.
   If we assume $\alpha \in \mathsf{lab}(s_1) \setminus \mathsf{lab}(t_1)$, rather than $\alpha \in \mathsf{lab}(t_1) \setminus \mathsf{lab}(s_1)$, we reach a contradiction in a similar way. Therefore, $\mathsf{lab}(t_1) = \mathsf{lab}(s_1)$. A similar argument shows that $\mathsf{lab}(t_2) = \mathsf{lab}(s_2)$.

9.2. $\prec_t = \prec_s$. Note that $\prec_{t_1} = \prec_{s_1}$ and $\prec_{t_2} = \prec_{s_2}$, follow immediately from $\prec_t = \prec_s$. We thus conclude from the IH applied twice.

10. $t = t_1[u^\alpha/t_2]$. Same as above.

Lemma D.27 (Well-definedness of Step Correspondence). *Let $t \in \mathsf{T}^{\mathcal{L}}_{\bullet}$ be well-named having labels exactly at the anchors of all the steps in t°. Let s be such that $t \overset{\pi}{\equiv} s$ and $t^\circ = s^\circ$. Then $t = s$.*

Proof. By induction on π we can prove that t and s are equally labeled. Indeed, $\equiv$ does not create nor erase steps nor labels. Moreover, from Lem. D.25, we obtain that s is well-named, $\mathsf{lab}(t) = \mathsf{lab}(s)$ and $\prec_t = \prec_s$. We then conclude from Lem. D.26.

As a consequence of Lem. D.27, we will henceforth drop the derivation π in $[\![t \overset{\pi}{\equiv} s]\!]$ and write simply $[\![t \equiv s]\!]$. The residual relation on flattening $[\![t \equiv s]\!]$ is in fact a bijection between steps in t and those in s. We shall name this bijection *step-correspondence*:

Definition D.13 (Step-correspondence). *Let $t \equiv s$. We define* step-correspondence*, $\phi_{t,s} \subseteq \mathsf{steps}(t) \times \mathsf{steps}(s)$, as follows: $\phi_{t,s}(\mathfrak{r}) = \mathfrak{s}$ iff $\mathfrak{r}[\![t \equiv s]\!]\mathfrak{s}$.*

Step-correspondence is well-defined in the sense that it does not depend on the proof of $t \equiv s$, a consequence of Lem. D.27.

Lemma D.28 (Well-definedness of step-correspondence). *Let t, s be labeled terms such that $t \equiv s$. Then $\phi_{t,s}$ is well-defined.*

Proof. Assume that all bound variables in t are distinct. Let t' be the lifting of all the steps in t. Note that t is well-named. Let π be any derivation such that $t \overset{\pi}{\equiv} s$. Consider $t' \overset{\pi}{\equiv} s'$. By induction on π it is easy to verify that s' has labels on the anchors of all steps in s. Therefore, each step in t' has a unique residual in s'. Moreover, by Lem. D.27, s' is unique for $t \equiv s$ and $s'^\circ = s$. Thus $\phi_{t,s}$ is well-defined.

Definition D.14 (Steps modulo flattening and their equivalence). *Let $\mathfrak{r} : t_1 \to s_1$ be a step in $\lambda^{!\bullet}$ and suppose $t_1' \overset{\pi_1}{\equiv} t_1$ and $s_1 \overset{\pi_2}{\equiv} s_1'$. We say that $\langle \pi_1, \mathfrak{r}, \pi_2 \rangle : [t_1]_\equiv \to_\bullet [s_1]_\equiv$ is a* step modulo flattening*:*

$$t_1' \overset{\pi_1}{\equiv} t_1 \to_\bullet^{\mathfrak{r}} s_1 \overset{\pi_2}{\equiv} s_1'$$

Suppose $\langle \pi_3, \mathfrak{s}, \pi_4 \rangle : [t_2]_\equiv \to_\bullet [s_2]_\equiv$. In other words,

$$t_2' \overset{\pi_3}{\equiv} t_2 \to_\bullet^{\mathfrak{s}} s_2 \overset{\pi_4}{\equiv} s_2'$$

Assume, moreover, that $t_1 \equiv t_2$ and $\phi_{t_1,t_2}(\mathfrak{r}) = \mathfrak{s}$. Then we say $\langle \pi_1, \mathfrak{r}, \pi_2 \rangle$ and $\langle \pi_3, \mathfrak{s}, \pi_4 \rangle$ are equivalent *and write $\langle \pi_1, \mathfrak{r}, \pi_2 \rangle \equiv \langle \pi_3, \mathfrak{s}, \pi_4 \rangle$. Notice that equivalent steps are coinitial and cofinal; the latter follows from Prop. D.3.*

Remark D.4. Equivalence on steps modulo flattening is an equivalence relation. We write $[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv$ for the equivalence class of steps of $\langle \pi_1, \mathfrak{r}, \pi_2 \rangle$.

Definition D.15 ($\lambda^{!\bullet}$ as an ARS). *$\lambda^{!\bullet}$ may be modeled as an Axiomatic Rewrite System $\mathcal{A}_{\lambda^{!\bullet}} = \langle \mathcal{O}, \mathcal{R}, \mathsf{src}, \mathsf{tgt}, [\![]\!] \rangle$ where*

– ***Objects***. *Objects are $\equiv$-equivalence classes of terms:* $\mathcal{O} := \{[t]_\equiv \mid t \in \mathcal{T}_\bullet\}$.
– ***Steps, Src, Tgt***. *Steps are $\lambda^{!\bullet}$-steps modulo flattening*

$$\mathcal{R} := \{[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv : [t]_\equiv \to [s]_\equiv \mid \mathfrak{r} : t \to s \text{ is a step in } \lambda^{!\bullet} \text{ and } t' \overset{\pi_1}{\equiv} t \text{ and } s \overset{\pi_2}{\equiv} s', \text{for some } t', s'\}$$

– ***Residuals***. *Consider two coinitial steps* $[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv : [t_1]_\equiv \to [s_1]_\equiv$ *and* $[\langle \pi_3, \mathfrak{s}, \pi_4 \rangle]_\equiv : [t_2]_\equiv \to [s_2]_\equiv$ *and* $t_1 \equiv t_2$*:*

$$t'_1 \overset{\pi_1}{\equiv} t_1 \to_\bullet^{\mathfrak{r}} s_1 \overset{\pi_2}{\equiv} s'_1$$
$$t'_2 \overset{\pi_3}{\equiv} t_2 \to_\bullet^{\mathfrak{s}} s_2 \overset{\pi_4}{\equiv} s'_2$$

The set of residuals of $[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv$ *after* $[\langle \pi_3, \mathfrak{s}, \pi_4 \rangle]_\equiv$ *is defined as follows*

$$\langle \pi_1, \mathfrak{r}, \pi_2 \rangle [\![\langle \pi_3, \mathfrak{s}, \pi_4 \rangle]\!] := \phi_{t_1,t_2}(\mathfrak{r})[\![\mathfrak{s}]\!]$$

Note that the notion of residual is well-defined as a consequence of the following result, whose proof is an immediate consequence of the proof of Prop. D.3 (that flattening is a strong $\to_\bullet$-bisimulation) whose diagrams are all closed using steps with the same label.

Lemma D.29. $\mathfrak{r}[\![\mathfrak{s}]\!]\mathfrak{r}'$ *iff* $\phi_{t_1,t_2}\mathfrak{r}[\![\phi_{t_1,t_2}\mathfrak{s}]\!]\xi_{s_1,s_2}\mathfrak{r}'$

$$\begin{tikzcd}
t''_1 & & & s''_1 \\
t'_1 \arrow[r, equal, "\pi_1"] & t_1 \arrow[ul, "\mathfrak{r}"'] \arrow[r, "\mathfrak{s}"] \arrow[d, equal, "\pi"] & s_1 \arrow[ur, "\mathfrak{r}'"] \arrow[r, equal, "\pi_2"] \arrow[d, equal, "\pi_5"] & s'_1 \\
t'_2 \arrow[r, equal, "\pi_3"] & t_2 \arrow[dl, "\phi_{t_1,t_2}\mathfrak{r}"] \arrow[r, "\phi_{t_1,t_2}\mathfrak{s}"] & s_2 \arrow[r, equal, "\pi_4"] \arrow[dr, "\xi_{s_1,s_2}\mathfrak{r}' \in \phi_{t_1,t_2}\mathfrak{r}[\![\phi_{t_1,t_2}\mathfrak{s}]\!]"'] & s'_2 \\
t''_2 & & & s''_2
\end{tikzcd}$$

Lemma D.30 (Auto-Erasure). $[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv [\![[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv]\!] = \emptyset$.

Proof. Recall from Def. D.15 that $[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv [\![[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv]\!]$ is defined as

$$\phi_{t_1,t_1}(\mathfrak{r})[\![\mathfrak{r}]\!] \tag{1}$$

By taking the trivial derivation of $t_1 \equiv t_1$ that uses reflexivity and resorting to Lem. D.27, (1) is just $\mathfrak{r}[\![\mathfrak{r}]\!]$. The latter is immediately seen to be the empty set.

Lemma D.31 (Finite Residuals). *For any pair of coinitial steps in* $\mathcal{A}_{\lambda^{!\bullet}}$, $[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv : [t_1]_\equiv \to [s_1]_\equiv$ *and* $[\langle \pi_3, \mathfrak{s}, \pi_4 \rangle]_\equiv : [t_1]_\equiv \to [s_2]_\equiv$, $[\langle \pi_1, \mathfrak{r}, \pi_2 \rangle]_\equiv [\![[\langle \pi_3, \mathfrak{s}, \pi_4 \rangle]_\equiv]\!]$ *is finite.*

Proof. This is immediate from the fact that $\phi_{t_1,t_2}(\mathfrak{r})[\![\mathfrak{s}]\!]$ is finite.

We next prove finite developments. Note that since step-correspondence is a bijection (Lem. D.27), it suffices to prove the following:

Lemma D.32 (Finite Developments). *Let $t \in \mathsf{T}^{\mathcal{L}}_{\bullet}$. Then $\twoheadrightarrow^{\alpha}$ is strongly-normalizing.*

Proof. This is immediate from Prop. D.2.

Lemma D.33 (Semantic Orthogonality). *Consider two coinitial steps in $\mathcal{A}_{\lambda^{!\bullet}}$, say $[\langle\pi_1, \mathfrak{r}, \pi_2\rangle]_{\equiv} : [t_1]_{\equiv} \to [s_1]_{\equiv}$ and $[\langle\pi_3, \mathfrak{s}, \pi_4\rangle]_{\equiv} : [t_1]_{\equiv} \to [s_2]_{\equiv}$. Then there exists r such that $[\langle\pi_1, \mathfrak{r}, \pi_2\rangle]_{\equiv}[\![[\langle\pi_3, \mathfrak{s}, \pi_4\rangle]_{\equiv}]\!] : [s_1]_{\equiv} \to [r]_{\equiv}$ and $[\langle\pi_3, \mathfrak{s}, \pi_4\rangle]_{\equiv}[\![[\langle\pi_1, \mathfrak{r}, \pi_2\rangle]_{\equiv}]\!] : [s_2]_{\equiv} \to [r]_{\equiv}$.*

Proof. This is an immediate consequence of Prop. D.4, Prop. D.3 and Lem. D.27.

D.2 Strong normalization

In this subsection, we show that $\to_{\bullet}$ is strongly normalizing for the typed fragment of $\lambda^{!\bullet}$ by means of the translation $[\![-]\!]$ to the Linear Substitution Calculus (LSC). Recall that the simply typed LSC *without* gc is given by the relation $\to_{\mathrm{lsci}} := \to_{\mathsf{db}} \cup \to_{\mathsf{ls}} \cup \to_{\mathsf{gc}}$ on the set of LSC terms.

Theorem D.2 (The typed LSC is strongly normalizing). *If $\Gamma \vdash_{\mathrm{lsc}} t : A$, there are no infinite reduction sequences $t \to_{\mathrm{lsci}} t_1 \to_{\mathrm{lsci}} t_2 \to_{\mathrm{lsci}} \ldots$.*

Proof. See [19].

Postponement of $\to_{\bullet\mathsf{gc}}$

Lemma D.34. *If $t \to_{\bullet\mathsf{gc}} (\lambda a.\, s)\mathsf{L}$, then one of the following hold:*

1. $t = (\lambda a.\, t_1)\mathsf{L}$ *and* $t_1 \to_{\bullet\mathsf{gc}} s$
2. $\mathsf{L} = \mathsf{L}_1[u/r_2]\mathsf{L}_2$ *and* $t = (\lambda a.\, s)\mathsf{L}_1[u/r_1]\mathsf{L}_2$ *and* $r_1 \to_{\bullet\mathsf{gc}} r_2$
3. $\mathsf{L} = \mathsf{L}_1\mathsf{L}_2\mathsf{L}_3$ *and* $t = (\lambda a.\, s)\mathsf{L}_1[u/(!r)\mathsf{L}_2]\mathsf{L}_3$ *and* $u \notin \mathsf{fv}((\lambda a.\, s)\mathsf{L}_1)$.

Proof. By induction on t.

1. $t = a$ or $t = u$. Not possible since there are no $\to_{\bullet\mathsf{gc}}$ steps from variables.
2. $t = \lambda a.\, s$. Then $\mathsf{L} = \Box$ and the $\to_{\bullet\mathsf{gc}}$ step must be in s and case 1 holds.
3. $t = t_1\, t_2$. Not possible since then the $\to_{\bullet\mathsf{gc}}$ step would have to be in t_1 or in t_2 and $(\lambda a.\, s)\mathsf{L}$ would have to be an application.
4. $t = \bullet t_1$. Not possible since then the $\to_{\bullet\mathsf{gc}}$ step would have to be in t_1 and $(\lambda a.\, s)\mathsf{L}$ would have to be of the form $\bullet r$.
5. $t = \mathtt{req}(t_1)$. Not possible since then the $\to_{\bullet\mathsf{gc}}$ step would have to be in t_1 and $(\lambda a.\, s)\mathsf{L}$ would have to be of the form $\mathtt{req}(r)$.
6. $t = !t_1$. Not possible since then the $\to_{\bullet\mathsf{gc}}$ step would have to be in t_1 and $(\lambda a.\, s)\mathsf{L}$ would have to be of the form $!r$.
7. $t = t_1[u/t_2]$. There are three possible cases.
   - 7.1 The $\to_{\bullet\mathsf{gc}}$ step is at the root. Then $t_2 = (!t_3)\mathsf{L}_2$ and $u \notin \mathsf{fv}(t_1)$ and $\mathsf{L} = \mathsf{L}_1\mathsf{L}_2$ and $t_1 = (\lambda a.\, s)\mathsf{L}_1$. Case 3 then holds.
   - 7.2 The $\to_{\bullet\mathsf{gc}}$ step is in t_1. Then $\mathsf{L} = \mathsf{L}_1[u/r]$ and $t_1 \to_{\bullet\mathsf{gc}} (\lambda a.\, s)\mathsf{L}_1$. We conclude from the IH.

7.3 The $\to_{\bullet\mathsf{gc}}$ step is in t_2. Then $\mathrm{L} = \mathrm{L}_1[u/r]$ and $t_2 \to_{\bullet\mathsf{gc}} r$. Case 2 holds.

Lemma D.35. *If $t \to_{\bullet\mathsf{gc}} (\bullet s)\mathrm{L}$, then one of the following hold:*

1. $t = (\bullet t_1)\mathrm{L}$ *and* $t_1 \to_{\bullet\mathsf{gc}} s$
2. $\mathrm{L} = \mathrm{L}_1[u/r_2]\mathrm{L}_2$ *and* $t = (\bullet s)\mathrm{L}_1[u/r_1]\mathrm{L}_2$ *and* $r_1 \to_{\bullet\mathsf{gc}} r_2$
3. $\mathrm{L} = \mathrm{L}_1\mathrm{L}_2\mathrm{L}_3$ *and* $t = (\bullet s)\mathrm{L}_1[u/(!r)\mathrm{L}_2]\mathrm{L}_3$ *and* $u \notin \mathsf{fv}((\bullet s)\mathrm{L}_1)$.

Proof. By induction on t. Similar to that of Lem. D.34.

Lemma D.36. *If $t \to_{\bullet\mathsf{gc}} (!(\bullet s)\mathrm{L}_1)\mathrm{L}_2$, then one of the following hold:*

1. $t = (!(\bullet t_1)\mathrm{L}_1)\mathrm{L}_2$ *and* $t_1 \to_{\bullet\mathsf{gc}} s$
2. $\mathrm{L}_1 = \mathrm{L}_{11}[u/r_2]\mathrm{L}_{12}$ *and* $t = (!(\bullet s)\mathrm{L}_{11}[u/r_1]\mathrm{L}_{12})\mathrm{L}_2$ *and* $r_1 \to_{\bullet\mathsf{gc}} r_2$
3. $\mathrm{L}_2 = \mathrm{L}_{21}[u/r_2]\mathrm{L}_{22}$ *and* $t = (!(\bullet s)\mathrm{L}_1)\mathrm{L}_{21}[u/r_1]\mathrm{L}_{22}$ *and* $r_1 \to_{\bullet\mathsf{gc}} r_2$
4. $\mathrm{L}_1 = \mathrm{L}_{11}\mathrm{L}_{12}\mathrm{L}_{13}$ *and* $t = (!(\bullet s)\mathrm{L}_{11}[u/(!r)\mathrm{L}_{12}]\mathrm{L}_{13})\mathrm{L}_2$ *and* $u \notin \mathsf{fv}((\bullet s)\mathrm{L}_{11})$.
5. $\mathrm{L}_2 = \mathrm{L}_{21}\mathrm{L}_{22}\mathrm{L}_{23}$ *and* $t = (!(\bullet s)\mathrm{L}_1)\mathrm{L}_{21}[u/(!r)\mathrm{L}_{22}]\mathrm{L}_{23}$ *and* $u \notin \mathsf{fv}((!(\bullet s)\mathrm{L}_1)\mathrm{L}_{21})$.

Proof. By induction on t. Similar to that of Lem. D.34.

Lemma D.37.

1. $t \to_{\bullet\mathsf{gc}} s \to_{\bullet\mathsf{db}} r$ *implies there exists* s' *such that* $t \to_{\bullet\mathsf{db}} s' \to^{*}_{\bullet\mathsf{gc}} r$
2. $t \to_{\bullet\mathsf{gc}} s \to_{\bullet\mathsf{ls}} r$ *implies there exists* s' *such that* $t \to_{\bullet\mathsf{ls}} s' \to^{+}_{\bullet\mathsf{gc}} r$
3. $t \to_{\bullet\mathsf{gc}} s \to_{\bullet\mathsf{req}} r$ *implies there exists* s' *such that* $t \to_{\bullet\mathsf{req}} s' \to_{\bullet\mathsf{gc}} r$

Proof. By induction on t.

1. $t = a$ or $t = u$. This case is not possible since no $\to_{\bullet\mathsf{gc}}$-steps are possible from variables.
2. $t = \lambda a. t_1$. Then it must be the case that $s = \lambda a. s_1$ and the step $t \to_{\bullet\mathsf{gc}} s$ follows from $t_1 \to_{\bullet\mathsf{gc}} s_1$, for some term s_1. Similarly, it must be the case that $r = \lambda a. r_1$ and $s \to_{\mathrm{r}} r$ follows from $s_1 \to_{\mathrm{r}} r_1$, for any $\mathrm{r} \in \{\bullet\mathsf{db}, \bullet\mathsf{ls}, \bullet\mathsf{req}\}$. We thus obtain the desired result by the IH.
3. $t = t_1\, t_2$. Two cases arise depending on whether the $\bullet\mathsf{gc}$ is internal to t_1 or to t_2.
   - 3.1. The $\bullet\mathsf{gc}$ is internal to t_1. Then $s = s_1\, t_2$ and $t_1\, t_2 \to_{\bullet\mathsf{gc}} s_1\, t_2$ follows from $t_1 \to_{\bullet\mathsf{gc}} s_1$. We next consider each item.
     - 3.1.1. Item 1. There are three possibilities.
       - 3.1.1.1. The $\bullet\mathsf{db}$ step is at the root. Then $s_1 = (\lambda a. s_{11})\mathrm{L}$ and $r = s_{11}\{a := t_2\}\mathrm{L}$. By Lem. D.34, there are three cases.
         - 3.1.1.1.1. $t_1 = (\lambda a. t_{11})\mathrm{L}$ and $t_{11} \to_{\bullet\mathsf{gc}} s_{11}$. Then $t_1\, t_2 = (\lambda a. t_{11})\mathrm{L}\, t_2 \to_{\bullet\mathsf{db}} t_{11}\{a := t_2\}\mathrm{L} \to_{\bullet\mathsf{gc}} s_{11}\{a := t_2\}\mathrm{L}$.
         - 3.1.1.1.2. $\mathrm{L} = \mathrm{L}_1[u/t'_{11}]\mathrm{L}_2$ and $t_1 = (\lambda a. s_{11})\mathrm{L}_1[u/t_{11}]\mathrm{L}_2$ and $t_{11} \to_{\bullet\mathsf{gc}} t'_{11}$. Then $t_1\, t_2 = (\lambda a. s_{11})\mathrm{L}_1[u/t_{11}]\mathrm{L}_2\, t_2 \to_{\bullet\mathsf{db}} s_{11}\{a := t_2\}\mathrm{L}_1[u/t_{11}]\mathrm{L}_2 \to_{\bullet\mathsf{gc}} s_{11}\{a := t_2\}\mathrm{L}_1[u/t'_{11}]\mathrm{L}_2$.
         - 3.1.1.1.3. $\mathrm{L} = \mathrm{L}_1\mathrm{L}_2\mathrm{L}_3$ and $t_1 = (\lambda a. s_{11})\mathrm{L}_1[u/(!t_{11})\mathrm{L}_2]\mathrm{L}_3$ and $u \notin \mathsf{fv}((\lambda a. s_{11})\mathrm{L}_1)$. Then $t_1\, t_2 = (\lambda a. s_{11})\mathrm{L}_1[u/(!t_{11})\mathrm{L}_2]\mathrm{L}_3\, t_2 \to_{\bullet\mathsf{db}} s_{11}\{a := t_2\}\mathrm{L}_1[u/(!t_{11})\mathrm{L}_2]\mathrm{L}_3 \to_{\bullet\mathsf{gc}} s_{11}\{a := t_2\}\mathrm{L}_1\mathrm{L}_2\mathrm{L}_3$. Note that we may assume that $u \notin \mathsf{fv}(t_2)$.
       - 3.1.1.2. The $\bullet\mathsf{db}$ step is internal to t_1. We use the IH.

3.1.1.3. The $\bullet\mathsf{db}$ step is internal to t_2. Then the $\bullet\mathsf{gc}$ step and the $\bullet\mathsf{db}$ steps are disjoint and we can conclude immediately by swapping them.

3.1.2. Item 2. There are two possibilities.

3.1.2.1. The $\bullet\mathsf{ls}$ step is internal to t_1. We use the IH.

3.1.2.2. The $\bullet\mathsf{ls}$ step is internal to t_2. Then the $\bullet\mathsf{gc}$ step and the $\bullet\mathsf{ls}$ steps are disjoint and we can conclude immediately by swapping them.

3.1.3. Item 3. There are two possibilities

3.1.3.1. The $\bullet\mathsf{req}$ step is internal to t_1. We use the IH.

3.1.3.2. The $\bullet\mathsf{req}$ step is internal to t_2. Then the $\bullet\mathsf{gc}$ step and the $\bullet\mathsf{req}$ steps are disjoint and we can conclude immediately by swapping them.

3.2. The $\bullet\mathsf{gc}$ is internal to t_2. Similar to the previous case.

4. $t = \bullet t_1$. Then it must be the case that $s = \bullet s_1$ and the step $t \to_{\bullet\mathsf{gc}} s$ follows from $t_1 \to_{\bullet\mathsf{gc}} s_1$, for some s_1. Similarly, it must be the case that $r = \bullet r_1$ and $s \to_{\mathrm{r}} r$ follows from $s_1 \to_{\mathrm{r}} r_1$, for any $\mathrm{r} \in \{\bullet\mathsf{db}, \bullet\mathsf{ls}, \bullet\mathsf{req}\}$. We thus obtain the desired result by the IH.

5. $t = \mathtt{req}(t_1)$. Then it must be the case that $s = \mathtt{req}(s_1)$ and the step $t \to_{\bullet\mathsf{gc}} s$ follows from $t_1 \to_{\bullet\mathsf{gc}} s_1$. We next consider each item.

5.1. Item 1. The $\bullet\mathsf{db}$ step must be internal to s_1. We use the IH.

5.2. Item 2. The $\bullet\mathsf{ls}$ step mut be internal to s_1. We use the IH.

5.3. Item 3. There are two possibilities

5.3.1. The $\bullet\mathsf{req}$ step is at the root. Then $s_1 = (\bullet s_{11})\mathtt{L}$ and $s = \mathtt{req}((\bullet s_{11})\mathtt{L}) \to_{\bullet\mathsf{req}} s_{11}\mathtt{L} = r$. By Lem. D.35, there are three cases.

5.3.1.1. $t_1 = \mathtt{req}((\bullet t_{11})\mathtt{L})$ and $t_{11} \to_{\bullet\mathsf{gc}} s_{11}$. Then $\mathtt{req}(t_1) = \mathtt{req}((\bullet t_{11})\mathtt{L}) \to_{\bullet\mathsf{req}} t_{11}\mathtt{L} \to_{\bullet\mathsf{gc}} s_{11}\mathtt{L}$.

5.3.1.2. $\mathtt{L} = \mathtt{L}_1[u/t'_{11}]\mathtt{L}_2$ and $t_1 = (\bullet s_{11})\mathtt{L}_1[u/t_{11}]\mathtt{L}_2$ and $t_{11} \to_{\bullet\mathsf{gc}} t'_{11}$. Then $\mathtt{req}(t_1) = \mathtt{req}((\bullet s_{11})\mathtt{L}_1[u/t_{11}]\mathtt{L}_2) \to_{\bullet\mathsf{req}} s_{11}\mathtt{L}_1[u/t_{11}]\mathtt{L}_2 \to_{\bullet\mathsf{gc}} s_{11}\mathtt{L}_1[u/t'_{11}]\mathtt{L}_2$.

5.3.1.3. $\mathtt{L} = \mathtt{L}_1\mathtt{L}_2\mathtt{L}_3$ and $t_1 = (\bullet s_{11})\mathtt{L}_1[u/(!t_{11})\mathtt{L}_2]\mathtt{L}_3$ and $u \notin \mathsf{fv}((\bullet s_{11})\mathtt{L}_1)$. Then $\mathtt{req}(t_1) = \mathtt{req}((\bullet s_{11})\mathtt{L}_1[u/(!t_{11})\mathtt{L}_2]\mathtt{L}_3) \to_{\bullet\mathsf{req}} s_{11}\mathtt{L}_1[u/(!t_{11})\mathtt{L}_2]\mathtt{L}_3 \to_{\bullet\mathsf{gc}} s_{11}\mathtt{L}_1\mathtt{L}_2\mathtt{L}_3$.

5.3.2. The $\bullet\mathsf{req}$ step is internal to s_1. We resort to the IH.

6. $t = !t_1$. Then it must be the case that $s = !s_1$ and the step $t \to_{\bullet\mathsf{gc}} s$ follows from $t_1 \to_{\bullet\mathsf{gc}} s_1$. We next consider each item.

6.1. Item 1. The $\bullet\mathsf{db}$ step must be internal to s_1. We use the IH.

6.2. Item 2. The $\bullet\mathsf{ls}$ step must be internal to s_1. We use the IH.

6.3. Item 3. The $\bullet\mathsf{req}$ step must be internal to s_1. We use the IH.

7. $t = t_1[u/t_2]$. There are three possibilities for the $\bullet\mathsf{gc}$-step.

7.1. The $\bullet\mathsf{gc}$ step is at the root. Then $t_2 = (!t_{21})\mathtt{L}$ and $u \notin \mathsf{fv}(t_1)$ and $s = t_1\mathtt{L}$ and $\mathsf{dom}(\mathtt{L}) \cap \mathsf{fv}(t_1) = \varnothing$. We next consider each item.

7.1.1. Item 1. Three further cases arise.

7.1.1.1. The $\bullet\mathsf{db}$ step is at the root. This is not possible unless $\mathtt{L} = \square$, in which case the following case applies.

7.1.1.2. The $\bullet\mathsf{db}$ step is internal to t_1. Then we conclude with $t = t_1[u/(!t_{21})\mathtt{L}] \to_{\bullet\mathsf{db}} t'_1[u/(!t_{21})\mathtt{L}] \to_{\bullet\mathsf{gc}} t'_1\mathtt{L}$.

7.1.1.3. The $\bullet\mathsf{db}$ step is internal to $\mathtt{L}$. Then we conclude with $t = t_1[u/(!t_{21})\mathtt{L}] \to_{\bullet\mathsf{db}} t_1[u/(!t_{21})\mathtt{L}'] \to_{\bullet\mathsf{gc}} t_1\mathtt{L}'$.

7.1.2. Item 2. Three further cases arise.

7.1.2.1. The $\bullet\mathsf{ls}$ step is at the root of $t_1\mathtt{L}_1[v/(!(\bullet t_{11})\mathtt{K}_1)\mathtt{K}_2]$ and $\mathtt{L} = \mathtt{L}_1[v/(!(\bullet t_{11})\mathtt{K}_1)\mathtt{K}_2]\,\mathtt{L}_2$. This case is not possible since $\mathsf{dom}(\mathtt{L}) \cap \mathsf{fv}(t_1) \neq \emptyset$.

7.1.2.2. The $\bullet\mathsf{ls}$ step is internal to t_1. We use the IH.

7.1.2.3. The $\bullet\mathsf{ls}$ step is internal to $\mathtt{L}$. The steps are disjoint and we can commute them.

7.1.3. Item 3. The $\bullet\mathsf{req}$ step must be in t_1. Then both steps are disjoint and we can commute them. That is $t = t_1[u/(!t_{21})\mathtt{L}] \to_{\bullet\mathsf{req}} t_1'[u/(!t_{21})\mathtt{L}] \to_{\bullet\mathsf{gc}} t_1'\mathtt{L}$

7.2. The $\bullet\mathsf{gc}$ step is internal to t_1. In other words, $s = s_1[u/t_2]$ and the step $t \to_{\bullet\mathsf{gc}} s$ follows from $t_1 \to_{\bullet\mathsf{gc}} s_1$. We next consider each item.

7.2.1. Item 1. Two further cases arise.

7.2.1.1. The $\bullet\mathsf{db}$ step is internal to s_1. We use the IH.

7.2.1.2. The $\bullet\mathsf{db}$ step is internal to t_2. Then both steps are disjoint and we can commute them.

7.2.2. Item 2.

7.2.2.1. The $\bullet\mathsf{ls}$ step is at the root. Then $s_1 = \mathtt{C}\langle\!\langle u\rangle\!\rangle$ and $t_2 = (!(\bullet t_{11})\mathtt{K}_1)\mathtt{K}_2$ and

$$s = \mathtt{C}\langle\!\langle u\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{K}_1)\mathtt{K}_2] \to_{\bullet\mathsf{ls}} \mathtt{C}\langle\!\langle(\bullet t_{11})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet t_{11})\mathtt{K}_1]\mathtt{K}_2 = r$$

From $t_1 \to_{\bullet\mathsf{gc}} s_1 = \mathtt{C}\langle\!\langle u\rangle\!\rangle$, it follows that $t_1 = \mathtt{D}\langle\!\langle u\rangle\!\rangle$, for some context $\mathtt{D}$. Then we have $t = t_1[u/t_2] = \mathtt{D}\langle\!\langle u\rangle\!\rangle[u/(!(\bullet t_{11})\mathtt{K}_1)\mathtt{K}_2] \to_{\bullet\mathsf{ls}} \mathtt{D}\langle\!\langle(\bullet t_{11})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet t_{11})\mathtt{K}_1]\mathtt{K}_2 \to_{\bullet\mathsf{gc}} \mathtt{C}\langle\!\langle(\bullet t_{11})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet t_{11})\mathtt{K}_1]\mathtt{K}_2$.

7.2.2.2. The $\bullet\mathsf{ls}$ step is internal to s_1. We use the IH.

7.2.2.3. The $\bullet\mathsf{ls}$ step is internal to t_2. Then both steps are disjoint and we can commute them.

7.2.3. Item 3. There are two possibilities

7.2.3.1. The $\bullet\mathsf{req}$ step is internal to s_1. We use the IH.

7.2.3.2. The $\bullet\mathsf{req}$ step is internal to t_2. Then both steps are disjoint and we can commute them.

7.3. The $\bullet\mathsf{gc}$ step is internal to t_2. In other words, $s = t_1[u/s_2]$ and the step $t \to_{\bullet\mathsf{gc}} s$ follows from $t_2 \to_{\bullet\mathsf{gc}} s_2$. We next consider each item.

7.3.1. Item 1.

7.3.1.1. The $\bullet\mathsf{db}$ step is internal to t_1. Then both steps are disjoint and we can commute them.

7.3.1.2. The $\bullet\mathsf{db}$ step is internal to s_2. We use the IH.

7.3.2. Item 2. There are three possibilities.

7.3.2.1. The $\bullet\mathsf{ls}$ step is at the root. Then $t_1 = \mathtt{C}\langle\!\langle u\rangle\!\rangle$ and $s_2 = (!(\bullet s_{21})\mathtt{K}_1)\mathtt{K}_2$ and

$$s = \mathtt{C}\langle\!\langle u\rangle\!\rangle[u/(!(\bullet s_{21})\mathtt{K}_1)\mathtt{K}_2] \to_{\bullet\mathsf{ls}} \mathtt{C}\langle\!\langle(\bullet s_{21})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet s_{21})\mathtt{K}_1]\mathtt{K}_2 = r$$

From $t_2 \to_{\bullet\mathsf{gc}} (!(\bullet s_{21})\mathtt{K}_1)\mathtt{K}_2$ and Lem. D.36 there are five cases to consider.

1. $t_2 = (!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2$ and $t_{21} \to_{\bullet\mathsf{gc}} s_{21}$. Then we have $t = t_1[u/t_2] = \mathtt{C}\langle\!\langle u\rangle\!\rangle[u/(!(\bullet t_{21})\mathtt{K}_1)\mathtt{K}_2] \to_{\bullet\mathsf{ls}} \mathtt{C}\langle\!\langle(\bullet t_{21})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet t_{21})\mathtt{K}_1]\mathtt{K}_2 \to_{\bullet\mathsf{gc}}\to_{\bullet\mathsf{gc}} \mathtt{C}\langle\!\langle(\bullet s_{21})\mathtt{K}_1\rangle\!\rangle[u/!(\bullet s_{21})\mathtt{K}_1]\mathtt{K}_2$.
2. $\mathtt{K}_1 = \mathtt{K}_{11}[v/r_2]\mathtt{K}_{12}$ and $t_2 = (!(\bullet s_{21})\mathtt{K}_{11}[v/r_1]\mathtt{K}_{12})\mathtt{K}_2$ and $r_1 \to_{\bullet\mathsf{gc}} r_2$. Similar to the previous case.

3. $\mathtt{K}_2 = \mathtt{K}_{21}[v/r_2]\mathtt{K}_{22}$ and $t_2 = (!(\bullet s_{21})\mathtt{K}_1)\mathtt{K}_{21}[v/r_1]\mathtt{K}_{22}$ and $r_1 \to_{\bullet\mathsf{gc}} r_2$.
   Similar to the previous case.
4. $\mathtt{K}_1 = \mathtt{K}_{11}\mathtt{K}_{12}\mathtt{K}_{13}$ and $t_2 = (!(\bullet s_{21})\mathtt{K}_{11}[v/(!r)\mathtt{K}_{12}]\mathtt{K}_{13})\mathtt{K}_2$ and $v \notin \mathsf{fv}((\bullet s_{21})\mathtt{K}_{11})$.
   Similar to the previous case.
5. $\mathtt{K}_2 = \mathtt{K}_{21}\mathtt{K}_{22}\mathtt{K}_{23}$ and $t_2 = (!(\bullet s_{21})\mathtt{K}_1)\mathtt{K}_{21}[v/(!r)\mathtt{K}_{22}]\mathtt{K}_{23}$ and $v \notin \mathsf{fv}((!(\bullet s_{21})\mathtt{K}_1)\mathtt{K}_{21})$.
   Similar to the previous case.

7.3.2.2. The $\bullet\mathsf{ls}$ step is internal to t_1. Then both steps are disjoint and we can commute them.
7.3.2.3. The $\bullet\mathsf{ls}$ step is internal to s_2. We resort to the IH.
7.3.3. Item 3. There are two possibilities
7.3.3.1. The $\bullet\mathsf{req}$ step is internal to t_1. Then both steps are disjoint and we can commute them.
7.3.3.2. The $\bullet\mathsf{req}$ step is internal to s_2. We use the IH.

Fusion

Definition D.16 (Fusion). *The binary relation of fusion $\Rrightarrow \subseteq \mathcal{T}_{\mathsf{LSC}} \times \mathcal{T}_{\mathsf{LSC}}$ between terms of the LSC, called* fusion*, is defined as $\Rrightarrow := (\Rrightarrow^1)^*$, where $\Rrightarrow^1$ in turn is the closure by compatibility under arbitrary contexts of the following rules:*

$(\Rrightarrow\mathtt{w})$	$t[x/s] \Rrightarrow^1 t$	*if* $x \notin \mathsf{fv}(t)$
$(\Rrightarrow\mathtt{c})$	$t[x/s][y/s] \Rrightarrow^1 t\{x := y\}[y/s]$	
$(\Rrightarrow\mathtt{abs})$	$\lambda x.\, t[y/s] \Rrightarrow^1 (\lambda x.\, t)[y/s]$	*if* $x \notin \mathsf{fv}(s)$
$(\Rrightarrow\mathtt{appL})$	$t[x/s]\, r \Rrightarrow^1 (t\, r)[x/s]$	*if* $x \notin \mathsf{fv}(r)$
$(\Rrightarrow\mathtt{appR})$	$t\, r[x/s] \Rrightarrow^1 (t\, r)[x/s]$	*if* $x \notin \mathsf{fv}(t)$
$(\Rrightarrow\mathtt{esL})$	$t[x/s][y/r] \Rrightarrow^1 t[y/r][x/s]$	*if* $x \notin \mathsf{fv}(r)$ *and* $y \notin \mathsf{fv}(s)$
$(\Rrightarrow\mathtt{esR})$	$t[x/s[y/r]] \Rrightarrow^1 t[x/s][y/r]$	*if* $y \notin \mathsf{fv}(t)$

Remark D.5. It is easy to check that the relation $\Rrightarrow$ as defined in Def. D.16 is equivalent to the definition of fusion given in the body of the paper, which replaces rules $\Rrightarrow\mathtt{abs}$, $\Rrightarrow\mathtt{appL}$, $\Rrightarrow\mathtt{appR}$, $\Rrightarrow\mathtt{esL}$, $\Rrightarrow\mathtt{esR}$ by a single rule of the form $\mathtt{C}\langle t[x/s]\rangle \Rrightarrow \mathtt{C}\langle t\rangle[x/s]$ provided that $\mathtt{C}$ does not bind s.

Lemma D.38 (Properties of fusion).

1. $\mathtt{C}\langle\!\langle t[x/s]\rangle\!\rangle \Rrightarrow \mathtt{C}\langle\!\langle t\rangle\!\rangle[x/s]$
2. $\mathtt{C}\langle\!\langle t\mathtt{L}\rangle\!\rangle[x/t\mathtt{L}] \Rrightarrow \mathtt{C}\langle\!\langle t\rangle\!\rangle[x/t]\mathtt{L}$

Proof. We prove each item separately:

1. By induction on $\mathtt{C}$:
   1.1 If $\mathtt{C} = \Box$, it is immediate.
   1.2 If $\mathtt{C} = \lambda x.\, \mathtt{C}'$, it follows from the IH and $\Rrightarrow\mathtt{abs}$.
   1.3 If $\mathtt{C} = \mathtt{C}'\, s$, it follows from the IH and $\Rrightarrow\mathtt{appL}$.
   1.4 If $\mathtt{C} = s\, \mathtt{C}'$, it follows from the IH and $\Rrightarrow\mathtt{appR}$.
   1.5 If $\mathtt{C} = \mathtt{C}'[x/s]$, it follows from the IH and $\Rrightarrow\mathtt{esL}$.
   1.6 If $\mathtt{C} = s[x/\mathtt{C}']$, it follows from the IH and $\Rrightarrow\mathtt{esR}$.

2. By induction on $\mathtt{L}$. If $\mathtt{L} = \square$, it is immediate, so let $\mathtt{L} = \mathtt{L}'[x/s]$. Then:

$$\begin{array}{lll}
\mathtt{C}\langle\!\langle t\mathtt{L}\rangle\!\rangle[x/t\mathtt{L}] & = \mathtt{C}\langle\!\langle t\mathtt{L}'[x/s]\rangle\!\rangle[x/t\mathtt{L}'[x/s]] & \\
& \Rrightarrow \mathtt{C}\langle\!\langle t\mathtt{L}'\rangle\!\rangle[x/s][x/t\mathtt{L}'[x/s]] & \text{by part 1. of this lemma} \\
& \Rrightarrow \mathtt{C}\langle\!\langle t\mathtt{L}'\rangle\!\rangle[x/s][x/t\mathtt{L}'][x/s] & \text{by } \Rrightarrow_{\mathtt{esR}} \\
& = \mathtt{C}\langle\!\langle (t\mathtt{L}')\{x := x'\}\rangle\!\rangle[x'/s][x/t\mathtt{L}'][x/s] & \text{by } \alpha\text{-conversion, where } x' \notin \mathsf{fv}(t\mathtt{L}') \\
& \Rrightarrow \mathtt{C}\langle\!\langle (t\mathtt{L}')\{x := x'\}\rangle\!\rangle[x/t\mathtt{L}'][x'/s][x/s] & \text{by } \Rrightarrow_{\mathtt{esL}} \\
& \Rrightarrow \mathtt{C}\langle\!\langle t\mathtt{L}'\rangle\!\rangle[x/t\mathtt{L}'][x/s] & \text{by } \Rrightarrow_{\mathtt{c}} \\
& \Rrightarrow \mathtt{C}\langle\!\langle t\rangle\!\rangle[x/t]\mathtt{L}'[x/s] & \text{by IH} \\
& = \mathtt{C}\langle\!\langle t\rangle\!\rangle[x/t]\mathtt{L} &
\end{array}$$

Lemma D.39 (Reduction before fusion of variables).

1. *Let x, y, z be different variables. If $\mathtt{C}\langle\!\langle z\rangle\!\rangle = t\{x := y\}$ then there exists a context $\mathtt{C}_0$ such that $t = \mathtt{C}_0\langle\!\langle z\rangle\!\rangle$ and $\mathtt{C} = \mathtt{C}_0\{x := y\}$.*
2. *Let $t\{x := y\} \to_{\mathrm{lsci}} s$. Then there exists a term s_0 such that $t \to_{\mathrm{lsci}} s_0$ and $s = s_0\{x := y\}$.*

Proof. The first item is by induction on $\mathtt{C}$:

1. If $\mathtt{C} = \square$. Then $\mathtt{C}\langle\!\langle z\rangle\!\rangle = z = t\{x := y\}$ and z distinct from y implies that $t = z$. We set $\mathtt{C}_0 := \square$ and conclude.
2. If $\mathtt{C} = \lambda x'.\,\mathtt{C}'$. Without loss of generality, we may assume that x' is distinct from x, y, z. Then $\mathtt{C}\langle\!\langle z\rangle\!\rangle = (\lambda x'.\,\mathtt{C}')\langle\!\langle z\rangle\!\rangle = \lambda x'.\,\mathtt{C}'\langle\!\langle z\rangle\!\rangle = t\{x := y\}$. Thus $t = \lambda x'.\,t_1$ and $\mathtt{C}'\langle\!\langle z\rangle\!\rangle = t_1\{x := y\}$. From the IH there exists a context $\mathtt{C}_0$ such that $t_1 = \mathtt{C}'_0\langle\!\langle z\rangle\!\rangle$ and $\mathtt{C}' = \mathtt{C}'_0\{x := y\}$. We set $\mathtt{C}_0 := \lambda x'.\,\mathtt{C}'_0$ and conclude.
3. If $\mathtt{C} = \mathtt{C}'\ s$ (the case $\mathtt{C} = s\,\mathtt{C}'$ is similar). Then $\mathtt{C}\langle\!\langle z\rangle\!\rangle = (\mathtt{C}'\ s)\langle\!\langle z\rangle\!\rangle = \mathtt{C}'\langle\!\langle z\rangle\!\rangle\ s = t\{x := y\}$. Thus $t = t_1\ t_2$ and $\mathtt{C}'\langle\!\langle z\rangle\!\rangle = t_1\{x := y\}$ and $s = t_2\{x := y\}$. From the IH there exists a context $\mathtt{C}_0$ such that $t_1 = \mathtt{C}'_0\langle\!\langle z\rangle\!\rangle$ and $\mathtt{C}' = \mathtt{C}'_0\{x := y\}$. We set $\mathtt{C}_0 := \mathtt{C}'_0\,t_2\{x := y\}$ and conclude.
4. If $\mathtt{C} = \mathtt{C}'[x'/s]$ (the case $\mathtt{C} = s[x/\mathtt{C}']$ is similar). Without loss of generality, we may assume that x' is distinct from x, y, z. Then $\mathtt{C}\langle\!\langle z\rangle\!\rangle = (\mathtt{C}'[x'/s])\langle\!\langle z\rangle\!\rangle = \mathtt{C}'\langle\!\langle z\rangle\!\rangle[x'/s] = t\{x := y\}$. Thus $t = t_1[x'/t_2]$ and $\mathtt{C}'\langle\!\langle z\rangle\!\rangle = t_1\{x := y\}$ and $s = t_2\{x := y\}$. From the IH there exists a context $\mathtt{C}_0$ such that $t_1 = \mathtt{C}'_0\langle\!\langle z\rangle\!\rangle$ and $\mathtt{C}' = \mathtt{C}'_0\{x := y\}$. We set $\mathtt{C}_0 := \mathtt{C}'_0[x'/t_2\{x := y\}]$ and conclude.

For the second item we proceed as follows. Suppose $t\{x := y\} \to_{\mathrm{lsci}} s$. Then two cases must apply depending on whether the $\to_{\mathrm{lsci}}$ step is a $\to_{\mathsf{db}}$-step or a $\to_{\mathsf{ls}}$-step.

1. $t\{x := y\} = \mathtt{C}\langle(\lambda x.\,s)\mathtt{L}\ r\rangle$. Then it must be the case that $t = \mathtt{C}'\langle(\lambda x.\,s')\mathtt{L}\ r'\rangle$ with $s = s'\{x := y\}$ and $r = r'\{x := y\}$ and $\mathtt{C} = \mathtt{C}'\{x := y\}$. We set $s_0 := \mathtt{C}'\langle(\lambda x.\,s')\mathtt{L}\ r'\rangle$ and conclude.
2. $t\{x := y\} = \mathtt{C}\langle\mathtt{D}\langle\!\langle x\rangle\!\rangle[x/r]\rangle$. Then it must be the case that $t = \mathtt{C}'\langle t_1[x/r']\rangle$ with $\mathtt{D}\langle\!\langle x\rangle\!\rangle = t_1\{x := y\}$ and $r = r'\{x := y\}$ and $\mathtt{C} = \mathtt{C}'\{x := y\}$. By item 1, there exists a context $\mathtt{D}_0$ such that $t_1 = \mathtt{D}_0\langle\!\langle x\rangle\!\rangle$ and $\mathtt{D} = \mathtt{D}_0\{x := y\}$. We set $s_0 := \mathtt{C}'\langle\mathtt{D}_0\langle\!\langle x\rangle\!\rangle[x/r']\rangle$ and conclude.

Lemma D.40 (Backwards preservation of abstractions). *If $t \Rrightarrow^1 (\lambda x.\,s)\mathtt{L}$ then t is of the form $t = (\lambda x.\,t_0)\mathtt{L}_0$ where for all r whose free variables are not bound by $\mathtt{L}_0$ we have that $t_0[x/r]\mathtt{L}_0 \Rrightarrow s[x/r]\mathtt{L}$.*

Proof. We proceed by induction on t:

1. Variable, $t = x$: this case is impossible, as there are no steps $x \Rrightarrow^1 (\lambda x.\, s)\mathsf{L}$.
2. Abstraction, $t = \lambda x.\, t'$: we consider two subcases, depending on whether the fusion step is internal to t' or derived from $\Rrightarrow\mathtt{abs}$ at the root:

   2.1 If the fusion step is internal to t', the situation is that we have that $t' \Rrightarrow^1 t''$ and $t = \lambda x.\, t' \Rrightarrow^1 \lambda x.\, t'' = (\lambda x.\, s)\mathsf{L}$ so L must be empty and $t'' = s$. Taking $t_0 := t'$ and $\mathsf{L}_0 := \Box$ we have that $t = (\lambda x.\, t_0)\mathsf{L}_0$. Moreover, given an arbitrary term r we have that $t_0[x/r]\mathsf{L}_0 = t'[x/r] \Rrightarrow^1 t''[x/r] = s[x/r] = s[x/r]\mathsf{L}$, as required.

   2.2 If the fusion step is derived from $\Rrightarrow\mathtt{abs}$ at the root, then $t' = t_1[y/t_2]$ with $x \notin \mathsf{fv}(t_2)$ and the step is of the form $t = \lambda x.\, t_1[y/t_2] \Rrightarrow^1 (\lambda x.\, t_1)[y/t_2] = (\lambda x.\, s)\mathsf{L}$, so $\mathsf{L} = \Box[y/t_2]$ and $s = t_1$. Taking $t_0 := t' = t_1[y/t_2]$ and $\mathsf{L}_0 = \Box$ we have that $t = (\lambda x.\, t_0)\mathsf{L}_0$. Moreover, given an arbitrary term r we have that $t_0[x/r]\mathsf{L}_0 = t_1[y/t_2][x/r] \Rrightarrow^1 t_1[x/r][y/t_2] = s[x/r]\mathsf{L}$ by $\Rrightarrow\mathtt{esL}$.

3. Application, $t = t_1\, t_2$: this case is impossible, as the fusion step can be either internal to t_1, internal to t_2, derived from $\Rrightarrow\mathtt{appL}$ at the root, or derived from $\Rrightarrow\mathtt{appR}$ at the root. In any of these cases, the right-hand side is not of the form $(\lambda x.\, s)\mathsf{L}$.
4. Substitution, $t = t_1[y/t_2]$: we consider six subcases, depending on whether the fusion step is internal to t_1, internal to t_2, or derived from any of the rules $\Rrightarrow\mathtt{w}$, $\Rrightarrow\mathtt{c}$, $\Rrightarrow\mathtt{esL}$, $\Rrightarrow\mathtt{esR}$ at the root:

   4.1 If the fusion step is internal to t_1, then $t = t_1[y/t_2] \Rrightarrow^1 t_1'[y/t_2] = (\lambda x.\, s)\mathsf{L}$ with $t_1 \Rrightarrow^1 t_1'$. Hence $\mathsf{L} = \mathsf{L}'[y/t_2]$ and $t_1' = (\lambda x.\, s)\mathsf{L}'$. Since $t_1 \Rrightarrow^1 t_1' = (\lambda x.\, s)\mathsf{L}'$ by IH we have that $t_1 = (\lambda x.\, t_0)\mathsf{L}_0'$. Taking $\mathsf{L}_0 := \mathsf{L}_0'[y/t_2]$ we have that $t = t_1[y/t_2] = (\lambda x.\, t_0)\mathsf{L}_0'[y/t_2] = (\lambda x.\, t_0)\mathsf{L}_0$. Moreover, if r is a term whose free variables are not bound by L_0, then $t_0[x/r]\mathsf{L}_0 = t_0[x/r]\mathsf{L}_0'[y/t_2] \Rrightarrow s[x/r]\mathsf{L}'[y/t_2] = s[x/r]\mathsf{L}$ using the fact that $t_0[x/r]\mathsf{L}_0' \Rrightarrow s[x/r]\mathsf{L}'$ holds by IH.

   4.2 If the fusion step is internal to t_2, then $t = t_1[y/t_2] \Rrightarrow^1 t_1[y/t_2'] = (\lambda x.\, s)\mathsf{L}$ with $t_2 \Rrightarrow^1 t_2'$. Then $\mathsf{L} = \mathsf{L}'[y/t_2']$ and $t_1 = (\lambda x.\, s)\mathsf{L}'$. Taking $t_0 := s$ and $\mathsf{L}_0 := \mathsf{L} = \mathsf{L}'[y/t_2]$, we have that $t = t_1[y/t_2] = (\lambda x.\, s)\mathsf{L}'[y/t_2] = (\lambda x.\, t_0)\mathsf{L}_0$. Moreover, if r is a term whose free variables are not bound by L_0, then $t_0[x/r]\mathsf{L}_0 = s[x/r]\mathsf{L}'[y/t_2] \Rrightarrow^1 s[x/r]\mathsf{L}'[y/t_2'] = s[x/r]\mathsf{L}$, using the fact that $t_2 \Rrightarrow^1 t_2'$.

   4.3 If the fusion step is derived from $\Rrightarrow\mathtt{w}$ at the root, then $t = t_1[y/t_2] \Rrightarrow^1 t_1 = (\lambda x.\, s)\mathsf{L}$, where $y \notin \mathsf{fv}(t_1)$. Taking $t_0 := s$ and $\mathsf{L}_0 := \mathsf{L}[y/t_2]$, we have that $t = t_1[y/t_2] = (\lambda x.\, s)\mathsf{L}[y/t_2] = (\lambda x.\, t_0)\mathsf{L}_0$. Moreover, if r is a term whose free variables are not bound by L, we have that $t_0[x/r]\mathsf{L}_0 = s[x/r]\mathsf{L}[y/t_2] \Rrightarrow^1 s[x/r]\mathsf{L}$. For the last step, note that $y \notin \mathsf{fv}(t_1)$ by hypothesis and $y \notin \mathsf{fv}(r)$ because the free variables of r are not bound by L. Hence $y \notin \mathsf{fv}(t_1) \cup \mathsf{fv}(r) = \mathsf{fv}((\lambda x.\, s)\mathsf{L}) \cup \mathsf{fv}(r) = \mathsf{fv}(s[x/r]\mathsf{L})$.

   4.4 If the fusion step is derived from $\Rrightarrow\mathtt{c}$ at the root, then $t = t_1[y/t_2][z/t_2] \Rrightarrow^1 t_1\{y := z\}[z/t_2] = (\lambda x.\, s)\mathsf{L}$, so $\mathsf{L} = \mathsf{L}'[z/t_2]$ and $t_1\{y := z\} = (\lambda x.\, s)\mathsf{L}'$. This means that $t_1 = (\lambda x.\, t_0)\mathsf{L}_0'$ where $t_0\{y := z\} = s$ and $\mathsf{L}_0'\{y := z\} = \mathsf{L}'$. Taking $\mathsf{L}_0 := \mathsf{L}_0'[y/t_2][z/t_2]$ we have that $t = t_1[y/t_2][z/t_2] = (\lambda x.\, t_0)\mathsf{L}_0'[y/t_2][z/t_2] = (\lambda x.\, t_0)\mathsf{L}_0$. Moreover, if r is a term whose free variables are not bound by L_0, then:

$$\begin{array}{rcll} t_0[x/r]\mathtt{L}_0 & = & t_0[x/r]\mathtt{L}'_0[y/t_2][z/t_2] & \\ & \Rrightarrow^1 & (t_0[x/r]\mathtt{L}'_0)\{y := z\}[z/t_2] & \\ & = & t_0\{y := z\}[x/r\{y := z\}]\mathtt{L}'_0\{y := z\}[z/t_2] & \\ & = & t_0\{y := z\}[x/r]\mathtt{L}'_0\{y := z\}[z/t_2] & \text{as } y \notin \mathsf{fv}(r) \\ & = & s[x/r]\mathtt{L}'[z/t_2] & \\ & = & s[x/r]\mathtt{L} & \end{array}$$

4.5 If the fusion step is derived from $\Rrightarrow\mathtt{esL}$ at the root, then $t_1 = t_{11}[z/t_{12}]$ where $y \notin \mathsf{fv}(t_{12})$ and $z \notin \mathsf{fv}(t_2)$, and the step is of the form $t = t_{11}[z/t_{12}][y/t_2] \Rrightarrow^1 t_{11}[y/t_2][z/t_{12}] = (\lambda x.\, s)\mathtt{L}$. Then $\mathtt{L} = \mathtt{L}'[y/t_2][z/t_{12}]$ and $t_{11} = (\lambda x.\, s)\mathtt{L}'$. Taking $t_0 := s$ and $\mathtt{L}_0 := \mathtt{L}'[z/t_{12}][y/t_2]$ we have that $t = t_{11}[z/t_{12}][y/t_2] = (\lambda x.\, s)\mathtt{L}'[z/t_{12}][y/t_2] = (\lambda x.\, t_0)\mathtt{L}_0$. Moreover, if r is a term whose free variables are not bound by $\mathtt{L}_0$, then $t_0[x/r]\mathtt{L}_0 = s[x/r]\mathtt{L}'[z/t_{12}][y/t_2] \Rrightarrow^1 s[x/r]\mathtt{L}'[y/t_2][z/t_{12}] = s[x/r]\mathtt{L}$.

4.6 If the fusion step is derived from $\Rrightarrow\mathtt{esR}$ at the root, then $t_2 = t_{21}[z/t_{22}]$ where $z \notin \mathsf{fv}(t_1)$, and the step is of the form $t = t_1[y/t_{21}[z/t_{22}]] \Rrightarrow^1 t_1[y/t_{21}][z/t_{22}] = (\lambda x.\, s)\mathtt{L}$. Then $\mathtt{L} = \mathtt{L}'[y/t_{21}][z/t_{22}]$ and $t_1 = (\lambda x.\, s)\mathtt{L}'$. Taking $t_0 := s$ and $\mathtt{L}_0 := \mathtt{L}'[y/t_{21}[z/t_{22}]]$ we have that $t = t_1[y/t_{21}[z/t_{22}]] = (\lambda x.\, s)\mathtt{L}'[y/t_{21}[z/t_{22}]] = (\lambda x.\, t_0)\mathtt{L}_0$. Moreover, if r is a term whose free variables are not bound by $\mathtt{L}_0$, then $t_0[x/r]\mathtt{L}_0 = s[x/r]\mathtt{L}'[y/t_{21}[z/t_{22}]] \Rrightarrow^1 s[x/r]\mathtt{L}'[y/t_{21}][z/t_{22}] = s[x/r]\mathtt{L}$.

Lemma D.41 (Backwards preservation of variables). *If $t \Rrightarrow^1 \mathtt{C}\langle\!\langle x\rangle\!\rangle$ then t is of the form $t = \mathtt{C}_0\langle\!\langle x, \dots, x\rangle\!\rangle$ where $\mathtt{C}_0$ is a context with either one or two holes, and for any term r whose free variables are not bound by $\mathtt{C}_0$ one has that $\mathtt{C}_0\langle\!\langle r, \dots, r\rangle\!\rangle \Rrightarrow \mathtt{C}\langle\!\langle r\rangle\!\rangle$.*

Proof. By induction on t:

1. Variable, $t = y$: this case is impossible, as there are no steps $y \Rrightarrow \mathtt{C}\langle\!\langle x\rangle\!\rangle$.
2. Abstraction, $t = \lambda y.\, t'$: we consider two subcases, depending on whether the fusion step is internal to t' or derived from $\Rrightarrow\mathtt{abs}$ at the root:
   2.1 If the fusion step is internal to t', then $t = \lambda y.\, t' \Rrightarrow^1 \lambda y.\, \mathtt{C}'\langle\!\langle x\rangle\!\rangle = \mathtt{C}\langle\!\langle x\rangle\!\rangle$ where $\mathtt{C} = \lambda y.\, \mathtt{C}'$ and $t' \Rrightarrow^1 \mathtt{C}'\langle\!\langle x\rangle\!\rangle$. By IH, we have that $t' = \mathtt{C}'_0\langle\!\langle x, \dots, x\rangle\!\rangle$ and $\mathtt{C}'_0\langle\!\langle r, \dots, r\rangle\!\rangle \Rrightarrow \mathtt{C}'\langle\!\langle r\rangle\!\rangle$. Taking $\mathtt{C}_0 := \lambda y.\, \mathtt{C}'_0$ we have that $t = \lambda y.\, t' = \lambda y.\, \mathtt{C}'_0\langle\!\langle x, \dots, x\rangle\!\rangle = \mathtt{C}_0\langle\!\langle x, \dots, x\rangle\!\rangle$, and $\mathtt{C}_0\langle\!\langle r, \dots, r\rangle\!\rangle = \lambda y.\, \mathtt{C}'_0\langle\!\langle r, \dots, r\rangle\!\rangle \Rrightarrow \lambda y.\, \mathtt{C}'\langle\!\langle r\rangle\!\rangle = \mathtt{C}\langle\!\langle r\rangle\!\rangle$.
   2.2 If the fusion step is derived from $\Rrightarrow\mathtt{abs}$ at the root, then $t' = t_1[z/t_2]$ with $y \notin \mathsf{fv}(t_2)$, and the step is of the form $t = \lambda y.\, t_1[z/t_2] \Rrightarrow^1 (\lambda y.\, t_1)[z/t_2] = \mathtt{C}\langle\!\langle x\rangle\!\rangle$. There are two subcases, depending on whether the hole of $\mathtt{C}$ lies inside t_1 or inside t_2. We only check the first of these cases, the other one being similar. Indeed, suppose that the hole of $\mathtt{C}$ lies inside t_1. Then $\mathtt{C} = (\lambda y.\, \mathtt{C}')[z/t_2]$ and $t_1 = \mathtt{C}'\langle\!\langle x\rangle\!\rangle$. Taking $\mathtt{C}_0 := \lambda y.\, \mathtt{C}'[z/t_2]$ we have that $t = \lambda y.\, t_1[z/t_2] = \lambda y.\, \mathtt{C}'\langle\!\langle x\rangle\!\rangle[z/t_2] = \mathtt{C}_0\langle\!\langle x\rangle\!\rangle$ and $\mathtt{C}_0\langle\!\langle r\rangle\!\rangle = \lambda y.\, \mathtt{C}'\langle\!\langle r\rangle\!\rangle[z/t_2] \Rrightarrow^1 (\lambda y.\, \mathtt{C}'\langle\!\langle r\rangle\!\rangle)[z/t_2] = \mathtt{C}\langle\!\langle r\rangle\!\rangle$.
3. Application, $t = t_1\, t_2$: we consider four subcases, depending on whether the fusion step is internal to t_1, internal to t_2, derived from $\Rrightarrow\mathtt{appL}$ at the root, or derived from $\Rrightarrow\mathtt{appR}$ at the root:

3.1 If the fusion step is internal to t_1, then $t = t_1\, t_2 \Rrightarrow^1 t_1'\, t_2 = \mathsf{C}\langle\!\langle x \rangle\!\rangle$ with $t_1 \Rrightarrow^1 t_1'$. There are two subcases, depending on whether the hole of C lies inside t_1' or inside t_2:

3.1.1 If the hole of C lies inside t_1', then $t_1' = \mathsf{C}'\langle\!\langle x \rangle\!\rangle$ and $\mathsf{C} = \mathsf{C}'\, t_2$ and $t_1 \Rrightarrow^1 \mathsf{C}'\langle\!\langle x \rangle\!\rangle$. By IH, we have that $t_1 = \mathsf{C}_0'\langle\!\langle x, \dots, x \rangle\!\rangle$ and $\mathsf{C}_0'\langle\!\langle r, \dots, r \rangle\!\rangle \Rrightarrow \mathsf{C}'\langle\!\langle r \rangle\!\rangle$. Taking $\mathsf{C}_0 := \mathsf{C}_0'\, t_2$ we have that $t = t_1\, t_2 = \mathsf{C}_0'\langle\!\langle x, \dots, x \rangle\!\rangle\, t_2 = \mathsf{C}_0\langle\!\langle x, \dots, x \rangle\!\rangle$, and $\mathsf{C}_0\langle\!\langle r, \dots, r \rangle\!\rangle = \mathsf{C}_0'\langle\!\langle r, \dots, r \rangle\!\rangle\, t_2 \Rrightarrow \mathsf{C}'\langle\!\langle r \rangle\!\rangle\, t_2 = \mathsf{C}\langle\!\langle r \rangle\!\rangle$.

3.1.2 If the hole of C lies inside t_2, then $t_2 = \mathsf{C}'\langle\!\langle x \rangle\!\rangle$ and $\mathsf{C} = t_1'\, \mathsf{C}'$. Taking $\mathsf{C}_0 := t_1\, \mathsf{C}'$ we have that $t = t_1\, t_2 = t_1\, \mathsf{C}'\langle\!\langle x \rangle\!\rangle = \mathsf{C}_0\langle\!\langle x \rangle\!\rangle$ and $\mathsf{C}_0\langle\!\langle r \rangle\!\rangle = t_1\, \mathsf{C}' r \Rrightarrow^1 t_1'\, \mathsf{C}' r = \mathsf{C}\langle\!\langle r \rangle\!\rangle$.

3.2 If the fusion step is internal to t_2, the proof is similar to case 3.1.

3.3 If the fusion step is derived from $\Rrightarrow\mathtt{appL}$ at the root, then $t_1 = t_{11}[y/t_{12}]$ where $y \notin \mathsf{fv}(t_2)$ and the step is of the form $t = t_{11}[y/t_{12}]\, t_2 \Rrightarrow (t_{11}\, t_2)[y/t_{12}] = \mathsf{C}\langle\!\langle x \rangle\!\rangle$. There are three similar subcases, depending on whether the hole of C lies inside t_{11}, inside t_2, or inside t_{12}. We only check the first of these cases, the other ones being similar. Indeed, suppose that the hole of C lies inside t_{11}. Then $\mathsf{C} = (\mathsf{C}'\, t_2)[y/t_{12}]$ and $t_{11} = \mathsf{C}'\langle\!\langle x \rangle\!\rangle$. Taking $\mathsf{C}_0 := \mathsf{C}'[y/t_{12}]\, t_2$ we have that $t = t_{11}[y/t_{12}]\, t_2 = \mathsf{C}'\langle\!\langle x \rangle\!\rangle[y/t_{12}]\, t_2 = \mathsf{C}_0\langle\!\langle x \rangle\!\rangle$ and that $\mathsf{C}_0\langle\!\langle r \rangle\!\rangle = \mathsf{C}'\langle\!\langle r \rangle\!\rangle[y/t_{12}]\, t_2 \Rrightarrow^1 (\mathsf{C}'\langle\!\langle r \rangle\!\rangle\, t_2)[y/t_{12}] = \mathsf{C}\langle\!\langle r \rangle\!\rangle$.

3.4 If the fusion step is derived from $\Rrightarrow\mathtt{appR}$ at the root, the proof is similar to case 3.3.

4. Substitution, $t = t_1[y/t_2]$: we consider six subcases, depending on whether the fusion step is internal to t_1, internal to t_2, or derived from one of the rules $\Rrightarrow\mathtt{w}$, $\Rrightarrow\mathtt{c}$, $\Rrightarrow\mathtt{esL}$, or $\Rrightarrow\mathtt{esR}$ at the root:

4.1 If the fusion step is internal to t_1, then $t = t_1[y/t_2] \Rrightarrow^1 t_1'[y/t_2] = \mathsf{C}\langle\!\langle x \rangle\!\rangle$ with $t_1 \Rrightarrow^1 t_1'$ and the proof is similar to case 3.1.

4.2 If the fusion step is internal to t_2, then $t = t_1[y/t_2] \Rrightarrow^1 t_1[y/t_2'] = \mathsf{C}\langle\!\langle x \rangle\!\rangle$ with $t_2 \Rrightarrow^1 t_2'$ and the proof is similar to case 3.1.

4.3 If the fusion step is derived from $\Rrightarrow\mathtt{w}$ at the root, then $y \notin \mathsf{fv}(t_1)$ and $t = t_1[y/t_2] \Rrightarrow^1 t_1 = \mathsf{C}\langle\!\langle x \rangle\!\rangle$. Taking $\mathsf{C}_0 := \mathsf{C}[y/t_2]$ we have that $t = t_1[y/t_2] = \mathsf{C}\langle\!\langle x \rangle\!\rangle[y/t_2] = \mathsf{C}_0\langle\!\langle x \rangle\!\rangle$ and $\mathsf{C}_0\langle\!\langle r \rangle\!\rangle = \mathsf{C}\langle\!\langle r \rangle\!\rangle[y/t_2] \Rrightarrow^1 \mathsf{C}\langle\!\langle r \rangle\!\rangle$. The last step an instance of the $\Rrightarrow\mathtt{w}$ rule. To justify that this step can be applied, note that $y \notin \mathsf{fv}(t_1) = \mathsf{fv}(\mathsf{C}\langle\!\langle x \rangle\!\rangle)$ so in particular $y \notin \mathsf{fv}(\mathsf{C})$, and the free variables of r are not bound by C_0 by hypothesis, so $y \notin \mathsf{fv}(r)$. Hence $y \notin \mathsf{fv}(\mathsf{C}) \cup \mathsf{fv}(r) = \mathsf{fv}(\mathsf{C}\langle\!\langle r \rangle\!\rangle)$.

4.4 If the fusion step is derived from $\Rrightarrow\mathtt{c}$ at the root, then $t_1 = t_1'[z/t_2]$ and the step is of the form $t = t_1'[z/t_2][y/t_2] \Rrightarrow^1 t_1'\{z := y\}[y/t_2] = \mathsf{C}\langle\!\langle x \rangle\!\rangle$. There are two subcases, depending on whether the hole of C lies inside $t_1'\{z := y\}$ or inside t_2:

4.4.1 If the hole of C lies inside $t_1'\{z := y\}$, then $\mathsf{C} = \mathsf{C}'[y/t_2]$ and $\mathsf{C}'\langle\!\langle x \rangle\!\rangle = t_1'\{z := y\}$. Since $x \neq y$, by Lem. D.39 there exists a context C_0' such that $t_1' = \mathsf{C}_0'\langle\!\langle x \rangle\!\rangle$ and $\mathsf{C}' = \mathsf{C}_0'\{z := y\}$. Taking $\mathsf{C}_0 := \mathsf{C}_0'[z/t_2][y/t_2]$ we have that

$t = t_1'[z/t_2][y/t_2] = \mathsf{C}_0'\langle\!\langle x\rangle\!\rangle[z/t_2][y/t_2] = \mathsf{C}_0\langle\!\langle x\rangle\!\rangle$ and:

$$\begin{array}{rcll}
\mathsf{C}_0\langle\!\langle r\rangle\!\rangle & = & \mathsf{C}_0'\langle\!\langle r\rangle\!\rangle[z/t_2][y/t_2] & \\
& \Rrightarrow^1 & \mathsf{C}_0'\langle\!\langle r\rangle\!\rangle\{z := y\}[y/t_2] & \\
& = & \mathsf{C}_0'\{z := y\}\langle\!\langle r\{z := y\}\rangle\!\rangle[y/t_2] & \\
& = & \mathsf{C}_0'\{z := y\}\langle\!\langle r\rangle\!\rangle[y/t_2] & \text{as } z \notin \mathsf{fv}(r) \\
& = & \mathsf{C}'\langle\!\langle r\rangle\!\rangle[y/t_2] & \\
& = & \mathsf{C}\langle\!\langle r\rangle\!\rangle &
\end{array}$$

4.4.2 If the hole of C lies inside t_2, then $\mathsf{C} = t_1'\{z := y\}[y/\mathsf{C}']$ and $\mathsf{C}'\langle\!\langle x\rangle\!\rangle = t_2$. Taking C_0 as the two-hole context $\mathsf{C}_0 := t_1'[z/\mathsf{C}'][y/\mathsf{C}']$ we have that:

$$\begin{array}{rl}
t & = t_1'[z/t_2][y/t_2] \\
& = t_1'[z/\mathsf{C}'\langle\!\langle x\rangle\!\rangle][y/\mathsf{C}'\langle\!\langle x\rangle\!\rangle] \\
& = (t_1'[z/\mathsf{C}'][y/\mathsf{C}'])\langle\!\langle x, x\rangle\!\rangle \\
& = \mathsf{C}_0\langle\!\langle x, x\rangle\!\rangle
\end{array}$$

and that:

$$\begin{array}{rcl}
\mathsf{C}_0\langle\!\langle r, r\rangle\!\rangle & = & (t_1'[z/\mathsf{C}'][y/\mathsf{C}'])\langle\!\langle r, r\rangle\!\rangle \\
& = & t_1'[z/\mathsf{C}'\langle\!\langle r\rangle\!\rangle][y/\mathsf{C}'\langle\!\langle r\rangle\!\rangle] \\
& \Rrightarrow^1 & t_1'\{z := y\}[y/\mathsf{C}'\langle\!\langle r\rangle\!\rangle] \\
& = & \mathsf{C}\langle\!\langle r\rangle\!\rangle
\end{array}$$

Note that this is the only base case in which a context C_0 with more than one hole is needed.

4.5 If the fusion step is derived from $\Rrightarrow\mathtt{esL}$ at the root, then $t_1 = t_{11}[z/t_{12}]$ with $y \notin \mathsf{fv}(t_{12})$ and $z \notin \mathsf{fv}(t_2)$, and the step is of the form $t = t_{11}[z/t_{12}][y/t_2] \Rrightarrow^1 t_{11}[y/t_2][z/t_{12}] = \mathsf{C}\langle\!\langle x\rangle\!\rangle$. There are three similar subcases, depending on whether the hole of C lies inside t_{11}, inside t_2, or inside t_{12}. We only check the first of these cases, the other ones being similar. Indeed, suppose that the hole of C lies inside t_{11}. Then $\mathsf{C} = \mathsf{C}'[y/t_2][z/t_{12}]$ and $t_{11} = \mathsf{C}'\langle\!\langle x\rangle\!\rangle$. Taking $\mathsf{C}_0 := \mathsf{C}'[z/t_{12}][y/t_2]$ we have that $t = t_{11}[z/t_{12}][y/t_2] = \mathsf{C}'\langle\!\langle x\rangle\!\rangle[z/t_{12}][y/t_2] = \mathsf{C}_0\langle\!\langle x\rangle\!\rangle$ and $\mathsf{C}_0\langle\!\langle r\rangle\!\rangle = \mathsf{C}'\langle\!\langle r\rangle\!\rangle[z/t_{12}][y/t_2] \Rrightarrow^1 \mathsf{C}'\langle\!\langle r\rangle\!\rangle[y/t_2][z/t_{12}] = \mathsf{C}\langle\!\langle r\rangle\!\rangle$.

4.6 If the fusion step is derived from $\Rrightarrow\mathtt{esR}$ at the root, then $t_2 = t_{21}[z/t_{22}]$ with $z \notin \mathsf{fv}(t_1)$, and the step is of the form $t = t_1[y/t_{21}[z/t_{22}]] \Rrightarrow^1 t_1[y/t_{21}][z/t_{22}] = \mathsf{C}\langle\!\langle x\rangle\!\rangle$. There are three similar subcases, depending on whether the hole of C lies inside t_1, inside t_{21}, or inside t_{22}. We only check the first of these cases, the other ones being similar. Indeed, suppose that the hole of C lies inside t_1. Then $\mathsf{C} = \mathsf{C}'[y/t_{21}][z/t_{22}]$ and $t_1 = \mathsf{C}'\langle\!\langle x\rangle\!\rangle$. Taking $\mathsf{C}_0 := \mathsf{C}'[y/t_{21}[z/t_{22}]]$ we have that $t = t_1[y/t_{21}[z/t_{22}]] = \mathsf{C}'\langle\!\langle x\rangle\!\rangle[y/t_{21}[z/t_{22}]] = \mathsf{C}_0\langle\!\langle x\rangle\!\rangle$ and $\mathsf{C}_0\langle\!\langle r\rangle\!\rangle = \mathsf{C}'\langle\!\langle r\rangle\!\rangle[y/t_{21}[z/t_{22}]] \Rrightarrow^1 \mathsf{C}'\langle\!\langle r\rangle\!\rangle[y/t_{21}][z/t_{22}] = \mathsf{C}\langle\!\langle r\rangle\!\rangle$.

Lemma D.42 (Postponement of single fusion step). $\Rrightarrow^1 \rightarrow_{\text{lsci}} \subseteq \rightarrow_{\text{lsci}}^+ \Rrightarrow$

Proof. Let $t \Rrightarrow^1 s \rightarrow_{\text{lsci}} r$ and let us check that there exists a term p such that $t \rightarrow_{\text{lsci}}^+ p \Rrightarrow r$. Graphically:

$$\begin{array}{ccc}
t & \Rrightarrow^1 & s \\
\downarrow{\scriptstyle \text{lsci}^+} & & \downarrow{\scriptstyle \text{lsci}} \\
p & \Rrightarrow & r
\end{array}$$

We proceed by induction on t:

1. Variable, $t = x$: note that there is no s such that $t \Rrightarrow s$, so this case is impossible.
2. Abstraction, $t = \lambda x.\, t'$: we consider two subcases, depending on whether the fusion step $t = \lambda x.\, t' \Rrightarrow^1 s$ is internal to t' or derived from $\Rrightarrow\mathtt{abs}$ at the root:
   2.1 If the fusion step is internal, then $t = \lambda x.\, t' \Rrightarrow^1 \lambda x.\, s' = s$ with $s \Rrightarrow^1 s'$. Moreover, the reduction step $s = \lambda x.\, s' \to_{\mathrm{lsci}} r$ must be internal, so $s = \lambda x.\, s' \to_{\mathrm{lsci}} \lambda x.\, r' = r$ with $s' \to_{\mathrm{lsci}} r'$. Then:

$$\begin{array}{ccc} t' & \Rrightarrow^1 & s' \\ {\scriptstyle \mathrm{lsci}^+}\downarrow & & \downarrow{\scriptstyle \mathrm{lsci}} \\ p' & \Rrightarrow & r' \end{array} \quad \text{by IH, so} \quad \begin{array}{ccc} \lambda x.\, t' & \Rrightarrow^1 & \lambda x.\, s' \\ {\scriptstyle \mathrm{lsci}^+}\downarrow & & \downarrow{\scriptstyle \mathrm{lsci}} \\ \lambda x.\, p' & \Rrightarrow & \lambda x.\, r' \end{array}$$

   2.2 If the fusion step is derived from $\Rrightarrow\mathtt{abs}$ at the root, then $t = \lambda x.\, t_1[y/t_2] \Rrightarrow (\lambda x.\, t_1)[y/t_2] = s$. There are three further subcases, depending on whether the reduction step $s = (\lambda x.\, t_1)[y/t_2] \to_{\mathrm{lsci}} r$ is internal to t_1, internal to t_2, or a $\to_{\mathsf{ls}}$ step at the root:
      2.2.1 If the reduction step is internal to t_1, then the situation is:

$$\begin{array}{ccc} \lambda x.\, t_1[y/t_2] & \Rrightarrow^1 & (\lambda x.\, t_1)[y/t_2] \\ \downarrow{\scriptstyle \mathrm{lsci}} & & \downarrow{\scriptstyle \mathrm{lsci}} \\ \lambda x.\, t_1'[y/t_2] & \Rrightarrow^1 & (\lambda x.\, t_1')[y/t_2] \end{array} \quad \text{where } t_1 \to_{\mathrm{lsci}} t_1'$$

      2.2.2 If the reduction step is internal to t_2, it is similar to the previous case.
      2.2.3 If the reduction step is derived from $\to_{\mathsf{ls}}$ at the root: then $t_1 = \mathsf{C}\langle\!\langle y\rangle\!\rangle$ and the situation is:

$$\begin{array}{ccc} \lambda x.\, \mathsf{C}\langle\!\langle y\rangle\!\rangle[y/t_2] & \Rrightarrow^1 & (\lambda x.\, \mathsf{C}\langle\!\langle y\rangle\!\rangle)[y/t_2] \\ \downarrow{\scriptstyle \mathrm{lsci}} & & \downarrow{\scriptstyle \mathrm{lsci}} \\ \lambda x.\, \mathsf{C}\langle\!\langle t_2\rangle\!\rangle[y/t_2] & \Rrightarrow^1 & (\lambda x.\, \mathsf{C}\langle\!\langle t_2\rangle\!\rangle)[y/t_2] \end{array}$$

3. Application, $t = t_1\, t_2$: we consider four subcases, depending on whether the fusion step $t = t_1\, t_2 \Rrightarrow^1 s$ is internal to t_1, internal to t_2, derived from $\Rrightarrow\mathtt{appL}$ at the root, or derived from $\Rrightarrow\mathtt{appR}$ at the root:
   3.1 If the fusion step is internal to t_1, then $t = t_1\, t_2 \Rrightarrow^1 s_1\, t_2 = s$ with $t_1 \Rrightarrow^1 s_1$. There are three subcases, depending on whether the reduction step is internal to s_1, internal to t_2, or a $\to_{\mathsf{db}}$ step at the root:
      3.1.1 If the reduction step is internal to s_1: then $s = s_1\, t_2 \to_{\mathrm{lsci}} r_1\, t_2 = r$ with $s_1 \to_{\mathrm{lsci}} r_1$. Then the diagram can be closed by resorting to the IH:

$$\begin{array}{ccc} t_1 & \Rrightarrow^1 & s_1 \\ {\scriptstyle \mathrm{lsci}^+}\downarrow & & \downarrow{\scriptstyle \mathrm{lsci}} \\ p_1 & \Rrightarrow & r_1 \end{array} \quad \text{by IH, so} \quad \begin{array}{ccc} t_1\, t_2 & \Rrightarrow^1 & s_1\, t_2 \\ {\scriptstyle \mathrm{lsci}^+}\downarrow & & \downarrow{\scriptstyle \mathrm{lsci}} \\ p_1\, t_2 & \Rrightarrow & r_1\, t_2 \end{array}$$

3.1.2 If the reduction step is internal to t_2: then $s = s_1\, t_2 \to_{\mathrm{lsci}} s_1\, r_2 = r$ with $s_1 \to_{\mathrm{lsci}} r_1$. Then the steps are disjoint and the diagram can be immediately closed as follows:

$$\begin{array}{ccc} t_1\, t_2 & \Rrightarrow^1 & s_1\, t_2 \\ \downarrow_{\mathrm{lsci}} & & \downarrow_{\mathrm{lsci}} \\ t_1\, r_2 & \Rrightarrow^1 & s_1\, r_2 \end{array}$$

3.1.3 If the reduction step is a $\to_{\mathsf{db}}$ step at the root: then $s_1 = (\lambda x.\, s_1')\mathrm{L}$. Since $t_1 \Rrightarrow^1 s_1$, by Lem. D.40 we know that t_1 must be of the form $t_1 = (\lambda x.\, t_0)\mathrm{L}_0$ in such a way that $t_0[x/t_2]\mathrm{L}_0 \Rrightarrow s_1'[x/t_2]\mathrm{L}$. Hence the situation is:

$$\begin{array}{ccc} (\lambda x.\, t_0)\mathrm{L}_0\, t_2 & \Rrightarrow^1 & (\lambda x.\, s_1')\mathrm{L}\, t_2 \\ \downarrow_{\mathrm{lsci}}^{+} & & \downarrow_{\mathrm{lsci}} \\ t_0[x/t_2]\mathrm{L}_0 & \Rrightarrow & s_1'[x/t_2]\mathrm{L} \end{array}$$

3.2 If the fusion step is internal to t_2, then $t = t_1\, t_2 \Rrightarrow^1 t_1\, s_2$ with $t_2 \Rrightarrow^1 s_2$. There are three subcases, depending on whether the reduction step is internal to t_1, internal to s_2, or a $\to_{\mathsf{db}}$ step at the root:

3.2.1 If the reduction step is internal to t_1: the steps are disjoint and the diagram can be closed similarly as for case 3.1.2.

3.2.2 If the reduction step is internal to s_1: then the diagram can be closed by resorting to the IH, similarly as for case 3.1.1.

3.2.3 If the reduction step is a $\to_{\mathsf{db}}$ step at the root: then $t_1 = (\lambda x.\, t_1')\mathrm{L}$. Hence the situation is:

$$\begin{array}{ccc} (\lambda x.\, t_1')\mathrm{L}\, t_2 & \Rrightarrow^1 & (\lambda x.\, t_1')\mathrm{L}\, s_2 \\ \downarrow_{\mathrm{lsci}} & & \downarrow_{\mathrm{lsci}} \\ t_1'[x/t_2]\mathrm{L} & \Rrightarrow^1 & t_1'[x/s_2]\mathrm{L} \end{array}$$

3.3 If the fusion step is derived from $\Rrightarrow\mathsf{appL}$ at the root, then $t_1 = t_{11}[x/t_{12}]$ where $x \notin \mathsf{fv}(t_2)$ and the step is of the form $t = t_{11}[x/t_{12}]\, t_2 \Rrightarrow^1 (t_{11}\, t_2)[x/t_{12}] = s$. We consider five subcases, depending on whether the reduction step $(t_{11}\, t_2)[x/t_{12}] \to_{\mathrm{lsci}} r$ is internal to t_{11}, internal to t_2, internal to t_{12}, a $\to_{\mathsf{ls}}$ step at the root, or a $\to_{\mathsf{db}}$ step involving the application $t_{11}\, t_2$:

3.3.1 If the reduction step is internal to t_{11}: then $t_{11} \to_{\mathrm{lsci}} r_{11}$ and the situation is:

$$\begin{array}{ccc} t_{11}[x/t_{12}]\, t_2 & \Rrightarrow^1 & (t_{11}\, t_2)[x/t_{12}] \\ \downarrow_{\mathrm{lsci}} & & \downarrow_{\mathrm{lsci}} \\ r_{11}[x/t_{12}]\, t_2 & \Rrightarrow^1 & (r_{11}\, t_2)[x/t_{12}] \end{array}$$

3.3.2 If the reduction step is internal to t_2: then $t_2 \to_{\mathrm{lsci}} r_2$ and the situation is:

$$\begin{array}{ccc} t_{11}[x/t_{12}]\, t_2 & \Rrightarrow^1 & (t_{11}\, t_2)[x/t_{12}] \\ \downarrow_{\mathrm{lsci}} & & \downarrow_{\mathrm{lsci}} \\ t_{11}[x/t_{12}]\, r_2 & \Rrightarrow^1 & (t_{11}\, r_2)[x/t_{12}] \end{array}$$

for the fusion step at the bottom of the diagram, observe that $x \notin \mathsf{fv}(r_2)$ because reduction does not create free variables.

3.3.3 If the reduction step is internal to t_{12}: similar to case 3.3.1.

3.3.4 If the reduction step is a $\to_{\mathsf{ls}}$ step at the root: note that $x \notin \mathsf{fv}(t_2)$ because the step $t_{11}[x/t_{12}]\,t_2 \Rrightarrow^1 (t_{11}\,t_2)[x/t_{12}]$ is an instance of the $\Rrightarrow\mathtt{appL}$ rule. Hence in the $\to_{\mathsf{ls}}$ step $(t_{11}\,t_2)[x/t_{12}] \to_{\mathsf{ls}} r$ the substituted variable must necessarily lie inside t_{11}. This means that t_{11} is of the form $t_{11} = \mathsf{C}\langle\!\langle x\rangle\!\rangle$ and the situation is:

$$\begin{tikzcd}
\mathsf{C}\langle\!\langle x\rangle\!\rangle[x/t_{12}]\,t_2 \arrow[r, phantom, "\Rrightarrow^1"] \arrow[d, "\mathsf{lsci}"] & (\mathsf{C}\langle\!\langle x\rangle\!\rangle\,t_2)[x/t_{12}] \arrow[d, "\mathsf{lsci}"] \\
\mathsf{C}\langle\!\langle t_{12}\rangle\!\rangle[x/t_{12}]\,t_2 \arrow[r, phantom, "\Rrightarrow^1"] & (\mathsf{C}\langle\!\langle t_{12}\rangle\!\rangle\,t_2)[x/t_{12}]
\end{tikzcd}$$

3.3.5 If the reduction step is a $\to_{\mathsf{db}}$ step on the left: then $t_{11} = (\lambda y.\,t'_{11})\mathsf{L}$ and the situation is:

$$\begin{tikzcd}
(\lambda y.\,t'_{11})\mathsf{L}[x/t_{12}]\,t_2 \arrow[r, phantom, "\Rrightarrow^1"] \arrow[d, "\mathsf{lsci}"] & ((\lambda y.\,t'_{11})\mathsf{L}\,t_2)[x/t_{12}] \arrow[d, "\mathsf{lsci}"] \\
t'_{11}[y/t_2]\mathsf{L}[x/t_{12}] \arrow[r, phantom, "="] & t'_{11}[y/t_2]\mathsf{L}[x/t_{12}]
\end{tikzcd}$$

3.4 If the fusion step is derived from $\Rrightarrow\mathtt{appR}$ at the root, then $t_2 = t_{21}[x/t_{22}]$ where $x \notin \mathsf{fv}(t_1)$ and the step is of the form $t = t_1\,t_{21}[x/t_{22}] \Rrightarrow^1 (t_1\,t_{21})[x/t_{22}] = s$. We consider five subcases, depending on whether the reduction step $(t_1\,t_{21})[x/t_{22}] \to_{\mathsf{lsci}} r$ is internal to t_1, internal to t_{21}, internal to t_{22}, a $\to_{\mathsf{ls}}$ step at the root, or a $\to_{\mathsf{db}}$ step involving the application $t_1\,t_{21}$:

3.4.1 If the reduction step is internal to t_1: similar to case 3.3.2.

3.4.2 If the reduction step is internal to t_{21}: similar to case 3.3.1.

3.4.3 If the reduction step is internal to t_{22}: similar to case 3.3.1.

3.4.4 If the reduction step is a $\to_{\mathsf{ls}}$ step at the root: note that $x \notin \mathsf{fv}(t_1)$ because the step $t_1\,t_{21}[x/t_{22}] \Rrightarrow^1 (t_1\,t_{21})[x/t_{22}]$ is an instance of the $\Rrightarrow\mathtt{appR}$ rule. Hence in the $\to_{\mathsf{ls}}$ step $(t_1\,t_{21})[x/t_{22}] \to_{\mathsf{ls}} r$ the substituted variable must necessarily lie inside t_{21}. This means that t_{21} is of the form $t_{21} = \mathsf{C}\langle\!\langle x\rangle\!\rangle$ and the situation is:

$$\begin{tikzcd}
t_1\,\mathsf{C}\langle\!\langle x\rangle\!\rangle[x/t_{22}] \arrow[r, phantom, "\Rrightarrow^1"] \arrow[d, "\mathsf{lsci}"] & (t_1\,\mathsf{C}\langle\!\langle x\rangle\!\rangle)[x/t_{22}] \arrow[d, "\mathsf{lsci}"] \\
t_1\,\mathsf{C}\langle\!\langle t_{22}\rangle\!\rangle[x/t_{22}] \arrow[r, phantom, "\Rrightarrow^1"] & (t_1\,\mathsf{C}\langle\!\langle t_{22}\rangle\!\rangle)[x/t_{22}]
\end{tikzcd}$$

3.4.5 If the reduction step is a $\to_{\mathsf{db}}$ step on the left: then $t_1 = (\lambda y.\,t'_1)\mathsf{L}$ and the situation is:

$$\begin{tikzcd}
(\lambda y.\,t'_1)\mathsf{L}\,t_{21}[x/t_{22}] \arrow[r, phantom, "\Rrightarrow^1"] \arrow[d, "\mathsf{lsci}"] & ((\lambda y.\,t'_1)\mathsf{L}\,t_{21})[x/t_{22}] \arrow[d, "\mathsf{lsci}"] \\
t'_1[y/t_{21}]\mathsf{L}[x/t_{22}] \arrow[r, phantom, "="] & t'_1[y/t_{21}]\mathsf{L}[x/t_{22}]
\end{tikzcd}$$

4. Substitution, $t = t_1[x/t_2]$: we consider six subcases, depending on whether the fusion step $t = t_1[x/t_2] \Rrightarrow^1 s$ is internal to t_1, internal to t_2, or derived from one of the rules $\Rrightarrow\mathtt{w}$, $\Rrightarrow\mathtt{c}$, $\Rrightarrow\mathtt{esL}$, or $\Rrightarrow\mathtt{esR}$ at the root:
   4.1 If the fusion step is internal to t_1, then $t = t_1[x/t_2] \Rrightarrow^1 s_1[x/t_2] = s$ with $t_1 \Rrightarrow^1 s_1$. There are three subcases, depending on whether the reduction step is internal to s_1, internal to t_2, or a $\to_{\mathsf{ls}}$ step at the root:
      4.1.1 If the reduction step is internal to s_1: then the diagram can be closed by resorting to the IH, similarly as for case 3.1.1.
      4.1.2 If the reduction step is internal to t_2: the steps are disjoint and the diagram can be closed similarly as for case 3.1.2.
      4.1.3 If the reduction step is a $\to_{\mathsf{ls}}$ step at the root: then $s_1 = \mathsf{C}\langle\langle x\rangle\rangle$ and $t_1 \Rrightarrow^1 \mathsf{C}\langle\langle x\rangle\rangle$. By Lem. D.41, we have that t_1 is of the form $\mathsf{C}_0\langle\langle x,\dots,x\rangle\rangle$, where C_0 is a context with either one or two holes such that $\mathsf{C}_0\langle\langle t_2,\dots,t_2\rangle\rangle \Rrightarrow \mathsf{C}\langle\langle t_2\rangle\rangle$. Then the situation is:

$$\begin{array}{ccc} \mathsf{C}_0\langle\langle x,\dots,x\rangle\rangle[x/t_2] & \Rrightarrow^1 & \mathsf{C}\langle\langle x\rangle\rangle[x/t_2] \\ \downarrow_{\mathsf{lsci}}{}^{+} & & \downarrow_{\mathsf{lsci}} \\ \mathsf{C}_0\langle\langle t_2,\dots,t_2\rangle\rangle[x/t_2] & \Rrightarrow & \mathsf{C}\langle\langle t\rangle\rangle[x/t_2] \end{array}$$

   4.2 If the fusion step is internal to t_2, then $t = t_1[x/t_2] \Rrightarrow^1 t_1[x/s_2]$ with $t_2 \Rrightarrow^1 s_2$. There are three subcases, depending on whether the reduction step is internal to t_1, internal to s_2, or a $\to_{\mathsf{ls}}$ step at the root:
      4.2.1 If the reduction step is internal to t_1: the steps are disjoint and the diagram can be closed similarly as for case 3.1.2.
      4.2.2 If the reduction step is internal to s_2: then the diagram can be closed by resorting to the IH, similarly as for case 3.1.1.
      4.2.3 If the reduction step is a $\to_{\mathsf{ls}}$ step at the root, then $t_1 = \mathsf{C}\langle\langle x\rangle\rangle$ and the situation is:

$$\begin{array}{ccc} \mathsf{C}\langle\langle x\rangle\rangle[x/t_2] & \Rrightarrow^1 & \mathsf{C}\langle\langle x\rangle\rangle[x/s_2] \\ \downarrow_{\mathsf{lsci}} & & \downarrow_{\mathsf{lsci}} \\ \mathsf{C}\langle\langle s_2\rangle\rangle[x/s_2] & \Rrightarrow & \mathsf{C}\langle\langle t_2\rangle\rangle[x/t_2] \end{array}$$

   4.3 If the fusion step is derived from $\Rrightarrow\mathtt{w}$ at the root, then $x \notin \mathsf{fv}(t_1)$ and $t = t_1[x/t_2] \Rrightarrow t_1 = s$. The situation is:

$$\begin{array}{ccc} s[x/t_2] & \Rrightarrow^1 & s \\ \downarrow_{\mathsf{lsci}} & & \downarrow_{\mathsf{lsci}} \\ r[x/t_2] & \Rrightarrow^1 & r \end{array}$$

   The fusion step at the bottom is an instance of the $\Rrightarrow\mathtt{w}$ rule, which can be applied because $x \notin \mathsf{fv}(r)$ holds, given that $x \notin \mathsf{fv}(s)$ and using the fact that the reduction step $s \to_{\mathsf{lsci}} r$ does not create free variables.
   4.4 If the fusion step is derived from $\Rrightarrow\mathtt{c}$ at the root, then $t_1 = t_1'[y/t_2]$ and $t = t_1'[y/t_2][x/t_2] \Rrightarrow^1 t_1'\{y := x\}[x/t_2] = s$. There are three subcases, depending on whether the step is reduction step is internal to $t_1'\{y := x\}$, internal to t_2, or a $\to_{\mathsf{ls}}$ step at the root:

4.4.1 If the reduction step is internal to $t'_1\{y := x\}$: then $t'_1\{y := x\} \to_{\mathrm{lsci}} r$. By Lem. D.39 there exists r_0 such that $t'_1 \to_{\mathrm{lsci}} r_0$ and $r = r_0\{y := x\}$. Hence the situation is:

$$\begin{array}{ccc} t'_1[y/t_2][x/t_2] & \Rrightarrow^1 & t'_1\{y := x\}[x/t_2] \\ \downarrow_{\mathrm{lsci}} & & \downarrow_{\mathrm{lsci}} \\ r_0[y/t_2][x/t_2] & \Rrightarrow^1 & r_0\{y := x\}[x/t_2] \end{array}$$

4.4.2 If the reduction step is internal to t_2: then $t_2 \to_{\mathrm{lsci}} r_2$ and the situation is:

$$\begin{array}{ccc} t'_1[y/t_2][x/t_2] & \Rrightarrow^1 & t'_1\{y := x\}[x/t_2] \\ \downarrow_{\mathrm{lsci}}^{+} & & \downarrow_{\mathrm{lsci}} \\ t'_1[y/r_2][x/r_2] & \Rrightarrow^1 & t'_1\{y := x\}[x/r_2] \end{array}$$

4.4.3 If the reduction step is a $\to_{\mathsf{ls}}$ step at the root: then $t'_1\{y := x\}[x/t_2] \to_{\mathsf{ls}} r$. We consider two further subcases, depending on whether the substituted occurrence of x in $t'_1\{y := x\}$ corresponds to an occurrence of x in t'_1 or to an occurrence of y in t'_1:

4.4.3.1 Substitution of an occurrence of x in t'_1: then $t'_1 = \mathsf{C}\langle\!\langle x\rangle\!\rangle$ and the situation is:

$$\begin{array}{ccc} \mathsf{C}\langle\!\langle x\rangle\!\rangle[y/t_2][x/t_2] & \Rrightarrow^1 & \mathsf{C}\{y := x\}\langle\!\langle x\rangle\!\rangle[x/t_2] \\ \downarrow_{\mathrm{lsci}} & & \downarrow_{\mathrm{lsci}} \\ \mathsf{C}\langle\!\langle t_2\rangle\!\rangle[y/t_2][x/t_2] & \Rrightarrow^1 & \mathsf{C}\{y := x\}\langle\!\langle t_2\rangle\!\rangle[x/t_2] \end{array}$$

4.4.3.2 Substitution of an occurrence of y in t'_1: then $t'_1 = \mathsf{C}\langle\!\langle y\rangle\!\rangle$ and the situation is:

$$\begin{array}{ccc} \mathsf{C}\langle\!\langle y\rangle\!\rangle[y/t_2][x/t_2] & \Rrightarrow^1 & \mathsf{C}\{y := x\}\langle\!\langle x\rangle\!\rangle[x/t_2] \\ \downarrow_{\mathrm{lsci}} & & \downarrow_{\mathrm{lsci}} \\ \mathsf{C}\langle\!\langle t_2\rangle\!\rangle[y/t_2][x/t_2] & \Rrightarrow^1 & \mathsf{C}\{y := x\}\langle\!\langle t_2\rangle\!\rangle[x/t_2] \end{array}$$

4.5 If the fusion step is derived from $\Rrightarrow\mathtt{esL}$ at the root, then $t_1 = t_{11}[y/t_{12}]$ where $y \notin \mathsf{fv}(t_2)$ and $x \notin \mathsf{fv}(t_{12})$, and the step is of the form $t = t_{11}[y/t_{12}][x/t_2] \Rrightarrow^1 t_{11}[x/t_2][y/t_{12}] = s$. There are five subcases, depending on whether the reduction step is internal to t_{11}, internal to t_2, internal to t_{12}, a $\to_{\mathsf{ls}}$ step contracting y, or a $\to_{\mathsf{ls}}$ step contracting x:

4.5.1 If the reduction step is internal to t_{11}: then $t_{11} \to_{\mathrm{lsci}} s_{11}$ and the situation is:

$$\begin{array}{ccc} t_{11}[y/t_{12}][x/t_2] & \Rrightarrow^1 & t_{11}[x/t_2][y/t_{12}] \\ \downarrow_{\mathrm{lsci}} & & \downarrow_{\mathrm{lsci}} \\ s_{11}[y/t_{12}][x/t_2] & \Rrightarrow^1 & s_{11}[x/t_2][y/t_{12}] \end{array}$$

4.5.2 If the reduction step is internal to t_2: similar to the previous case.

4.5.3 If the reduction step is internal to t_{12}: similar to the previous case.

4.5.4 If the reduction step is a $\to_{\mathsf{ls}}$ step contracting y: since $y \notin \mathsf{fv}(t_2)$ we have that $t_{11} = \mathsf{C}\langle\!\langle y\rangle\!\rangle$ and the situation is:

$$\begin{array}{ccc}
\mathsf{C}\langle\!\langle y\rangle\!\rangle[y/t_{12}][x/t_2] & \Rrightarrow^1 & \mathsf{C}\langle\!\langle y\rangle\!\rangle[x/t_2][y/t_{12}] \\
\downarrow_{\mathsf{lsci}} & & \downarrow_{\mathsf{lsci}} \\
\mathsf{C}\langle\!\langle t_{12}\rangle\!\rangle[y/t_{12}][x/t_2] & \Rrightarrow^1 & \mathsf{C}\langle\!\langle t_{12}\rangle\!\rangle[x/t_2][y/t_{12}]
\end{array}$$

4.5.5 If the reduction step is a $\to_{\mathsf{ls}}$ step contracting x: then $t_{11} = \mathsf{C}\langle\!\langle x\rangle\!\rangle$ and the situation is:

$$\begin{array}{ccc}
\mathsf{C}\langle\!\langle x\rangle\!\rangle[y/t_{12}][x/t_2] & \Rrightarrow^1 & \mathsf{C}\langle\!\langle x\rangle\!\rangle[x/t_2][y/t_{12}] \\
\downarrow_{\mathsf{lsci}} & & \downarrow_{\mathsf{lsci}} \\
\mathsf{C}\langle\!\langle t_2\rangle\!\rangle[y/t_{12}][x/t_2] & \Rrightarrow^1 & \mathsf{C}\langle\!\langle t_2\rangle\!\rangle[x/t_2][y/t_{12}]
\end{array}$$

4.6 If the fusion step is derived from $\Rrightarrow\mathtt{esR}$ at the root, then $t_2 = t_{21}[y/t_{22}]$ where $y \notin \mathsf{fv}(t_1)$ and the step is of the form $t = t_1[x/t_{21}[y/t_{22}]] = s$. There are five subcases, depending on whether the reduction step is internal to t_1, internal to t_{21}, internal to t_{22}, a $\to_{\mathsf{ls}}$ step contracting y, or a $\to_{\mathsf{ls}}$ step contracting x:

4.6.1 If the reduction step is internal to t_1: then $t_1 \to_{\mathsf{lsci}} s_1$ and the situation is:

$$\begin{array}{ccc}
t_1[x/t_{21}[y/t_{22}]] & \Rrightarrow^1 & t_1[x/t_{21}][y/t_{22}] \\
\downarrow_{\mathsf{lsci}} & & \downarrow_{\mathsf{lsci}} \\
s_1[x/t_{21}[y/t_{22}]] & \Rrightarrow^1 & s_1[x/t_{21}][y/t_{22}]
\end{array}$$

The step on the bottom is an instance of the $\Rrightarrow\mathtt{esR}$ rule, which can be applied because $y \notin \mathsf{fv}(s_1)$ holds, given that $y \notin \mathsf{fv}(t_1)$ and using the fact that the reduction step $t_1 \to_{\mathsf{lsci}} s_1$ does not create free variables.

4.6.2 If the reduction step is internal to t_{21}: similar to the previous case.

4.6.3 If the reduction step is internal to t_{22}: similar to the previous case.

4.6.4 If the reduction step is a $\to_{\mathsf{ls}}$ step contracting y: since $y \notin \mathsf{fv}(t_1)$, we have that $t_{21} = \mathsf{C}\langle\!\langle y\rangle\!\rangle$, and the situation is:

$$\begin{array}{ccc}
t_1[x/\mathsf{C}\langle\!\langle y\rangle\!\rangle[y/t_{22}]] & \Rrightarrow^1 & t_1[x/\mathsf{C}\langle\!\langle y\rangle\!\rangle][y/t_{22}] \\
\downarrow_{\mathsf{lsci}} & & \downarrow_{\mathsf{lsci}} \\
t_1[x/\mathsf{C}\langle\!\langle t_{22}\rangle\!\rangle[y/t_{22}]] & \Rrightarrow^1 & t_1[x/\mathsf{C}\langle\!\langle t_{22}\rangle\!\rangle][y/t_{22}]
\end{array}$$

4.6.5 If the reduction step is a $\to_{\mathsf{ls}}$ step contracting x: then $t_1 = \mathsf{C}\langle\!\langle x\rangle\!\rangle$ and the situation is:

$$\begin{array}{ccc}
\mathsf{C}\langle\!\langle x\rangle\!\rangle[x/t_{21}[y/t_{22}]] & \Rrightarrow^1 & \mathsf{C}\langle\!\langle x\rangle\!\rangle[x/t_{21}][y/t_{22}] \\
\downarrow_{\mathsf{lsci}} & & \downarrow_{\mathsf{lsci}} \\
\mathsf{C}\langle\!\langle t_{21}[y/t_{22}]\rangle\!\rangle[x/t_{21}[y/t_{22}]] & \Rrightarrow & \mathsf{C}\langle\!\langle t_{21}\rangle\!\rangle[x/t_{21}][y/t_{22}]
\end{array}$$

The step on the bottom is justified by Lem. D.38.

Lemma D.43 (Postponement of fusion). *If t is $\to_{\mathrm{lsci}}$-SN and $t \Rrightarrow \to_{\mathrm{lsci}} s$ then $t \to^{+}_{\mathrm{lsci}} \Rrightarrow s$.*

Proof. We prove the statement by first proving two auxiliary claims:

1. First claim: $\Rrightarrow\to_{\mathrm{lsci}} \subseteq \to_{\mathrm{lsci}} (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*}$.
   Since $\Rrightarrow$ is the reflexive–transitive closure of $\Rrightarrow^{1}$, by definition, this is equivalent to showing that $(\Rrightarrow^{1})^{n} \to_{\mathrm{lsci}} \subseteq \to_{\mathrm{lsci}} (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*}$ holds for all $n \in \mathbb{N}_0$. We proceed by induction on n. If $n = 0$, it is immediate to conclude. Now assume the property holds for a given $n \in \mathbb{N}_0$. Then:

$$\begin{array}{rll} (\Rrightarrow^{1})^{n+1} \to_{\mathrm{lsci}} & = (\Rrightarrow^{1})^{n} \Rrightarrow^{1}\to_{\mathrm{lsci}} & \\ & \subseteq (\Rrightarrow^{1})^{n} \to^{+}_{\mathrm{lsci}}\Rrightarrow & \text{by Lem. D.42} \\ & = (\Rrightarrow^{1})^{n} \to_{\mathrm{lsci}}\to^{*}_{\mathrm{lsci}}\Rrightarrow & \text{since } \to^{+}_{\mathrm{lsci}} = \to_{\mathrm{lsci}}\to^{*}_{\mathrm{lsci}} \\ & \subseteq \to_{\mathrm{lsci}} (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} \to^{*}_{\mathrm{lsci}}\Rrightarrow & \text{by IH} \\ & = \to_{\mathrm{lsci}} (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} \Rrightarrow & \text{since } \to^{*}_{\mathrm{lsci}} \subseteq (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} \text{ and transitivity} \\ & = \to_{\mathrm{lsci}} (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} & \text{since } \Rrightarrow = (\Rrightarrow^{1})^{*} \subseteq (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} \text{ and transitivity} \end{array}$$

2. Second claim: if t is $\to_{\mathrm{lsci}}$-SN and $t\ (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*}\ s$ then $t \to^{*}_{\mathrm{lsci}}\Rrightarrow s$. The relation $\to_{\mathrm{lsci}}$ restricted to SN terms is obviously SN, so we may proceed by well-founded induction on t with respect to $\to_{\mathrm{lsci}}$.
   We know that $t\ (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*}\ s$ holds. We consider two cases, depending on whether $t \Rrightarrow s$ or not:
   2.1 If $t \Rrightarrow s$, it is immediate to conclude that $t \to^{*}_{\mathrm{lsci}}\Rrightarrow s$.
   2.2 Otherwise, in the reduction $t\ (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*}\ s$ there must be at least one $\to_{\mathrm{lsci}}$ step. Considering the first such step, this means that $t \Rrightarrow\to_{\mathrm{lsci}} (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} s$. By the previous claim we have that $t \to_{\mathrm{lsci}} (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*}(\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} s$ so, by transitivity, $t \to_{\mathrm{lsci}} (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} s$. Let t' be a term such that $t \to_{\mathrm{lsci}} t'(\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} s$, and note that t' is $\to_{\mathrm{lsci}}$-SN, because it is a reduct of t which is itself $\to_{\mathrm{lsci}}$-SN. By IH we have that $t' \to^{*}_{\mathrm{lsci}}\Rrightarrow s$, so $t \to_{\mathrm{lsci}} t' \to^{*}_{\mathrm{lsci}}\Rrightarrow s$. This means that $t \to^{*}_{\mathrm{lsci}}\Rrightarrow s$, as required.
3. Finally, we prove the main statement: if t is $\to_{\mathrm{lsci}}$-SN and $t \Rrightarrow \to_{\mathrm{lsci}} s$ then $t \to^{+}_{\mathrm{lsci}} \Rrightarrow s$.
   Suppose that $t \Rrightarrow \to_{\mathrm{lsci}} s$. By the first claim, $t \to_{\mathrm{lsci}} (\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*}\ s$. Let t' be a term such that $t \to_{\mathrm{lsci}} t'(\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} s$, and note that t' is $\to_{\mathrm{lsci}}$-SN, because it is a reduct of t which is itself $\to_{\mathrm{lsci}}$-SN. Since $t'(\to_{\mathrm{lsci}} \cup \Rrightarrow^{1})^{*} s$, by the second claim we have that $t' \to^{*}_{\mathrm{lsci}}\Rrightarrow s$. Hence $t \to_{\mathrm{lsci}} t' \to^{*}_{\mathrm{lsci}}\Rrightarrow s$, which means in particular that $t \to^{+}_{\mathrm{lsci}}\Rrightarrow s$, as required.

Lemma D.44 (Fusion preserves SN). *If t is $\to_{\mathrm{lsci}}$-SN then t is $(\to_{\mathrm{lsci}} \Rrightarrow)$-SN.*

Proof. We first prove an auxiliary claim: if t is a $\to_{\mathrm{lsci}}$-SN term which is not $(\to_{\mathrm{lsci}} \Rrightarrow)$-SN, then it has a reduct $t \to_{\mathrm{lsci}} s$ such that s is not $(\to_{\mathrm{lsci}} \Rrightarrow)$-SN. Indeed, suppose that t is not $(\to_{\mathrm{lsci}} \Rrightarrow)$-SN. Then there is an infinite $(\to_{\mathrm{lsci}} \Rrightarrow)$-reduction sequence starting from t. Consider in particular a prefix of the sequence of the form $t \to_{\mathrm{lsci}} \Rrightarrow\to_{\mathrm{lsci}} \Rrightarrow r$. Let t' be a term such that $t \to_{\mathrm{lsci}} t' \Rrightarrow\to_{\mathrm{lsci}} \Rrightarrow r$. By hypothesis, t is $\to_{\mathrm{lsci}}$-SN so its reduct t' must also be $\to_{\mathrm{lsci}}$-SN, Which means that we may apply Lem. D.43 to postpone the fusion step and obtain that $t \to_{\mathrm{lsci}} \to^{+}_{\mathrm{lsci}}\Rrightarrow\Rrightarrow r$. Since fusion is transitive, we have that $t \to_{\text{lsci}} \to^{+}_{\text{lsci}} \Rrightarrow r$. Since $\Rrightarrow$ is reflexive, $\to_{\text{lsci}} \subseteq \to_{\text{lsci}}\Rrightarrow$ and, moreover, $\to^{+}_{\text{lsci}} \subseteq (\to_{\text{lsci}}\Rrightarrow)^{+}$, which means that $t \to_{\text{lsci}} (\to_{\text{lsci}}\Rrightarrow)^{+} \Rrightarrow r$. The last fusion step can be absorbed to the immediately preceding one by transitivity, so we have that $t \to_{\text{lsci}} (\to_{\text{lsci}}\Rrightarrow)^{+} r$. Let s be a term such that $t \to_{\text{lsci}} s (\to_{\text{lsci}}\Rrightarrow)^{+} r$. Then $t \to_{\text{lsci}} s$ and s is not $(\to_{\text{lsci}}\Rrightarrow)$-SN because $s(\to_{\text{lsci}}\Rrightarrow)^{+}r$ where r in turn is not $(\to_{\text{lsci}}\Rrightarrow)$-SN. This concludes the proof of the claim.

To prove the statement of the lemma, let t be $\to_{\text{lsci}}$-SN and suppose by contradiction that it is not $(\to_{\text{lsci}}\Rrightarrow)$-SN. By the claim, it has a reduct $t \to_{\text{lsci}} t_1$ which is not $(\to_{\text{lsci}}\Rrightarrow)$-SN. Iterating this argument, we construct an infinite reduction sequence $t \to_{\text{lsci}} t_1 \to_{\text{lsci}} t_2 \ldots$, meaning that t is not $\to_{\text{lsci}}$-SN, contradicting the hypothesis.

D.3 Translation of $\lambda^{!\bullet}$ to LSC

Definition D.17 (From $\lambda^{!\bullet}$ to LSC). *We assume that* $\mathbf{1}$ *stands for some inhabited type in the LSC, and* $*$ *for a closed inhabitant of* $\mathbf{1}$ *in normal form. Types, terms, and typing contexts, are translated as follows:*

$$\begin{array}{rcl} [\![\alpha]\!] & := & \alpha \\ [\![A \multimap B]\!] & := & [\![A]\!] \to [\![B]\!] \\ [\![\bullet A]\!] & := & \mathbf{1} \to [\![A]\!] \\ [\![!A]\!] & := & [\![A]\!] \end{array} \qquad \begin{array}{l} \\ [\![u_1 : A_1, \ldots, u_n : A_n]\!] := u_1 : \mathbf{1} \to [\![A_1]\!], \ldots, u_n : \mathbf{1} \to [\![A_n]\!] \\ [\![a_1 : A_1, \ldots, a_n : A_n]\!] := a_1 : [\![A_1]\!], \ldots, a_n : [\![A_n]\!] \\ \end{array}$$

$$\begin{array}{rclrcl} [\![a]\!] & := & a & [\![\bullet t]\!] & := & \lambda z. [\![t]\!] \quad \textit{where } z \notin \mathsf{fv}(t) \\ [\![u]\!] & := & u & [\![\mathtt{req}(t)]\!] & := & [\![t]\!]\, * \\ [\![\lambda a. t]\!] & := & \lambda a. [\![t]\!] & [\![!t]\!] & := & [\![t]\!] \\ [\![t\, s]\!] & := & [\![t]\!]\, [\![s]\!] & [\![t[x/s]]\!] & := & [\![t]\!][x/[\![s]\!]] \end{array}$$

The translation is extended to contexts by declaring that $[\![\Box]\!] = \Box$. *In particular, the translation of a substitution context* $\mathtt{L} = \Box[x_1/t_1] \ldots [x_n/t_n]$ *is a substitution context* $[\![\mathtt{L}]\!] = \Box[x_1/[\![t_1]\!]] \ldots [x_n/[\![t_n]\!]]$.

Remark D.6. $\mathsf{fv}([\![t]\!]) = \mathsf{fv}(t)$

Lemma D.45 ($[\![-]\!]$ commutes with substitution). *The following hold:*

1. $[\![t\mathtt{L}]\!] = [\![t]\!][\![\mathtt{L}]\!]$
2. $[\![\mathtt{C}\langle t\rangle]\!] = [\![\mathtt{C}]\!]\langle[\![t]\!]\rangle$
3. $[\![t\{x := s\}]\!] = [\![t]\!]\{x := [\![s]\!]\}$

Proof. The first item is by induction on $\mathtt{L}$. If $\mathtt{L} = \Box$, then $[\![t\mathtt{L}]\!] = [\![t]\!] = [\![t]\!][\![\mathtt{L}]\!]$ since $[\![\Box]\!] = \Box$. If $\mathtt{L} = \mathtt{L}'[x/s]$, we reason as follows.

$$\begin{array}{ll} & [\![t\mathtt{L}]\!] \\ = & [\![t\mathtt{L}'[x/s]]\!] \\ = & [\![t\mathtt{L}']\!][x/[\![s]\!]] \quad \text{by definition} \\ = & [\![t]\!][\![\mathtt{L}']\!][x/[\![s]\!]] \quad (IH) \\ = & [\![t]\!][\![\mathtt{L}'[x/s]]\!] \quad \text{by definition} \\ = & [\![t]\!][\![\mathtt{L}]\!] \end{array}$$

The second item is by induction on $\mathtt{C}$.

1. $\mathsf{C} = \Box$. Then $[\![\mathsf{C}\langle t\rangle]\!] = [\![t]\!] = [\![\mathsf{C}]\!]\langle[\![t]\!]\rangle$, since $[\![\Box]\!] = \Box$.
2. $\mathsf{C} = \lambda a.\,\mathsf{C}_1$. Then we reason as follows:

$$\begin{array}{lll} & [\![\mathsf{C}\langle t\rangle]\!] & \\ = & [\![\lambda a.\,\mathsf{C}_1\langle t\rangle]\!] & \\ = & \lambda a.\,[\![\mathsf{C}_1\langle t\rangle]\!] & \text{by definition} \\ = & \lambda a.\,[\![\mathsf{C}_1]\!]\langle[\![t]\!]\rangle & (IH) \\ = & [\![\mathsf{C}]\!]\langle[\![t]\!]\rangle & \text{by definition} \end{array}$$

3. $\mathsf{C} = \mathsf{C}_1\,s$ (the case $\mathsf{C} = s\,\mathsf{C}_1$ is similar). Then we reason as follows:

$$\begin{array}{lll} & [\![\mathsf{C}\langle t\rangle]\!] & \\ = & [\![\mathsf{C}_1\langle t\rangle\,s]\!] & \\ = & [\![\mathsf{C}_1\langle t\rangle]\!]\,[\![s]\!] & \text{by definition} \\ = & [\![\mathsf{C}_1]\!]\langle[\![t]\!]\rangle\,[\![s]\!] & (IH) \\ = & [\![\mathsf{C}]\!]\langle[\![t]\!]\rangle & \text{by definition} \end{array}$$

4. $\mathsf{C} = \mathsf{C}_1[x/s]$ (the case $\mathsf{C} = s[x/\mathsf{C}_1]$ is similar). Then we reason as follows:

$$\begin{array}{lll} & [\![\mathsf{C}\langle t\rangle]\!] & \\ = & [\![\mathsf{C}_1\langle t\rangle[x/s]]\!] & \\ = & [\![\mathsf{C}_1\langle t\rangle]\!][x/[\![s]\!]] & \text{by definition} \\ = & [\![\mathsf{C}_1]\!]\langle[\![t]\!]\rangle[x/[\![s]\!]] & (IH) \\ = & [\![\mathsf{C}]\!]\langle[\![t]\!]\rangle & \text{by definition} \end{array}$$

The third item is by induction on t.

1. $t = y$. We consider two cases.
   1.1 $y = x$. Then $[\![t\{x := s\}]\!] = [\![s]\!] = x\{x := [\![s]\!]\} = [\![t]\!]\{x := [\![s]\!]\}$.
   1.2 $y \neq x$. Then $[\![t\{x := s\}]\!] = [\![y]\!] = y = y\{x := [\![s]\!]\} = [\![t]\!]\{x := [\![s]\!]\}$.
2. $t = \lambda y.\,r$. Then

$$\begin{array}{lll} & [\![t\{x := s\}]\!] & \\ = & [\![(\lambda y.\,r)\{x := s\}]\!] & \\ = & [\![\lambda y.\,r\{x := s\}]\!] & \\ = & \lambda y.\,[\![r\{x := s\}]\!] & \text{by definition} \\ = & \lambda y.\,[\![r]\!]\{x := [\![s]\!]\} & (IH) \\ = & (\lambda y.\,[\![r]\!])\{x := [\![s]\!]\} & \\ = & [\![\lambda y.\,r]\!]\{x := [\![s]\!]\} & \text{by definition} \end{array}$$

3. $t = t_1\,t_2$ (the case $t = t_1[x/t_2]$ is similar). Then

$$\begin{array}{lll} & [\![t\{x := s\}]\!] & \\ = & [\![(t_1\,t_2)\{x := s\}]\!] & \\ = & [\![t_1\{x := s\}\,t_2\{x := s\}]\!] & \\ = & [\![t_1\{x := s\}]\!]\,[\![t_2\{x := s\}]\!] & \text{by definition} \\ = & [\![t_1]\!]\{x := [\![s]\!]\}\,[\![t_2]\!]\{x := [\![s]\!]\} & (IH) \\ = & ([\![t_1]\!]\,[\![t_2]\!])\{x := [\![s]\!]\} & \\ = & [\![t_1\,t_2]\!]\{x := [\![s]\!]\} & \text{by definition} \end{array}$$

Lemma D.46 ($[\![-]\!]$ preserves typing). *If $\Delta;\Gamma \vdash t : A$ then $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : [\![A]\!]$.*

Proof. By induction on the derivation of $\Delta;\Gamma \vdash t : A$:

1. `lvar`: Let $\Delta; a : A \vdash a : A$. Then $[\![\Delta]\!], a : [\![A]\!] \vdash_{\mathrm{lsc}} a : [\![A]\!]$ by `l-var`.
2. `uvar`: Let $\Delta, u : A; \cdot \vdash u : \bullet A$. Then we have that $[\![\Delta]\!], u : \mathbf{1} \to [\![A]\!] \vdash_{\mathrm{lsc}} u : \mathbf{1} \to [\![A]\!]$, so indeed $[\![\Delta, u : A]\!] \vdash_{\mathrm{lsc}} u : [\![\bullet A]\!]$ by `l-var`.
3. `abs`: Let $\Delta;\Gamma \vdash \lambda a.t : A \multimap B$ be derived from $\Delta;\Gamma, a : A \vdash t : B$. By IH $[\![\Delta]\!], [\![\Gamma]\!], a : [\![A]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : [\![B]\!]$, so $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} \lambda a. [\![t]\!] : [\![A]\!] \to [\![B]\!]$ by `l-abs`, which means that $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} [\![\lambda a. t]\!] : [\![A \multimap B]\!]$, as required.
4. `app`: Let $\Delta;\Gamma_1,\Gamma_2 \vdash t\,s : B$ be derived from $\Delta;\Gamma_1 \vdash t : A \multimap B$ and $\Delta;\Gamma_2 \vdash s : B$. By IH we have that $[\![\Delta]\!], [\![\Gamma_1]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : [\![A]\!] \to [\![B]\!]$ and $[\![\Delta]\!], [\![\Gamma_2]\!] \vdash_{\mathrm{lsc}} [\![s]\!] : [\![B]\!]$. By weakening we have that $[\![\Delta]\!], [\![\Gamma_1]\!], [\![\Gamma_2]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : [\![A]\!] \to [\![B]\!]$ and $[\![\Delta]\!], [\![\Gamma_1]\!], [\![\Gamma_2]\!] \vdash_{\mathrm{lsc}} [\![s]\!] : [\![B]\!]$. Hence, by the `l-app` rule, we have that $[\![\Delta]\!], [\![\Gamma_1]\!], [\![\Gamma_2]\!] \vdash_{\mathrm{lsc}} [\![t]\!]\,[\![s]\!] : [\![B]\!]$, as required.
5. `grant`: Let $\Delta;\Gamma \vdash \bullet t : \bullet A$ be derived from $\Delta;\Gamma \vdash t : A$. By IH we have that $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : [\![A]\!]$. By weakening we have that $[\![\Delta]\!], [\![\Gamma]\!], z : \mathbf{1} \vdash_{\mathrm{lsc}} [\![t]\!] : [\![A]\!]$, where z is a fresh variable, not occurring free in Δ, Γ, nor t. Then, by the `l-abs` rule, $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} \lambda z. [\![t]\!] : \mathbf{1} \to [\![A]\!]$. Hence $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} [\![\bullet t]\!] : [\![\bullet A]\!]$.
6. `request`: Let $\Delta;\Gamma \vdash \mathtt{req}(t) : A$ be derived from $\Delta;\Gamma \vdash t : \bullet A$. By IH, $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : \mathbf{1} \to [\![A]\!]$. By the `l-app` rule we have that $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} [\![t]\!]\,* : [\![A]\!]$, given that $*$ is assumed to be a closed term of type $\mathbf{1}$. So indeed $[\![\Delta]\!], [\![\Gamma]\!] \vdash_{\mathrm{lsc}} [\![\mathtt{req}(t)]\!] : [\![A]\!]$.
7. `prom`: Let $\Delta; \cdot \vdash\, !t : !A$ be derived from $\Delta; \cdot \vdash t : A$. Then by IH $[\![\Delta]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : [\![A]\!]$, and we are done because $[\![!t]\!] = [\![t]\!]$ and $[\![!A]\!] = [\![A]\!]$ by definition.
8. `sub`: Let $\Delta;\Gamma_1,\Gamma_2 \vdash t[u/s] : B$ be derived from $\Delta, u : A;\Gamma_1 \vdash t : B$ and $\Delta;\Gamma_2 \vdash s : !\bullet A$. By IH we have that $[\![\Delta]\!], u : \mathbf{1} \to [\![A]\!], [\![\Gamma_1]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : [\![B]\!]$ and $[\![\Delta]\!], [\![\Gamma_2]\!] \vdash_{\mathrm{lsc}} [\![s]\!] : [\![!\bullet A]\!]$, where, by definition, $[\![!\bullet A]\!] = \mathbf{1} \to [\![A]\!]$. By weakening we have that $[\![\Delta]\!], u : \mathbf{1} \to [\![A]\!], [\![\Gamma_1]\!], [\![\Gamma_2]\!] \vdash_{\mathrm{lsc}} [\![t]\!] : [\![B]\!]$ and $[\![\Delta]\!], [\![\Gamma_1]\!], [\![\Gamma_2]\!] \vdash_{\mathrm{lsc}} [\![s]\!] : \mathbf{1} \to [\![A]\!]$. Hence by the `l-es` rule $[\![\Delta]\!], [\![\Gamma_1]\!], [\![\Gamma_2]\!] \vdash_{\mathrm{lsc}} [\![t]\!][u/[\![s]\!]] : [\![B]\!]$. To conclude, it suffices to observe that $[\![t[u/s]]\!] = [\![t]\!][u/[\![s]\!]]$.

Lemma D.47 (Simulation of $\lambda^{!\bullet}$ in LSC, up to fusion). *If $t \to_{\bullet\mathrm{i}} t'$ then $[\![t]\!] \to^{+}_{\mathrm{lsci}}\Rrightarrow [\![t']\!]$.*

Proof. By induction on the derivation of the step $t \to_{\bullet\mathrm{i}} t'$:

1. Root $\to_{\bullet\mathsf{db}}$ step: let $(\lambda a.t)\mathtt{L}\,s \mapsto_{\bullet\mathsf{db}} t\{a := s\}\mathtt{L}$. Then:

$$\begin{array}{rll} [\![(\lambda a.t)\mathtt{L}\,s]\!] = & (\lambda a.[\![t]\!])[\![\mathtt{L}]\!]\,[\![s]\!] & \\ \to_{\mathsf{db}} & [\![t]\!][a/[\![s]\!]][\![\mathtt{L}]\!] & \\ \to^{*}_{\mathsf{ls}} & [\![t]\!]\{a := [\![s]\!]\}[a/[\![s]\!]][\![\mathtt{L}]\!] & \\ \Rrightarrow & [\![t]\!]\{a := [\![s]\!]\}[\![\mathtt{L}]\!] & \text{by } \Rrightarrow\mathtt{w} \\ = & [\![t\{a := s\}\mathtt{L}]\!] & \text{by Lem. D.45} \end{array}$$

2. Root $\to_{\bullet\mathsf{req}}$ step: let $\mathtt{req}((\bullet t)\mathtt{L}) \mapsto_{\bullet\mathsf{req}} t\mathtt{L}$. Then, for $z \notin \mathsf{fv}(r)$:

$$\begin{array}{rll} [\![\mathtt{req}((\bullet t)\mathtt{L})]\!] = & (\lambda z.[\![t]\!])[\![\mathtt{L}]\!]\,* & \\ \to_{\mathsf{db}} & [\![t]\!][z/*][\![\mathtt{L}]\!] & \\ \Rrightarrow & [\![t]\!][\![\mathtt{L}]\!] & \text{by } \Rrightarrow\mathtt{w}\text{, as } z \notin \mathsf{fv}(t) = \mathsf{fv}([\![t]\!]) \\ = & [\![t\mathtt{L}]\!] & \end{array}$$

3. Root $\to_{\bullet\mathsf{ls}}$ step: let $\mathtt{C}\langle\!\langle u\rangle\!\rangle[u/(!(\bullet t)\mathtt{L}_1)\mathtt{L}_2] \mapsto_{\bullet\mathsf{ls}} \mathtt{C}\langle\!\langle(\bullet t)\mathtt{L}_1\rangle\!\rangle[u/!(\bullet t)\mathtt{L}_1]\mathtt{L}_2$. Then:

$$\begin{array}{rll} [\![\mathtt{C}\langle\!\langle u\rangle\!\rangle[u/(!(\bullet t)\mathtt{L}_1)\mathtt{L}_2]]\!] = & [\![\mathtt{C}]\!]\langle\!\langle u\rangle\!\rangle[u/(\lambda z.\,[\![t]\!])[\![\mathtt{L}_1]\!][\![\mathtt{L}_2]\!]] & \\ \to_{\mathsf{ls}} & [\![\mathtt{C}]\!]\langle\!\langle(\lambda z.\,[\![t]\!])[\![\mathtt{L}_1]\!][\![\mathtt{L}_2]\!]\rangle\!\rangle[u/(\lambda z.\,[\![t]\!])[\![\mathtt{L}_1]\!][\![\mathtt{L}_2]\!]] & \\ \Rrightarrow & [\![\mathtt{C}]\!]\langle\!\langle(\lambda z.\,[\![t]\!])[\![\mathtt{L}_1]\!]\rangle\!\rangle[u/(\lambda z.\,[\![t]\!])[\![\mathtt{L}_1]\!]][\![\mathtt{L}_2]\!] & \text{by Lem. D.38} \\ = & [\![\mathtt{C}\langle\!\langle(\bullet t)\mathtt{L}_1\rangle\!\rangle[u/!(\bullet t)\mathtt{L}_1]\mathtt{L}_2]\!] & \end{array}$$

4. Congruence closure below contexts of any of the following forms:

$$\lambda a.\,\Box \qquad \Box\, s \qquad s\,\Box \qquad \bullet\,\Box \qquad \mathtt{req}(\Box) \qquad !\Box \qquad \Box\, s \qquad s\,\Box$$

are all immediate by IH because both $\to_{\mathsf{lsci}}$ and $\Rrightarrow$ are closed by compatibility under arbitrary contexts. For example, suppose that $\bullet t \to_{\bullet\mathsf{i}} \bullet t'$ is derived from $t \to_{\bullet\mathsf{i}} t'$. Then by IH we have that $[\![t]\!] \to^{+}_{\mathsf{lsci}}\Rrightarrow [\![t']\!]$. Hence $[\![\bullet t]\!] = \lambda z.\,[\![t]\!] \to^{+}_{\mathsf{lsci}}\Rrightarrow \lambda z.\,[\![t']\!] = [\![\bullet t']\!]$.

Lemma D.48 (Abstract postponement lemma). *Let $\mathcal{X} = (X, \to)$ be an ARS. Suppose that reduction can be written as a union $\to\, = \,\to_1 \cup \to_2$, where $\to_2$ is SN and it can be postponed in the sense that $\to_2\to_1 \subseteq \to_1\to_2^*$. Then an arbitrary object $x \in X$ is $\to$-SN if and only if it is $\to_1$-SN.*

Proof. ($\Rightarrow$) Immediate. ($\Leftarrow$) We prove the statement by first proving two auxiliary claims:

1. First claim: $\to_2^*\to_1 \subseteq \to_1\to_2^*$.
Since $\to^*$ is the reflexive–transitive closure of $\to$, by definition, this is equivalent to showing that $\to_2^n\to_1 \subseteq \to_1\to_2^*$ holds for all $n \in \mathbb{N}_0$. We proceed by induction on n. If $n = 0$, it is immediate to conclude. Now assume the property holds for a given $n \in \mathbb{N}_0$. Then:

$$\begin{array}{rll} \to_2^{n+1}\to_1 & = \to_2^n\to_2\to_1 & \\ & \subseteq \to_2^n\to_1\to_2^* & \text{by the hypothesis that } \to_2\to_1 \subseteq \to_1\to_2^* \\ & \subseteq \to_1\ \to_2^*\to_2^* & \text{by IH} \\ & \subseteq \to_1\to_2^* & \text{by transitivity} \end{array}$$

2. Second claim: if $x \in X$ is not $\to$-SN, there exists $y \in X$ such that $x \to_1 y$ and y is not $\to$-SN.
Suppose that there is an infinite reduction sequence $x = x_0 \to x_1 \to x_2 \to \ldots$. By hypothesis, $\to_2$ is SN, so the reduction sequence must contain at least one $\to_1$ step. Let i be the first index such that $x \to_2^* x_i \to_1 x_{i+1}$. By the first claim we have that $\to_2^*\to_1 \subseteq \to_1\to_2^*$, so $x \to_1 y \to_2^* x_{i+1}$ for some $y \in X$. Note there is an infinite $\to$-sequence starting on y since $y \to_2^* x_{i+1} \to x_{i+2} \to x_{i+3} \ldots$, so y is not $\to$-SN, as required.
3. Finally, we prove the main statement: if x is $\to_1$-SN then x is $\to$-SN.
Suppose that x is $\to_1$-SN. By contradiction, suppose that x is not $\to$-SN. Then by the second claim there exists $x_1 \in X$ such that $x \to_1 x_1$ and x_1 is not $\to$-SN. Iterating this argument, we construct an infinite reduction sequence $x \to_1 x_1 \to_1 x_2 \ldots$, contradicting the fact that x is $\to_1$-SN.

Theorem D.3 (Termination [proof of Thm. 4.1]). *If t is typable in $\lambda^{!\bullet}$ then t is $\to_\bullet$-SN.*

Proof. Suppose that t is a typable $\lambda^{!\bullet}$-term, *i.e.* $\Delta; \Gamma \vdash t : A$ holds for some Δ, Γ, A. By the fact that gc steps can be postponed (Lem. D.37) and Lem. D.48, showing that t is $\to_\bullet$-SN is equivalent to showing that t is $\to_{\bullet\mathsf{i}}$-SN, where we recall that $\to_{\bullet\mathsf{i}} = \to_\bullet \setminus \to_{\bullet\mathsf{gc}}$.

Suppose that there is an infinite reduction sequence $t \to_{\bullet\mathsf{i}} t_1 \to_{\bullet\mathsf{i}} t_2 \ldots$ (without $\to_{\bullet\mathsf{gc}}$ steps). By Lem. D.47 we obtain an infinite reduction sequence $[\![t]\!] \to^+_{\mathsf{lsci}}\Rrightarrow [\![t_1]\!] \to^+_{\mathsf{lsci}}\Rrightarrow [\![t_2]\!] \ldots$ where we know that $[\![t]\!]$ is simply typable in LSC, as the translation preserves typing (Lem. D.46). Since $[\![t]\!]$ is simply typable in LSC, then $[\![t]\!]$ in particular $\to_{\mathsf{lsci}}$-SN by Thm. D.2. To conclude, apply Lem. D.44 to obtain that $[\![t]\!]$ must be $(\to^+_{\mathsf{lsci}}\Rrightarrow)$-SN, in contradiction with the fact that we have an infinite sequence of $(\to^+_{\mathsf{lsci}}\Rrightarrow)$-steps starting from $[\![t]\!]$.

D.4 Normal forms

We present an inductive characterization of the $\to_\bullet$-normal forms. It is used in the proof of preservation of normal forms of our translations. Let $\mathcal{A} := \{\mathtt{var}, \lambda, @, \mathtt{req}, \bullet, !, !\bullet\}$. The set of $\to_\bullet$-normal forms is characterized by $\mathcal{N} := \bigcup_{\alpha \in \mathcal{A}} \mathcal{N}_\alpha$, where $\mathcal{N}_\alpha$ is defined as follows:

$$\frac{}{a \in \mathcal{N}_{\mathtt{var}}} \qquad \frac{t \in \mathcal{N}_\alpha \quad s \in \mathcal{N}_\beta \quad u \in \mathsf{fv}(t) \quad \beta \neq !\bullet}{t[u/s] \in \mathcal{N}_\alpha}$$

$$\frac{}{u \in \mathcal{N}_{\mathtt{var}}} \qquad \frac{t \in \mathcal{N}_\alpha \quad s \in \mathcal{N}_\beta \quad u \notin \mathsf{fv}(t) \quad \beta \notin \{!\bullet, !\}}{t[u/s] \in \mathcal{N}_\alpha}$$

$$\frac{t \in \mathcal{N}_\alpha}{\lambda a. t \in \mathcal{N}_\lambda} \qquad \frac{t \in \mathcal{N}_\alpha}{\bullet t \in \mathcal{N}_\bullet} \qquad \frac{t \in \mathcal{N}_\alpha \quad \alpha \neq \bullet}{!t \in \mathcal{N}_!} \qquad \frac{t \in \mathcal{N}_\bullet}{!t \in \mathcal{N}_{!\bullet}}$$

$$\frac{t \in \mathcal{N}_\alpha \quad s \in \mathcal{N}_\beta \quad \alpha \neq \lambda}{t\, s \in \mathcal{N}_@} \qquad \frac{t \in \mathcal{N}_\alpha \quad \alpha \neq \bullet}{\mathtt{req}(t) \in \mathcal{N}_{\mathtt{req}}}$$

E Appendix: Embedding CBN, CBV and CBS

E.1 CBN

Remark E.1. $\mathsf{fv}(t^{\mathsf{N}}) = \mathsf{fv}(t)$.

Remark E.2. $\mathsf{C}\langle t\rangle^{\mathsf{N}} = \mathsf{C}^{\mathsf{N}}\langle t^{\mathsf{N}}\rangle$ and $\mathsf{C}\langle\!\langle t\rangle\!\rangle^{\mathsf{N}} = \mathsf{C}^{\mathsf{N}}\langle\!\langle t^{\mathsf{N}}\rangle\!\rangle$. In particular, $(t\mathsf{L})^{\mathsf{N}} = t^{\mathsf{N}}\mathsf{L}^{\mathsf{N}}$.

Proposition E.1 (CBN typing [proof of Prop. 5.1]). *If $\Gamma \vdash t : A$ then $\Gamma^{\mathsf{N}}; \cdot \vdash t^{\mathsf{N}} : A^{\mathsf{N}}$.*

Proof. By induction on the derivation of $\Gamma \vdash t : A$:

1. $\mathtt{l\text{-}var}$: Let $\Gamma, x : A \vdash x : A$ be derived from the $\mathtt{l\text{-}var}$ rule. Then:

$$\dfrac{\dfrac{}{\Gamma^{\mathsf{N}}, x : A^{\mathsf{N}}; \cdot \vdash x : \bullet A^{\mathsf{N}}}\ \mathtt{uvar}}{\Gamma^{\mathsf{N}}, x : A^{\mathsf{N}}; \cdot \vdash \mathtt{req}(x) : A^{\mathsf{N}}}\ \mathtt{request}$$

2. $\mathtt{l\text{-}abs}$: Let $\Gamma \vdash \lambda x.t : A \to B$ be derived from $\Gamma, x : A \vdash t : B$. Let a be a fresh linear variable, such that $a \notin \mathsf{fv}(t^{\mathsf{N}})$. Then:

$$\dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{N}}, x : A^{\mathsf{N}}; \cdot \vdash t^{\mathsf{N}} : B^{\mathsf{N}}} \quad \dfrac{}{\Gamma^{\mathsf{N}}; a : !\bullet A^{\mathsf{N}} \vdash a : !\bullet A^{\mathsf{N}}}\,\mathtt{lvar}}{\Gamma^{\mathsf{N}}; a : !\bullet A^{\mathsf{N}} \vdash t^{\mathsf{N}}[x/a] : B^{\mathsf{N}}}\,\mathtt{sub}}{\Gamma^{\mathsf{N}}; \cdot \vdash \lambda a.t^{\mathsf{N}}[x/a] : !\bullet A^{\mathsf{N}} \multimap B^{\mathsf{N}}}\,\mathtt{abs}$$

3. $\mathtt{l\text{-}app}$: Let $\Gamma \vdash t\,s : B$ be derived from $\Gamma \vdash t : A \to B$ and $\Gamma \vdash s : A$. Then:

$$\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{N}}; \cdot \vdash t^{\mathsf{N}} : !\bullet A^{\mathsf{N}} \multimap B^{\mathsf{N}}} \quad \dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{N}}; \cdot \vdash s^{\mathsf{N}} : A^{\mathsf{N}}}}{\Gamma^{\mathsf{N}}; \cdot \vdash \bullet s^{\mathsf{N}} : \bullet A^{\mathsf{N}}}\,\mathtt{grant}}{\Gamma^{\mathsf{N}}; \cdot \vdash !\bullet s^{\mathsf{N}} : !\bullet A^{\mathsf{N}}}\,\mathtt{prom}}{\Gamma^{\mathsf{N}}; \cdot \vdash t^{\mathsf{N}}\, !\bullet s^{\mathsf{N}} : B^{\mathsf{N}}}\,\mathtt{app}$$

4. $\mathtt{l\text{-}es}$: Let $\Gamma \vdash t[x/s] : B$ be derived from $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. Then:

$$\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{N}}, x : A^{\mathsf{N}}; \cdot \vdash t^{\mathsf{N}} : B^{\mathsf{N}}} \quad \dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{N}}; \cdot \vdash s^{\mathsf{N}} : A^{\mathsf{N}}}}{\Gamma^{\mathsf{N}}; \cdot \vdash \bullet s^{\mathsf{N}} : \bullet A^{\mathsf{N}}}\,\mathtt{grant}}{\Gamma^{\mathsf{N}}; \cdot \vdash !\bullet s^{\mathsf{N}} : !\bullet A^{\mathsf{N}}}\,\mathtt{prom}}{\Gamma^{\mathsf{N}}; \cdot \vdash t^{\mathsf{N}}[x/!\bullet s^{\mathsf{N}}] : B^{\mathsf{N}}}$$

Definition E.1. *We define a subset* $\mathsf{Ctxs}^{\mathsf{N}}_{\bullet} \subseteq \mathsf{Ctxs}_{\bullet}$*, called* CBN contexts*, by the following grammar:*

$$\underline{\mathsf{C}} ::= \mathtt{req}(\Box) \mid \lambda a.\underline{\mathsf{C}}[u/a] \mid \underline{\mathsf{C}}\,!\bullet\underline{t} \mid \underline{t}\,!\bullet\underline{\mathsf{C}} \mid \underline{\mathsf{C}}[u/!\bullet\underline{t}] \mid \underline{t}[u/!\bullet\underline{\mathsf{C}}]$$

where, in the production $\underline{\mathsf{C}} ::= \lambda a.\underline{\mathsf{C}}[u/a]$ *we assume that* a *is fresh, that is* $a \notin \mathsf{fv}(\underline{\mathsf{C}})$*. Furthermore, we define a subset* $\mathsf{SCtxs}^{\mathsf{N}}_{\bullet} \subseteq \mathsf{SCtxs}_{\bullet}$ *of the set of substitution contexts, called* CBN substitution contexts*:*

$$\underline{\mathsf{L}} ::= \Box \mid \underline{\mathsf{L}}[u/!\bullet\underline{t}]$$

The inverse translation can be extended to CBN *contexts and substitution contexts, setting* $\mathtt{req}(\Box)^{-\mathsf{N}} := \Box$ *for* CBN *contexts, and* $\Box^{-\mathsf{N}} := \Box$ *for* CBN *substitution contexts.*

Remark E.3. $\mathsf{fv}(\underline{t}^{-\mathsf{N}}) = \mathsf{fv}(\underline{t})$.

Lemma E.1 (Context decomposition for the inverse CBN translation).

1. $t\mathsf{L} \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ *if and only if* $t \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ *and* $\mathsf{L} \in \mathsf{SCtxs}^{\mathsf{N}}_{\bullet}$.
2. $\mathsf{C}\langle\!\langle x \rangle\!\rangle \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ *if and only if* $\mathsf{C} \in \mathsf{Ctxs}^{\mathsf{N}}_{\bullet}$.
3. *If* $\underline{\mathsf{C}} \in \mathsf{Ctxs}^{\mathsf{N}}_{\bullet}$ *and* $\underline{t} \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ *then* $\underline{\mathsf{C}}\langle\!\langle \bullet\underline{t} \rangle\!\rangle \in \mathcal{T}^{\mathsf{N}}_{\bullet}$.
4. *If* $\underline{t} \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ *and* $\underline{\mathsf{L}} \in \mathsf{SCtxs}^{\mathsf{N}}_{\bullet}$*, then* $(\underline{t}\,\underline{\mathsf{L}})^{-\mathsf{N}} = \underline{t}^{-\mathsf{N}}\underline{\mathsf{L}}^{-\mathsf{N}}$.

5. *If* $\underline{\mathsf{C}} \in \mathsf{Ctxs}^{\mathsf{N}}_{\bullet}$ *then* $\underline{\mathsf{C}}\langle\!\langle x\rangle\!\rangle^{-\mathsf{N}} = \underline{\mathsf{C}}^{-\mathsf{N}}\langle\!\langle x\rangle\!\rangle$.
6. *If* $\underline{\mathsf{C}} \in \mathsf{Ctxs}^{\mathsf{N}}_{\bullet}$ *and* $\underline{t} \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, *then* $\underline{\mathsf{C}}\langle\!\langle \bullet\underline{t}\rangle\!\rangle^{-\mathsf{N}} = \underline{\mathsf{C}}^{-\mathsf{N}}\langle\!\langle \underline{t}^{-\mathsf{N}}\rangle\!\rangle$.

Proof. By induction on the first judgement in the statement of each item, except the first and fourth items which are by induction on $\mathtt{L}$ and $\underline{\mathtt{L}} \in \mathsf{SCtxs}^{\mathsf{N}}_{\bullet}$, resp.

Lemma E.2 (Inverse CBN simulation [PROOF OF LEMMA 5.2]). *Let* $\underline{t} \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ *and* $s \in \mathcal{T}_{\bullet}$ *such that* $\underline{t} \to_{\bullet} s$. *Then* $s \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ *and* $\underline{t}^{-\mathsf{N}} \to^{=}_{\mathsf{N}} s^{-\mathsf{N}}$.

Proof. By induction on the (unique) derivation of $\underline{t} \in \mathcal{T}^{\mathsf{N}}_{\bullet}$:

1. $\underline{t} = \mathtt{req}(x)$: Impossible, as there are no steps $\underline{t} \to_{\bullet} s$.
2. $\underline{t} = \mathtt{req}(\bullet\underline{t}')$: We consider two subcases, depending on whether the step is at the root of the term or internal to $\bullet\underline{t}'$:
   2.1 If the step is at the root, we have that $\underline{t} = \mathtt{req}(\bullet\underline{t}') \mapsto_{\bullet\mathsf{req}} \underline{t}' = s$, so $s = \underline{t}' \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ and $\underline{t}^{-\mathsf{N}} = \mathtt{req}(\bullet\underline{t}')^{-\mathsf{N}} = \underline{t}'^{-\mathsf{N}} = s^{-\mathsf{N}}$.
   2.2 If the step is internal to $\bullet\underline{t}'$, note that it cannot be at the root of $\bullet\underline{t}'$, since this term does not match the left-hand side of any rewriting rule. Then the step must be internal to $\underline{t}'$, that is, $\underline{t} = \mathtt{req}(\bullet\underline{t}') \to_{\bullet} \mathtt{req}(\bullet s') = s$ with $\underline{t}' \to_{\bullet} s'$. By IH, $s' \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, so $s = \mathtt{req}(\bullet s') \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, and $\underline{t}^{-\mathsf{N}} = \mathtt{req}(\bullet\underline{t}')^{-\mathsf{N}} = \underline{t}'^{-\mathsf{N}} \to^{=}_{\mathsf{N}} s'^{-\mathsf{N}} = \mathtt{req}(\bullet s')^{-\mathsf{N}} = s^{-\mathsf{N}}$.
3. $\underline{t} = \lambda a.\, \underline{t}'[x/a]$: Note that the step cannot be at the root, and that that there cannot be a $\to_{\bullet\mathsf{ls}}$ nor a $\to_{\bullet\mathsf{gc}}$ step involving the substitution $[x/a]$, since these rules would require that a be of the form $(!r)\mathtt{L}$, but a is a linear variable. This means that the step must be internal to $\underline{t}'$, that is, $\underline{t} = \lambda a.\, \underline{t}'[x/a] \to_{\bullet} \lambda a.\, s'[x/a] = s$ with $\underline{t}' \to_{\bullet} s'$. By IH, $s' \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, so $s = \lambda a.\, s'[x/a] \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, and $\underline{t}^{-\mathsf{N}} = (\lambda a.\, \underline{t}'[x/a])^{-\mathsf{N}} = \lambda x.\, \underline{t}'^{-\mathsf{N}} \to^{=}_{\mathsf{N}} \lambda x.\, s'^{-\mathsf{N}} = (\lambda a.\, s'[x/a])^{-\mathsf{N}} = s^{-\mathsf{N}}$.
4. $\underline{t} = \underline{t}_1\, !\bullet\underline{t}_2$: We consider three subcases, depending on whether the step is at the root of the term, internal to $\underline{t}_1$, or internal to $!\bullet\underline{t}_2$:
   4.1 If the step is at the root of the term, it must be a $\bullet\mathsf{db}$ step, that is, $\underline{t}_1$ must be of the form $(\lambda a.\, t)\mathtt{L}$. By Lem. E.1, this means that $\lambda a.\, t \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ and $\mathtt{L} \in \mathsf{SCtxs}^{\mathsf{N}}_{\bullet}$. In particular, t must be of the form $t = \underline{t}_1'[x/a]$. Then we have that $\underline{t} = (\lambda a.\, \underline{t}_1'[x/a])\mathtt{L}\, !\bullet\underline{t}_2 \mapsto_{\bullet\mathsf{db}} \underline{t}_1'[x/!\bullet\underline{t}_2]\mathtt{L} = s$. Note that $s = \underline{t}_1'[x/!\bullet\underline{t}_2]\mathtt{L} \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ by Lem. E.1. By IH, we have that
$$\begin{array}{rl} \underline{t}^{-\mathsf{N}} = & ((\lambda a.\, \underline{t}_1'[x/a])\mathtt{L}\, !\bullet\underline{t}_2)^{-\mathsf{N}} \\ = & (\lambda x.\, \underline{t}_1'^{-\mathsf{N}})\mathtt{L}^{-\mathsf{N}}\, \underline{t}_2^{-\mathsf{N}} \\ \mapsto_{\mathsf{db}} & \underline{t}_1'^{-\mathsf{N}}[x/\underline{t}_2^{-\mathsf{N}}]\mathtt{L}^{-\mathsf{N}} \\ = & (\underline{t}_1'[x/!\bullet\underline{t}_2]\mathtt{L})^{-\mathsf{N}} \\ = & s^{-\mathsf{N}} \end{array}$$
   4.2 If the step is internal to $\underline{t}_1$, we have that $\underline{t} = \underline{t}_1\, !\bullet\underline{t}_2 \to_{\bullet} s_1\, !\bullet\underline{t}_2 = s$ with $\underline{t}_1 \to_{\bullet} s_1$. By IH, $s_1 \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, so $s = s_1\, !\bullet\underline{t}_2 \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ and $\underline{t}^{-\mathsf{N}} = \underline{t}_1^{-\mathsf{N}}\, \underline{t}_2^{-\mathsf{N}} \to^{=}_{\mathsf{N}} s_1^{-\mathsf{N}}\, \underline{t}_2^{-\mathsf{N}} = s^{-\mathsf{N}}$.
   4.3 If the step is internal to $!\bullet\underline{t}_2$, note that it cannot be at the root of $!\bullet\underline{t}_2$ nor at the root of $\bullet\underline{t}_2$, since these terms do not match the left-hand side of any rewriting rule. So the step must be internal to $\underline{t}_2$, that is, we have that $\underline{t} = \underline{t}_1\, !\bullet\underline{t}_2 \to_{\bullet} \underline{t}_1\, !\bullet s_2 = s$ with $\underline{t}_2 \to_{\bullet} s_2$. By IH, $s_2 \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, so $s = \underline{t}_1\, !\bullet s_2 \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ and $\underline{t}^{-\mathsf{N}} = \underline{t}_1^{-\mathsf{N}}\, \underline{t}_2^{-\mathsf{N}} \to^{=}_{\mathsf{N}} \underline{t}_1^{-\mathsf{N}}\, s_2^{-\mathsf{N}} = s^{-\mathsf{N}}$.

5. $\underline{t} = \underline{t}_1[x/!\bullet\underline{t}_2]$: We consider three subcases, depending on whether the step is at the root of the term, internal to $\underline{t}_1$, or internal to $!\bullet\underline{t}_2$:
   5.1 If the step is at the root of the term, it must be either a $\bullet\mathsf{ls}$ or a $\bullet\mathsf{gc}$ step. We consider two further subcases:
      5.1.1 If the step is a $\bullet\mathsf{ls}$ step, we have that $\underline{t}_1 = \mathsf{C}\langle\langle x\rangle\rangle \in \mathcal{T}_\bullet^{\mathsf{N}}$ and $\underline{t} = \mathsf{C}\langle\langle x\rangle\rangle[x/!\bullet\underline{t}_2] \mapsto_{\bullet\mathsf{ls}} \mathsf{C}\langle\langle\bullet\underline{t}_2\rangle\rangle[x/!\bullet\underline{t}_2] = s$. Note that, by Lem. E.1, $\mathsf{C} \in \mathsf{Ctxs}_\bullet^{\mathsf{N}}$, so, again by Lem. E.1, we have $s = \mathsf{C}\langle\langle\bullet\underline{t}_2\rangle\rangle[x/!\bullet\underline{t}_2] \in \mathcal{T}_\bullet^{\mathsf{N}}$. Moreover, also using Lem. E.1, we have that $\underline{t}^{-\mathsf{N}} = (\mathsf{C}\langle\langle x\rangle\rangle[x/!\bullet\underline{t}_2])^{-\mathsf{N}} = \mathsf{C}^{-\mathsf{N}}\langle\langle x\rangle\rangle[x/\underline{t}_2^{-\mathsf{N}}] \mapsto_{\mathsf{ls}} \mathsf{C}^{-\mathsf{N}}\langle\langle\underline{t}_2^{-\mathsf{N}}\rangle\rangle[x/\underline{t}_2^{-\mathsf{N}}] = (\mathsf{C}\langle\langle\bullet\underline{t}_2\rangle\rangle[x/!\bullet\underline{t}_2])^{-\mathsf{N}} = s^{-\mathsf{N}}$.
      5.1.2 If the step is a $\bullet\mathsf{gc}$ step, we have that $x \notin \mathsf{fv}(\underline{t}_1)$ and $\underline{t} = \underline{t}_1[x/!\bullet\underline{t}_2] \mapsto_{\bullet\mathsf{gc}} \underline{t}_1 = s$. Note that $s = \underline{t}_1 \in \mathcal{T}_\bullet^{\mathsf{N}}$. Moreover, $\underline{t}^{-\mathsf{N}} = \underline{t}_1[x/!\bullet\underline{t}_2]^{-\mathsf{N}} = \underline{t}_1^{-\mathsf{N}}[x/\underline{t}_2^{-\mathsf{N}}] \mapsto_{\mathsf{gc}} \underline{t}_1^{-\mathsf{N}} = s^{-\mathsf{N}}$. Note that $x \notin \underline{t}_1^{-\mathsf{N}}$ because $\mathsf{fv}(\underline{t}_1^{-\mathsf{N}}) = \mathsf{fv}(\underline{t}_1)$, as noted in Rem. E.3.
   5.2 If the step is internal to $\underline{t}_1$, then $\underline{t} = \underline{t}_1[x/!\bullet\underline{t}_2] \to_\bullet s_1[x/!\bullet\underline{t}_2] = s$ with $\underline{t}_1 \to_\bullet s_1$. By IH, $s_1 \in \mathcal{T}_\bullet^{\mathsf{N}}$, so $s = s_1[x/!\bullet\underline{t}_2] \in \mathcal{T}_\bullet^{\mathsf{N}}$ and $\underline{t}^{-\mathsf{N}} = \underline{t}_1^{-\mathsf{N}}[x/\underline{t}_2^{-\mathsf{N}}] \to_{\mathsf{N}}^{=} s_1^{-\mathsf{N}}[x/\underline{t}_2^{-\mathsf{N}}] = s^{-\mathsf{N}}$.
   5.3 If the step is internal to $!\bullet\underline{t}_2$, note that it cannot be at the root of $!\bullet\underline{t}_2$ nor at the root of $\bullet\underline{t}_2$, since these terms do not match the left-hand side of any rewriting rule. So the step must be internal to $\underline{t}_2$, that is, we have that $\underline{t} = \underline{t}_1[x/!\bullet\underline{t}_2] \to_\bullet \underline{t}_1[x/!\bullet s_2] = s$ with $\underline{t}_2 \to_\bullet s_2$. By IH, $s_2 \in \mathcal{T}_\bullet^{\mathsf{N}}$, so $s = \underline{t}_1[x/!\bullet s_2] \in \mathcal{T}_\bullet^{\mathsf{N}}$ and $\underline{t}^{-\mathsf{N}} = \underline{t}_1^{-\mathsf{N}}[x/\underline{t}_2^{-\mathsf{N}}] \to_{\mathsf{N}}^{=} \underline{t}_1^{-\mathsf{N}}[x/s_2^{-\mathsf{N}}] = s^{-\mathsf{N}}$.

E.2 CBV

Proposition E.2 (CBV **typing** [PROOF OF PROP. 5.2]). *If $\Gamma \vdash t : A$ then $\Gamma^{\mathsf{V}}; \cdot \vdash t^{\mathsf{V}} : !\bullet A^{\mathsf{V}}$.*

Proof. By induction on the derivation of $\Gamma \vdash t : A$:

1. `l-var`: Let $\Gamma, x : A \vdash x : A$ be derived from the `l-var` rule. Then:

$$\dfrac{\dfrac{}{\Gamma^{\mathsf{V}}, x : A^{\mathsf{V}}; \cdot \vdash x : \bullet A^{\mathsf{V}}}\,\texttt{uvar}}{\Gamma^{\mathsf{V}}, x : A^{\mathsf{V}}; \cdot \vdash !x : !\bullet A^{\mathsf{V}}}\,\texttt{prom}$$

2. `l-abs`: Let $\Gamma \vdash \lambda x.t : A \to B$ be derived from $\Gamma, x : A \vdash t : B$. Let a be a fresh linear variable, such that $a \notin \mathsf{fv}(t^{\mathsf{V}})$. Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{V}}, x : A^{\mathsf{V}}; \cdot \vdash t^{\mathsf{V}} : !\bullet B^{\mathsf{V}}} \qquad \dfrac{}{\Gamma^{\mathsf{V}}; a : !\bullet A^{\mathsf{V}} \vdash a : !\bullet A^{\mathsf{V}}}\,\texttt{lvar}}{\Gamma^{\mathsf{V}}; a : !\bullet A^{\mathsf{V}} \vdash t^{\mathsf{V}}[x/a] : !\bullet B^{\mathsf{V}}}\,\texttt{sub}}{\Gamma^{\mathsf{V}}; \cdot \vdash \lambda a.t^{\mathsf{V}}[x/a] : !\bullet A^{\mathsf{V}} \multimap !\bullet B^{\mathsf{V}}}\,\texttt{abs}}{\Gamma^{\mathsf{V}}; \cdot \vdash \bullet\lambda a.t^{\mathsf{V}}[x/a] : \bullet(!\bullet A^{\mathsf{V}} \multimap !\bullet B^{\mathsf{V}})}\,\texttt{grant}}{\Gamma^{\mathsf{V}}; \cdot \vdash !\bullet\lambda a.t^{\mathsf{V}}[x/a] : !\bullet(!\bullet A^{\mathsf{V}} \multimap !\bullet B^{\mathsf{V}})}\,\texttt{prom}$$

3. `l-app`: Let $\Gamma \vdash t\,s : B$ be derived from $\Gamma \vdash t : A \to B$ and $\Gamma \vdash s : A$. Then:

$$\dfrac{\dfrac{\pi}{\Gamma^{\mathsf{V}}; \cdot \vdash \texttt{req}(u)[u/t^{\mathsf{V}}] : !\bullet A^{\mathsf{V}} \multimap !\bullet B^{\mathsf{V}}}\,\texttt{sub} \qquad \dfrac{\text{IH}}{\Gamma^{\mathsf{V}}; \cdot \vdash s^{\mathsf{V}} : !\bullet A^{\mathsf{V}}}}{\Gamma^{\mathsf{V}}; \cdot \vdash \texttt{req}(u)[u/t^{\mathsf{V}}]\,s^{\mathsf{V}} : !\bullet B^{\mathsf{V}}}\,\texttt{app}$$

where π is given by:

$$\dfrac{\dfrac{\overline{\Gamma^{\vee}, u : !\bullet A^{\vee} \multimap !\bullet B^{\vee}; \cdot \vdash u : \bullet(!\bullet A^{\vee} \multimap !\bullet B^{\vee})}\ \texttt{uvar}}{\Gamma^{\vee}, u : !\bullet A^{\vee} \multimap !\bullet B^{\vee}; \cdot \vdash \texttt{req}(u) : !\bullet A^{\vee} \multimap !\bullet B^{\vee}}\ \texttt{request} \qquad \dfrac{\text{IH}}{\Gamma^{\vee}; \cdot \vdash t^{\vee} : !\bullet(!\bullet A^{\vee} \multimap !\bullet B^{\vee})}}{\Gamma^{\vee}; \cdot \vdash \texttt{req}(u)[u/t^{\vee}] : !\bullet A^{\vee} \multimap !\bullet B^{\vee}}\ \texttt{sub}$$

4. `l-es`: Let $\Gamma \vdash t[x/s] : B$ be derived from $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. Then:

$$\dfrac{\dfrac{\text{IH}}{\Gamma^{\vee}, x : A^{\vee}; \cdot \vdash t^{\vee} : !\bullet B^{\vee}} \qquad \dfrac{\text{IH}}{\Gamma^{\vee}; \cdot \vdash s^{\vee} : !\bullet A^{\vee}}}{\Gamma^{\vee}; \cdot \vdash t^{\vee}[x/s^{\vee}] : !\bullet B^{\vee}}\ \texttt{sub}$$

The translation can be extended to operate on contexts, by declaring $\Box^{\vee} := \Box$. Note that the translation of a substitution context is a substitution context:

$$(\Box[x_1/t_1]\ldots[x_n/t_n])^{\vee} = \Box[x_1/t_1^{\vee}]\ldots[x_n/t_n^{\vee}]$$

Remark E.4. $\mathsf{fv}(t^{\vee}) = \mathsf{fv}(t)$

Remark E.5. $\mathtt{C}\langle t\rangle^{\vee} = \mathtt{C}^{\vee}\langle t^{\vee}\rangle$ and $\mathtt{C}\langle\langle t\rangle\rangle^{\vee} = \mathtt{C}^{\vee}\langle\langle t^{\vee}\rangle\rangle$. In particular, $(t\mathtt{L})^{\vee} = t^{\vee}\mathtt{L}^{\vee}$.

Lemma E.3 (CBV **simulation** [PROOF OF LEMMA 5.3]). *If $t \to_{\mathsf{V}} s$ then $t^{\vee} \to_{\bullet}^{*} s^{\vee}$. Furthermore, the reduction uses at least one, and at most four $\to_{\bullet}$ steps.*

Proof. By induction on the derivation of $t \to_{\mathsf{V}} s$. The interesting cases are when there is a db, lsv, or gcv+ step at the root. The closure by compatibility under arbitrary contexts is straightforward by resorting to the IH:

1. $\mapsto_{\mathsf{db}}$: Let $(\lambda x.\, t)\mathtt{L}\ s \mapsto_{\mathsf{db}} t[x/s]\mathtt{L}$. Then:

$$\begin{array}{rl}
((\lambda x.\, t)\mathtt{L}\ s)^{\vee} = & \texttt{req}(u)[u/(!\bullet\lambda a.\, t^{\vee}[x/a])\mathtt{L}^{\vee}]\ s^{\vee} \\
\to_{\bullet\mathsf{ls}} & \texttt{req}(\bullet\lambda a.\, t^{\vee}[x/a])[u/!\bullet\lambda a.\, t^{\vee}[x/a]]\mathtt{L}^{\vee}\ s^{\vee} \\
\to_{\bullet\mathsf{req}} & (\lambda a.\, t^{\vee}[x/a])[u/!\bullet\lambda a.\, t^{\vee}[x/a]]\mathtt{L}^{\vee}\ s^{\vee} \\
\to_{\bullet\mathsf{db}} & t^{\vee}[x/s^{\vee}][u/!\bullet\lambda a.\, t^{\vee}[x/a]]\mathtt{L}^{\vee} \\
\to_{\bullet\mathsf{gc}} & t^{\vee}[x/s^{\vee}]\mathtt{L}^{\vee} \\
= & (t[x/s]\mathtt{L})^{\vee}
\end{array}$$

   To perform the •gc step, observe that in the second term in the sequence, namely the term $\texttt{req}(u)[u/(!\bullet\lambda a.\, t^{\vee}[x/a])\mathtt{L}^{\vee}]\ s^{\vee}$, the subterms $t^{\vee}$ and $s^{\vee}$ lie outside the scope of the bound variable u, so by α-conversion we may assume that $u \notin \mathsf{fv}(t^{\vee}[x/s^{\vee}])$.
2. $\mapsto_{\mathsf{lsv}}$: Let $\mathtt{C}\langle\langle x\rangle\rangle[x/\mathtt{v}\mathtt{L}] \mapsto_{\mathsf{lsv}} \mathtt{C}\langle\langle \mathtt{v}\rangle\rangle[x/\mathtt{v}]\mathtt{L}$. Recall that values are abstractions, so $\mathtt{v}$ is of the form $\mathtt{v} = \lambda y.\, t$, which means that in turn $\mathtt{v}^{\vee} = !\bullet s$, where $s = \lambda a.\, t^{\vee}[y/a]$ for a fresh linear variable a. Then:

$$\begin{array}{rl}
(\mathtt{C}\langle\langle x\rangle\rangle[x/\mathtt{v}\mathtt{L}])^{\vee} = & \mathtt{C}^{\vee}\langle\langle !x\rangle\rangle[x/\mathtt{v}^{\vee}\mathtt{L}^{\vee}] \\
= & \mathtt{C}^{\vee}\langle\langle !x\rangle\rangle[x/(!\bullet s)\mathtt{L}^{\vee}] \\
\to_{\bullet\mathsf{ls}} & \mathtt{C}^{\vee}\langle\langle !\bullet s\rangle\rangle[x/!\bullet s]\mathtt{L}^{\vee} \\
= & \mathtt{C}^{\vee}\langle\langle \mathtt{v}^{\vee}\rangle\rangle[x/\mathtt{v}^{\vee}]\mathtt{L}^{\vee} \\
= & (\mathtt{C}\langle\langle \mathtt{v}\rangle\rangle[x/\mathtt{v}]\mathtt{L})^{\vee}
\end{array}$$

3. $\mapsto_{\mathsf{gcv+}}$: Let $t[x/\mathsf{v}^{+}\mathsf{L}] \mapsto_{\mathsf{gcv+}} t\mathsf{L}$, where $x \notin \mathsf{fv}(t)$. Recall that lax values are either variables abstractions, so the lax value v^{+} is of either of the forms $\mathsf{v}^{+} = y$ or $\mathsf{v}^{+} = \lambda y.\, s$. In either case, this means that $\mathsf{v}^{+\mathsf{V}}$ must be of the form $\mathsf{v}^{+\mathsf{V}} = !r$ for some term $r \in \mathcal{T}_{\bullet}$. Then:
$$\begin{array}{rl} (t[x/\mathsf{v}^{+}\mathsf{L}])^{\mathsf{V}} = & t^{\mathsf{V}}[x/\mathsf{v}^{+\mathsf{V}}\mathsf{L}^{\mathsf{V}}] \\ = & t^{\mathsf{V}}[x/(!r)\mathsf{L}^{\mathsf{V}}] \\ \to_{\bullet\mathsf{gc}} & t^{\mathsf{V}}\mathsf{L}^{\mathsf{V}} \\ = & (t\mathsf{L})^{\mathsf{V}} \end{array}$$
To perform the $\bullet\mathsf{gc}$ step, note that $x \notin \mathsf{fv}(t^{\mathsf{V}})$ because $\mathsf{fv}(t^{\mathsf{V}}) = \mathsf{fv}(t)$, as noted in Rem. E.4.

Definition E.2. *We define a subset* $\mathsf{Ctxs}_{\bullet}^{\mathsf{V}} \subseteq \mathsf{Ctxs}_{\bullet}$*, called* CBV contexts *by the following grammar:*
$$\begin{array}{rl} \underline{\mathsf{C}} ::= & !\Box \\ & \mid\ !\bullet\lambda a.\, \underline{\mathsf{C}}[x/a] \\ & \mid\ \mathtt{req}(u)[u/\underline{\mathsf{C}}] \\ & \mid\ \mathtt{req}(\bullet\lambda a.\, \underline{\mathsf{C}}[x/a]) \\ & \mid\ \lambda a.\, \underline{\mathsf{C}}[x/a] \\ & \mid\ \underline{\mathsf{C}}\, \underline{t} \mid \underline{t}\, \underline{\mathsf{C}} \\ & \mid\ \underline{\mathsf{C}}[u/\underline{t}] \mid \underline{t}[u/\underline{\mathsf{C}}] \end{array}$$

Remark E.6. $\mathsf{fv}(\underline{t}^{-\mathsf{V}}) = \mathsf{fv}(\underline{t})$.

Remark E.7.

1. If $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ is of the form $\underline{t} = (\lambda a.\, s)\mathsf{L}$, then s is of the form $s = \underline{s}_1[u/a]$ where $\underline{s}_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and $a \notin \mathsf{fv}(\underline{s}_1)$ and $\mathsf{L} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$.
2. If $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ is of the form $\underline{t} = (!s)\mathsf{L}$, then exactly one of the following two holds:
   2.1 $\underline{t} = (!x)\underline{\mathsf{L}}$, where $\underline{\mathsf{L}} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$.
   2.2 $\underline{t} = (!\bullet\lambda a.\, \underline{t}'[x/a])\underline{\mathsf{L}}$, where $\underline{t}' \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, $\underline{\mathsf{L}} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$, and $a \notin \mathsf{fv}(\underline{t}')$.
3. If $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ is of the form $\underline{t} = (!(\bullet s)\mathsf{L}_1)\mathsf{L}_2$, then it is in fact of the form $\underline{t} = (!\bullet\lambda a.\, \underline{t}'[x/a])\underline{\mathsf{L}}$, where $\underline{t}' \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, $\underline{\mathsf{L}} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$, and $a \notin \mathsf{fv}(\underline{t}')$.

Lemma E.4 (Context decomposition for the inverse CBV translation).

1. $t\mathsf{L} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ *if and only if* $t \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ *and* $\mathsf{L} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$.
2. $\mathsf{C}\langle\!\langle x\rangle\!\rangle \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ *if and only if* $\mathsf{C} \in \mathsf{Ctxs}_{\bullet}^{\mathsf{V}}$.
3. *If* $\underline{\mathsf{C}} \in \mathsf{Ctxs}_{\bullet}^{\mathsf{V}}$ *and* $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ *and* $a \notin \mathsf{fv}(\underline{t})$*, then* $\underline{\mathsf{C}}\langle\!\langle\bullet\lambda a.\, \underline{t}[x/a]\rangle\!\rangle \in \mathcal{T}_{\bullet}^{\mathsf{V}}$.
4. *If* $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ *and* $\underline{\mathsf{L}} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$*, then* $(\underline{t}\underline{\mathsf{L}})^{-\mathsf{V}} = \underline{t}^{-\mathsf{V}}\, \underline{\mathsf{L}}^{-\mathsf{V}}$.
5. *If* $\underline{\mathsf{C}} \in \mathsf{Ctxs}_{\bullet}^{\mathsf{V}}$ *then* $\underline{\mathsf{C}}\langle\!\langle x\rangle\!\rangle^{-\mathsf{V}} = \underline{\mathsf{C}}^{-\mathsf{V}}\langle\!\langle x\rangle\!\rangle$.
6. *If* $\underline{\mathsf{C}} \in \mathsf{Ctxs}_{\bullet}^{\mathsf{V}}$ *and* $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ *and* $a \notin \mathsf{fv}(\underline{t})$*, then* $\underline{\mathsf{C}}\langle\!\langle\bullet\lambda a.\, \underline{t}[x/a]\rangle\!\rangle^{-\mathsf{V}} = \underline{\mathsf{C}}^{-\mathsf{V}}\langle\!\langle\lambda x.\, \underline{t}^{-\mathsf{V}}\rangle\!\rangle$.

Proof. By induction on the first judgement in the statement of each item, except the first and fourth items which are by induction on L and $\underline{\mathsf{L}} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{N}}$, resp.

Lemma E.5 (Inverse CBV simulation, up to gcv+ [PROOF OF LEMMA 5.4]). *Let* $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ *and* $s \in \mathcal{T}_{\bullet}$ *such that* $\underline{t} \to_{\bullet} s$*. Then* $s \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ *and* $\underline{t}^{-\mathsf{V}} \rhd_{\mathsf{V}}^{*} s^{-\mathsf{V}}$*, where* $\rhd_{\mathsf{V}} := (\to_{\mathsf{V}} \cup \to_{\mathsf{gcv+}}^{-1})$.

Proof. By induction on the (unique) derivation of $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$:

1. $\underline{t} = !x$: Impossible, as there are no steps $\underline{t} \to_{\bullet} s$.
2. $\underline{t} = !\bullet\lambda a.\,\underline{t}'[x/a]$ with $a \notin \mathsf{fv}(\underline{t}')$: Note that the step cannot be at the root of $\underline{t}$, nor at the root of $\bullet\lambda a.\,\underline{t}'[x/a]$, and that there cannot be a $\bullet\mathsf{ls}$ nor a $\bullet\mathsf{gc}$ step involving the substitution $[x/a]$, since these rules would require that a be of the form $(!r)\mathtt{L}$, but a is a linear variable. This means that the step must be internal to $\underline{t}'$, that is, $\underline{t} = !\bullet\lambda a.\,\underline{t}'[x/a] \to_{\bullet} !\bullet\lambda a.\,s'[x/a] = s$ with $\underline{t}' \to_{\bullet} s'$. By IH, $s' \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, so $s = !\bullet\lambda a.\,s'[x/a] \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, and $\underline{t}^{-\mathsf{V}} = \lambda x.\,\underline{t}'^{-\mathsf{V}} \rhd_{\mathsf{V}}^{*} \lambda x.\,s'^{-\mathsf{V}} = s^{-\mathsf{V}}$.
3. $\underline{t} = \mathtt{req}(u)[u/\underline{t}_1]$: Note that the step cannot be at the root of $\mathtt{req}(u)$, since this term does not match the left-hand side of the $\bullet\mathsf{req}$ rule. We consider two subcases, depending on whether the step is at the root of $\mathtt{req}(u)[u/\underline{t}_1]$ or internal to $\underline{t}_1$:
   3.1 If the step is at the root, note that it cannot be a $\to_{\bullet\mathsf{gc}}$ step, because there are free occurrences of u in $\mathtt{req}(u)$. Hence the step must be a $\to_{\bullet\mathsf{ls}}$ step. This in turn means that $\underline{t}_1$ must be of the form $\underline{t}_1 = (!(\bullet r)\mathtt{L}_1)\mathtt{L}_2$. But by construction of $\underline{t}$ we know that $\underline{t}_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, so by Rem. E.7 we know that it must be of the form $\underline{t}_1 = (!\bullet\lambda a.\,\underline{t}_1'[x/a])\underline{\mathtt{L}}$. Then the step is of the form:

$$\begin{array}{ll} & \underline{t} \\ = & \mathtt{req}(u)[u/(!\bullet\lambda a.\,\underline{t}_1'[x/a])\underline{\mathtt{L}}] \\ \to_{\bullet\mathsf{ls}} & \mathtt{req}(\bullet\lambda a.\,\underline{t}_1'[x/a])[u/!\bullet\lambda a.\,\underline{t}_1'[x/a]]\underline{\mathtt{L}} \\ = & s \end{array}$$

   Observe that $s \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and that:

$$\begin{array}{lll} & \underline{t}^{-\mathsf{V}} & \\ = & (\mathtt{req}(u)[u/(!\bullet\lambda a.\,\underline{t}_1'[x/a])\underline{\mathtt{L}}])^{-\mathsf{V}} & \\ = & (\lambda x.\,\underline{t}_1'^{-\mathsf{V}})\underline{\mathtt{L}}^{-\mathsf{V}} & \text{by Lem. E.4} \\ \to_{\mathsf{gcv+}}^{-1} & (\lambda x.\,\underline{t}_1'^{-\mathsf{V}})[u/\lambda x.\,\underline{t}_1'^{-\mathsf{V}}]\underline{\mathtt{L}}^{-\mathsf{V}} & \\ = & (\mathtt{req}(\bullet\lambda a.\,\underline{t}_1'[x/a])[u/!\bullet\lambda a.\,\underline{t}_1'[x/a]]\underline{\mathtt{L}})^{-\mathsf{V}} & \\ = & s^{-\mathsf{V}} & \end{array}$$

   To be able to perform the $\to_{\mathsf{gcv+}}^{-1}$ step, note that in the second term in the sequence, namely the term $(\mathtt{req}(u)[u/(!\bullet\lambda a.\,\underline{t}_1'[x/a])\underline{\mathtt{L}}])^{-\mathsf{V}}$, the subterm $\underline{t}_1'$ lies outside the scope of the bound variable u, so by α-conversion we may assume that $u \notin \mathsf{fv}(\underline{t}_1')$. This in turn implies that $u \notin \mathsf{fv}(\underline{t}_1'^{-\mathsf{V}})$, because $\mathsf{fv}(\underline{t}_1'^{-\mathsf{V}}) = \mathsf{fv}(\underline{t}_1')$ by Rem. E.6. Moreover, the $\to_{\mathsf{gcv+}}^{-1}$ step can be applied because $\lambda x.\,\underline{t}_1'^{-\mathsf{V}}$ is indeed a lax value.
   Remark: this is the only point in the proof in which a $\to_{\mathsf{gcv+}}^{-1}$ step is explicitly applied.
   3.2 If the step is internal to $\underline{t}_1$, then $\underline{t} = \mathtt{req}(u)[u/\underline{t}_1] \to_{\bullet} \mathtt{req}(u)[u/s_1] = s$ with $\underline{t}_1 \to_{\bullet} s_1$. By IH, $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, so $s = \mathtt{req}(u)[u/s_1] \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and $\underline{t}^{-\mathsf{V}} = (\mathtt{req}(u)[u/\underline{t}_1])^{-\mathsf{V}} = \underline{t}_1^{-\mathsf{V}} \rhd_{\mathsf{V}}^{*} s_1^{-\mathsf{V}} = (\mathtt{req}(u)[u/s_1])^{-\mathsf{V}} = s^{-\mathsf{V}}$.
4. $\underline{t} = \mathtt{req}(\bullet\lambda a.\,\underline{t}_1[x/a])$, with $a \notin \mathsf{fv}(\underline{t}_1)$: Note that there cannot be a $\bullet\mathsf{ls}$ nor a $\bullet\mathsf{gc}$ step involving the substitution $[x/a]$ since, since these rules would require that a be of the form $(!r)\mathtt{L}$, but a is a linear variable.
   We consider two subcases, depending on whether an $\bullet\mathsf{req}$ step is performed at the root or internal to $\underline{t}_1$:

4.1 If the step is an $\bullet\mathsf{req}$ step at the root, then $\underline{t} = \mathtt{req}(\bullet\lambda a.\, \underline{t}_1[x/a]) \to_{\bullet\mathsf{req}} \lambda a.\, \underline{t}_1[x/a] = s$, so $s \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and we have that

$$\begin{aligned} \underline{t}^{-\mathsf{V}} &= (\mathtt{req}(\bullet\lambda a.\, \underline{t}_1[x/a]))^{-\mathsf{V}} \\ &= \lambda x.\, \underline{t}_1^{-\mathsf{V}} \\ &= (\lambda a.\, \underline{t}_1[x/a])^{-\mathsf{V}} \\ &= s^{-\mathsf{V}} \end{aligned}$$

4.2 If the step is internal to $\underline{t}_1$, then $\underline{t} = \mathtt{req}(\bullet\lambda a.\, \underline{t}_1[x/a]) \to_{\bullet} \mathtt{req}(\bullet\lambda a.\, s_1[x/a]) = s$ with $\underline{t}_1 \to_{\bullet} s_1$. By IH, $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, so $s = \mathtt{req}(\bullet\lambda a.\, s_1[x/a]) \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and we have that

$$\begin{aligned} &\underline{t}^{-\mathsf{V}} \\ &= (\mathtt{req}(\bullet\lambda a.\, \underline{t}_1[x/a]))^{-\mathsf{V}} \\ &= \lambda x.\, \underline{t}_1^{-\mathsf{V}} \\ &\rhd_{\mathsf{V}}^{*} \lambda x.\, s_1^{-\mathsf{V}} \\ &= (\mathtt{req}(\bullet\lambda a.\, s_1[x/a]))^{-\mathsf{V}} \\ &= s^{-\mathsf{V}} \end{aligned}$$

5. $\underline{t} = \lambda a.\, \underline{t}_1[x/a]$, with $a \notin \mathsf{fv}(\underline{t}')$: Note that there cannot be a $\bullet\mathsf{ls}$ or a $\bullet\mathsf{gc}$ step involving the substitution $[x/a]$, since these rules would require that a be of the form $(!\underline{r})\mathsf{L}$, but a is a linear variable. Hence the step must be internal to $\underline{t}_1$. Then $\underline{t} = (\lambda a.\, \underline{t}_1[x/a])\underline{\mathsf{L}}\, \underline{t}_2 \to_{\bullet} (\lambda a.\, s_1[x/a])\underline{\mathsf{L}}\, \underline{t}_2 = \underline{s}$ with $\underline{t}_1 \to_{\bullet} s_1$ By IH, $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, so $\underline{s} = (\lambda a.\, s_1[x/a])\underline{\mathsf{L}}\, \underline{t}_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and we have that

$$\underline{t}^{-\mathsf{V}} = \lambda x.\, \underline{t}_1^{-\mathsf{V}} \rhd_{\mathsf{V}}^{*} \lambda x.\, s_1^{-\mathsf{V}} = \underline{s}^{-\mathsf{V}}$$

6. $\underline{t} = \underline{t}_1\, \underline{t}_2$: We consider three subcases, depending on whether the step is a $\bullet\mathsf{db}$ step at the root, internal to $\underline{t}_1$ or internal to $\underline{t}_2$:

6.1 If the step is a $\bullet\mathsf{db}$ step at the root of the term, then $\underline{t}_1$ must be of the form $\underline{t}_1 = \lambda a.\, \underline{t}_{11}[x/a])$ and $\underline{t} = (\lambda a.\, \underline{t}_{11}[x/a])\underline{\mathsf{L}}\, \underline{t}_2 \mapsto_{\bullet\mathsf{db}} \underline{t}_{11}[x/\underline{t}_2]\underline{\mathsf{L}} = s$, so $s \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and

$$\begin{aligned} &\underline{t}^{-\mathsf{V}} \\ &= (\lambda x.\, \underline{t}_{11}^{-\mathsf{V}})\underline{\mathsf{L}}^{-\mathsf{V}}\, \underline{t}_2^{-\mathsf{V}} \\ &\mapsto_{\mathsf{db}} \underline{t}_{11}^{-\mathsf{V}}[x/\underline{t}_2^{-\mathsf{V}}]\underline{\mathsf{L}}^{-\mathsf{V}} \\ &= (\underline{t}_{11}[x/\underline{t}_2]\underline{\mathsf{L}})^{-\mathsf{V}} \\ &= s^{-\mathsf{V}} \end{aligned}$$

6.2 If the step is internal to $\underline{t}_1$, then $\underline{t} = \underline{t}_1\, \underline{t}_2 \to_{\bullet} s_1\, \underline{t}_2 = s$ with $\underline{t}_1 \to_{\bullet} s_1$. By IH, $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and $\underline{t}_1^{-\mathsf{V}} \rhd_{\mathsf{V}}^{*} s_1^{-\mathsf{V}}$, so $s = s_1\, \underline{t}_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and we have that

$$\underline{t}^{-\mathsf{V}} = \underline{t}_1^{-\mathsf{V}}\, \underline{t}_2^{-\mathsf{V}} \rhd_{\mathsf{V}}^{*} s_1^{-\mathsf{V}}\, \underline{t}_2^{-\mathsf{V}} = s^{-\mathsf{V}}$$

6.3 If the step is internal to $\underline{t}_2$ then $\underline{t} = \underline{t}_1\, \underline{t}_2 \to_{\bullet} \underline{t}_1\, s_2 = s$ with $\underline{t}_2 \to_{\bullet} s_2$. By IH, $s_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and $\underline{t}_2^{-\mathsf{V}} \rhd_{\mathsf{V}}^{*} s_2^{-\mathsf{V}}$, so $s = \underline{t}_1\, s_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and we have that

$$\underline{t}^{-\mathsf{V}} = \underline{t}_1^{-\mathsf{V}}\, \underline{t}_2^{-\mathsf{V}} \rhd_{\mathsf{V}}^{*} \underline{t}_1^{-\mathsf{V}}\, s_2^{-\mathsf{V}} = s^{-\mathsf{V}}$$

7. $\underline{t} = \underline{t}_1[u/\underline{t}_2]$: We consider four cases, depending on whether there is a $\bullet\mathsf{ls}$ step at the root, or a $\bullet\mathsf{gc}$ step at the root, or whether the step is internal to $\underline{t}_1$ or $\underline{t}_2$:

7.1 If there is a $\bullet\mathsf{ls}$ step at the root, then $\underline{t}_2$ is of the form $\underline{t}_2 = (!(\bullet\underline{t}_2'')\mathsf{L}_1)\mathsf{L}_2$. By Rem. E.7, this implies that $\underline{t}_2$ must actually be of the form $\underline{t}_2 = (!\bullet\lambda a.\,\underline{t}_2'[x/a])\underline{\mathsf{L}}$ where $\underline{t}_2' \in \mathcal{T}_\bullet^{\mathsf{V}}$, $\underline{\mathsf{L}} \in \mathsf{SCtxs}_\bullet^{\mathsf{V}}$, and $a \notin \mathsf{fv}(\underline{t}_2')$. Moreover, $\underline{t}_1$ must be of the form $\underline{t}_1 = \mathsf{C}\langle\!\langle u\rangle\!\rangle$. The step is of the form:

$$\begin{array}{rl} & \underline{t} \\ = & \mathsf{C}\langle\!\langle u\rangle\!\rangle[u/(!\bullet\lambda a.\,\underline{t}_2'[x/a])\underline{\mathsf{L}}] \\ \mapsto_{\bullet\mathsf{ls}} & \mathsf{C}\langle\!\langle\bullet\lambda a.\,\underline{t}_2'[x/a]\rangle\!\rangle[u/!\bullet\lambda a.\,\underline{t}_2'[x/a]]\underline{\mathsf{L}} \\ = & s \end{array}$$

Note that $\underline{t}_1 = \mathsf{C}\langle\!\langle u\rangle\!\rangle \in \mathcal{T}_\bullet^{\mathsf{V}}$ so, by Lem. E.4, we have that $\mathsf{C} \in \mathsf{Ctxs}_\bullet^{\mathsf{V}}$ and, again by Lem. E.4, $\mathsf{C}\langle\!\langle\bullet\lambda a.\,\underline{t}_2'[x/a]\rangle\!\rangle \in \mathcal{T}_\bullet^{\mathsf{V}}$. This in turn implies that $s \in \mathcal{T}_\bullet^{\mathsf{V}}$. Moreover, using Lem. E.4, we have that:

$$\begin{array}{rl} & \underline{t}^{-\mathsf{V}} \\ = & (\mathsf{C}\langle\!\langle u\rangle\!\rangle[u/(!\bullet\lambda a.\,\underline{t}_2'[x/a])\underline{\mathsf{L}}])^{-\mathsf{V}} \\ = & \mathsf{C}^{-\mathsf{V}}\langle\!\langle u\rangle\!\rangle[u/(\lambda x.\,\underline{t}_2'^{-\mathsf{V}})\underline{\mathsf{L}}^{-\mathsf{V}}] \\ \mapsto_{\mathsf{lsv}} & \mathsf{C}^{-\mathsf{V}}\langle\!\langle\lambda x.\,\underline{t}_2'^{-\mathsf{V}}\rangle\!\rangle[u/\lambda x.\,\underline{t}_2'^{-\mathsf{V}}]\underline{\mathsf{L}}^{-\mathsf{V}} \\ = & (\mathsf{C}\langle\!\langle\bullet\lambda a.\,\underline{t}_2'[x/a]\rangle\!\rangle[u/!\bullet\lambda a.\,\underline{t}_2'[x/a]]\underline{\mathsf{L}})^{-\mathsf{V}} \\ = & s^{-\mathsf{V}} \end{array}$$

7.2 If there is a $\bullet\mathsf{gc}$ step at the root, we have that $u \notin \mathsf{fv}(\underline{t}_1)$, and that $\underline{t}_2$ is of the form $\underline{t}_2 = (!\underline{t}_2'')\mathsf{L}$. By Rem. E.7, this implies that $\underline{t}_2$ must actually be of either the form $\underline{t}_2 = (!y)\underline{\mathsf{L}}$ with $\underline{\mathsf{L}} \in \mathsf{SCtxs}_\bullet^{\mathsf{V}}$, or of the form $\underline{t}_2 = (!\bullet\lambda a.\,\underline{t}_2'[x/a])\underline{\mathsf{L}}$ with $\underline{t}_2' \in \mathcal{T}_\bullet^{\mathsf{V}}$, $\underline{\mathsf{L}} \in \mathsf{SCtxs}_\bullet^{\mathsf{V}}$, and $a \notin \mathsf{fv}(\underline{t}_2')$. In either case, $\underline{t}_2$ is of the form $\underline{t}_2 = (!\underline{t}_2')\underline{\mathsf{L}}$ where $!\underline{t}_2' \in \mathcal{T}_\bullet^{\mathsf{V}}$, $\underline{\mathsf{L}} \in \mathsf{SCtxs}_\bullet^{\mathsf{V}}$, and $(!\underline{t}_2')^{-\mathsf{V}}$ is a lax value, since both $(!y)^{-\mathsf{V}} = y$ and $(!\bullet\lambda a.\,\underline{t}_2'[x/a])^{-\mathsf{V}} = \lambda x.\,\underline{t}_2'^{-\mathsf{V}}$ are lax values.
Then the step is of the form:

$$\begin{array}{rl} & \underline{t} \\ = & \underline{t}_1[u/(!\underline{t}_2')\mathsf{L}'] \\ \mapsto_{\bullet\mathsf{gc}} & \underline{t}_1\mathsf{L}' \\ = & s \end{array}$$

So $s \in \mathcal{T}_\bullet^{\mathsf{V}}$ and:

$$\begin{array}{rl} & \underline{t}^{-\mathsf{V}} \\ = & (\underline{t}_1[u/(!\underline{t}_2')\mathsf{L}'])^{-\mathsf{V}} \\ = & \underline{t}_1^{-\mathsf{V}}[u/(!\underline{t}_2')^{-\mathsf{V}}\mathsf{L}'^{-\mathsf{V}}] \\ \mapsto_{\mathsf{gcv+}} & \underline{t}_1^{-\mathsf{V}}\mathsf{L}'^{-\mathsf{V}} \\ = & s^{-\mathsf{V}} \end{array}$$

To be able to perform the $\mathsf{gcv+}$ step, note that $u \notin \mathsf{fv}(\underline{t}_1^{-\mathsf{V}})$ because we know that $u \notin \mathsf{fv}(\underline{t}_1) = \mathsf{fv}(\underline{t}_1^{-\mathsf{V}})$ by Rem. E.6. To be able to perform the $\mathsf{gcv+}$ step, it must be also noted that, as already remarked, $(!\underline{t}_2')^{-\mathsf{V}}$ is a lax value.

7.3 If the step is internal to $\underline{t}_1$, then $\underline{t} = \underline{t}_1[u/\underline{t}_2] \to_\bullet s_1[u/\underline{t}_2] = s$ with $\underline{t}_1 \to_\bullet s_1$. By IH, $s_1 \in \mathcal{T}_\bullet^{\mathsf{V}}$, so $s = s_1[u/\underline{t}_2] \in \mathcal{T}_\bullet^{\mathsf{V}}$ and $\underline{t}^{-\mathsf{V}} = \underline{t}_1^{-\mathsf{V}}[u/\underline{t}_2^{-\mathsf{V}}] \rhd_{\mathsf{V}}^* s_1^{-\mathsf{V}}[u/\underline{t}_2^{-\mathsf{V}}] = s^{-\mathsf{V}}$.

7.4 If the step is internal to $\underline{t}_2$, then $\underline{t} = \underline{t}_1[u/\underline{t}_2] \to_\bullet \underline{t}_1[u/s_2] = s$ with $\underline{t}_2 \to_\bullet s_2$. By IH, $s_2 \in \mathcal{T}_\bullet^{\mathsf{V}}$, so $s = \underline{t}_1[u/s_2] \in \mathcal{T}_\bullet^{\mathsf{V}}$ and $\underline{t}^{-\mathsf{V}} = \underline{t}_1^{-\mathsf{V}}[u/\underline{t}_2^{-\mathsf{V}}] \rhd_{\mathsf{V}}^* s_1^{-\mathsf{V}}[u/s_2^{-\mathsf{V}}] = s^{-\mathsf{V}}$.

E.3 CBS

Proposition E.3 (CBS **typing** [proof of Prop. 5.3]). *If* $\Gamma \vdash t : A$ *then* $\Gamma^{\mathsf{S}}; \cdot \vdash t^{\mathsf{S}} : \bullet A^{\mathsf{S}}$.

Proof. By induction on the derivation of $\Gamma \vdash t : A$:

1. `l-var`: Let $\Gamma, x : A \vdash x : A$ be derived from the `l-var` rule. Then:

$$\frac{}{\Gamma^{\mathsf{S}}, x : A^{\mathsf{S}}; \cdot \vdash x : \bullet A^{\mathsf{S}}}\ \texttt{uvar}$$

2. `l-abs`: Let $\Gamma \vdash \lambda x.\, t : A \to B$ be derived from $\Gamma, x : A \vdash t : B$. Let a be a fresh linear variable, such that $a \notin \mathsf{fv}(t^{\mathsf{S}})$. Then:

$$\frac{\dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{S}}, x : A^{\mathsf{S}}; \cdot \vdash t^{\mathsf{S}} : \bullet B^{\mathsf{S}}} \qquad \dfrac{}{\Gamma^{\mathsf{S}}; a : !\bullet A^{\mathsf{S}} \vdash a : !\bullet A^{\mathsf{S}}}\ \texttt{lvar}}{\Gamma^{\mathsf{S}}; a : !\bullet A^{\mathsf{S}} \vdash t^{\mathsf{S}}[x/a] : \bullet B^{\mathsf{S}}}\ \texttt{sub}}{\Gamma^{\mathsf{S}}; \cdot \vdash \lambda a.\, t^{\mathsf{S}}[x/a] : !\bullet A^{\mathsf{S}} \multimap \bullet B^{\mathsf{S}}}\ \texttt{abs}}{\Gamma^{\mathsf{S}}; \cdot \vdash \bullet\lambda a.\, t^{\mathsf{S}}[x/a] : \bullet(!\bullet A^{\mathsf{S}} \multimap \bullet B^{\mathsf{S}})}\ \texttt{g}$$

3. `l-app`: Let $\Gamma \vdash t\, s : B$ be derived from $\Gamma \vdash t : A \to B$ and $\Gamma \vdash s : A$. Then:

$$\frac{\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{S}}; \cdot \vdash t^{\mathsf{S}} : \bullet(!\bullet A^{\mathsf{S}} \multimap \bullet B^{\mathsf{S}})}}{\Gamma^{\mathsf{S}}; \cdot \vdash \texttt{req}(t^{\mathsf{S}}) : !\bullet A^{\mathsf{S}} \multimap \bullet B^{\mathsf{S}}}\ \texttt{request} \qquad \dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{S}}; \cdot \vdash s^{\mathsf{S}} : \bullet A^{\mathsf{S}}}}{\Gamma^{\mathsf{S}}; \cdot \vdash !s^{\mathsf{S}} : !\bullet A^{\mathsf{S}}}\ \texttt{prom}}{\Gamma^{\mathsf{S}}; \cdot \vdash \texttt{req}(t^{\mathsf{S}})\, !s^{\mathsf{S}} : \bullet B^{\mathsf{S}}}\ \texttt{app}$$

4. `l-es`: Let $\Gamma \vdash t[x/s] : B$ be derived from $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. Then:

$$\frac{\dfrac{\text{IH}}{\Gamma^{\mathsf{S}}, x : A^{\mathsf{S}}; \cdot \vdash t^{\mathsf{S}} : \bullet B^{\mathsf{S}}} \qquad \dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{S}}; \cdot \vdash s^{\mathsf{S}} : \bullet A^{\mathsf{S}}}}{\Gamma^{\mathsf{S}}; \cdot \vdash !s^{\mathsf{S}} : !\bullet A^{\mathsf{S}}}\ \texttt{prom}}{\Gamma^{\mathsf{S}}; \cdot \vdash t^{\mathsf{S}}[x/!s^{\mathsf{S}}] : \bullet B^{\mathsf{S}}}\ \texttt{sub}$$

The translation can be extended to operate on contexts, by declaring $\Box^{\mathsf{S}} := \Box$. Note that the translation of a substitution context is a substitution context:

$$(\Box[x_1/t_1]\ldots[x_n/t_n])^{\mathsf{S}} = \Box[x_1/!t_1^{\mathsf{S}}]\ldots[x_n/!t_n^{\mathsf{S}}]$$

Remark E.8. $\mathsf{fv}(t^{\mathsf{S}}) = \mathsf{fv}(t)$

Remark E.9. $\mathsf{C}\langle t\rangle^{\mathsf{S}} = \mathsf{C}^{\mathsf{S}}\langle t^{\mathsf{S}}\rangle$ and $\mathsf{C}\langle\!\langle t\rangle\!\rangle^{\mathsf{S}} = \mathsf{C}^{\mathsf{S}}\langle\!\langle t^{\mathsf{S}}\rangle\!\rangle$. In particular, $(t\mathsf{L})^{\mathsf{S}} = t^{\mathsf{S}}\mathsf{L}^{\mathsf{S}}$.

Definition E.3. *We define a subset* $\mathsf{Ctxs}^{\mathsf{S}}_{\bullet} \subseteq \mathsf{Ctxs}_{\bullet}$, *called* CBS contexts*:*

$$\begin{array}{rl}\underline{\mathsf{C}} ::= & \Box \\ & \mid \bullet\lambda a.\, \underline{\mathsf{C}}[x/a] \\ & \mid \texttt{req}(\underline{\mathsf{C}}) \\ & \mid \lambda a.\, \underline{\mathsf{C}}[x/a] \\ & \mid \underline{\mathsf{C}}[u/!\underline{t}] \mid \underline{t}[u/!\underline{\mathsf{C}}] \\ & \mid \underline{\mathsf{C}}\, !\underline{t} \mid \underline{t}\, !\underline{\mathsf{C}}\end{array}$$

Remark E.10. $\mathsf{fv}(\underline{t}^{-\mathsf{S}}) = \mathsf{fv}(\underline{t})$.

Remark E.11.

1. If $t \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ is such that $t = (\lambda a. t')\mathtt{L}$, then $\mathtt{L} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{S}}$ and $t' = \underline{s}[u/a]$ for some $\underline{s} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $a \notin \mathsf{fv}(\underline{s})$.
2. If $\bullet t \in \mathcal{T}_{\bullet}^{\mathsf{S}}$, then $t = \lambda a. \underline{s}[x/a]$, for some $\underline{s} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $a \notin \mathsf{fv}(\underline{s})$.

Lemma E.6 (Context decomposition for the inverse CBS translation).

1. *$t\mathtt{L} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ if and only if $t \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\mathtt{L} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{S}}$.*
2. *$\mathtt{C}\langle\langle x\rangle\rangle \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ if and only if $\mathtt{C} \in \mathsf{Ctxs}_{\bullet}^{\mathsf{S}}$.*
3. *If $\underline{\mathtt{C}} \in \mathsf{Ctxs}_{\bullet}^{\mathsf{S}}$ and $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ then $\underline{\mathtt{C}}\langle\langle \underline{t}\rangle\rangle \in \mathcal{T}_{\bullet}^{\mathsf{S}}$.*
4. *If $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\underline{\mathtt{L}} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{S}}$, then $(\underline{t}\underline{\mathtt{L}})^{-\mathsf{S}} = \underline{t}^{-\mathsf{S}}\,\underline{\mathtt{L}}^{-\mathsf{S}}$.*
5. *If $\underline{\mathtt{C}} \in \mathsf{Ctxs}_{\bullet}^{\mathsf{S}}$ then $\underline{\mathtt{C}}\langle\langle x\rangle\rangle^{-\mathsf{S}} = \underline{\mathtt{C}}^{-\mathsf{S}}\langle\langle x\rangle\rangle$.*
6. *If $\underline{\mathtt{C}} \in \mathsf{Ctxs}_{\bullet}^{\mathsf{S}}$ and $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\underline{\mathtt{L}} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{S}}$ and $a \notin \mathsf{fv}(\underline{t})$, then $\underline{\mathtt{C}}\langle\langle(\bullet\lambda a. \underline{t}[x/a])\underline{\mathtt{L}}\rangle\rangle^{-\mathsf{S}} = \underline{\mathtt{C}}^{-\mathsf{S}}\langle\langle(\lambda x. \underline{t}^{-\mathsf{S}})\underline{\mathtt{L}}^{-\mathsf{S}}\rangle\rangle$.*

Proof. By induction on the first judgement in the statement of each item, except the first and fourth items which are by induction on $\mathtt{L}$ and $\underline{\mathtt{L}} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{N}}$, resp.

Lemma E.7 (Inverse CBS simulation [proof of lemma 5.6]). *Let $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $s \in \mathcal{T}_{\bullet}$ such that $\underline{t} \to_{\bullet} s$. Then $s \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\underline{t}^{-\mathsf{S}} \to_{\mathsf{Nd}}^{=} s^{-\mathsf{S}}$.*

Proof. By induction on the (unique) derivation of $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$:

1. $\underline{t} = x$: Impossible, as there are no steps $\underline{t} \to_{\bullet} s$.
2. $\underline{t} = \bullet\lambda a. \underline{t}'[x/a]$ with $a \notin \mathsf{fv}(\underline{t}')$: Note that the step cannot be at the root of $\underline{t}$, and that there cannot be a $\bullet\mathsf{ls}$ nor a $\bullet\mathsf{gc}$ step involving the substitution $[x/a]$, since these rules would require that a be of the form $(!r)\mathtt{L}$, but a is a (linear) variable. This means that the step must be internal to $\underline{t}'$, that is, $\underline{t} = \bullet\lambda a. \underline{t}'[x/a] \to_{\bullet} \bullet\lambda a. s'[x/a] = s$ with $\underline{t}' \to_{\bullet} s'$. By IH, $s' \in \mathcal{T}_{\bullet}^{\mathsf{S}}$, so $s = \bullet\lambda a. s'[x/a] \in \mathcal{T}_{\bullet}^{\mathsf{S}}$, and $\underline{t}^{-\mathsf{S}} = \lambda x. \underline{t}'^{-\mathsf{S}} \to_{\mathsf{Nd}}^{=} \lambda x. s'^{-\mathsf{S}} = s^{-\mathsf{S}}$.
3. $\underline{t} = \mathtt{req}(\underline{t}_1)$: We consider two subcases, depending on whether the step is at the root or internal to $\underline{t}_1$:
   3.1 If the step is at the root of $\mathtt{req}(\underline{t}_1)$, then $\underline{t}_1$ must be of the form $(\bullet t_1')\underline{\mathtt{L}}$ and, by Rem. E.11, t_1' must be of the form $\lambda a. \underline{t}_1''[x/a]$. Thus the step is
   $$\begin{aligned} & \underline{t} \\ &= \mathtt{req}((\bullet\lambda a. \underline{t}_1''[x/a])\underline{\mathtt{L}}) \\ \to_{\bullet\mathsf{req}}\ & (\lambda a. \underline{t}_1''[x/a])\underline{\mathtt{L}} \\ &= s \end{aligned}$$
   so $s \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and, using Lem. E.6, we have that
   $$\begin{aligned} \underline{t}^{-\mathsf{S}} &= (\mathtt{req}((\bullet\lambda a. \underline{t}_1''[x/a])\underline{\mathtt{L}}))^{-\mathsf{S}} \\ &= (\lambda x. \underline{t}_1''^{-\mathsf{S}})\underline{\mathtt{L}}^{-\mathsf{S}} \\ &= ((\lambda a. \underline{t}_1''[x/a])\underline{\mathtt{L}})^{-\mathsf{S}} \\ &= s^{-\mathsf{S}} \end{aligned}$$
   3.2 If the step is internal to $\underline{t}_1$, then $\underline{t} = \mathtt{req}(\underline{t}_1) \to_{\bullet} \mathtt{req}(s_1) = s$ with $\underline{t}_1 \to_{\bullet} s_1$. By IH, $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$, so $s = \mathtt{req}(s_1) \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\underline{t}^{-\mathsf{S}} = \mathtt{req}(\underline{t}_1)^{-\mathsf{S}} = \underline{t}_1^{-\mathsf{S}} \to_{\mathsf{Nd}}^{=} s_1^{-\mathsf{S}} = \mathtt{req}(s_1)^{-\mathsf{S}} = s^{-\mathsf{S}}$.

4. $\underline{t} = \lambda a.\underline{t}_1[x/a]$, with $a \notin \mathsf{fv}(\underline{t}_1)$: Note that there cannot be a $\bullet\mathsf{ls}$ or a $\bullet\mathsf{gc}$ step involving the substitution $[x/a]$, since these rules would require that a be of the form $!\underline{r}\mathsf{L}$, but a is a (linear) variable. Hence the step must be internal to $\underline{t}_1$. Then $\underline{t} = \lambda a.\underline{t}_1[x/a] \to_\bullet \lambda a.s_1[x/a] = \underline{s}$ with $\underline{t}_1 \to_\bullet s_1$. By IH, $s_1 \in \mathcal{T}^{\mathsf{S}}_\bullet$, so $\underline{s} = (\lambda a.s_1[x/a])\underline{\mathsf{L}}\ !\underline{t}_2 \in \mathcal{T}^{\mathsf{S}}_\bullet$ and we have that $\underline{t}^{-\mathsf{S}} = \lambda x.\underline{t}_1^{-\mathsf{S}} \to^{=}_{\mathsf{Nd}} \lambda x.s_1^{-\mathsf{S}} = \underline{s}^{-\mathsf{S}}$.
5. $\underline{t} = \underline{t}_1[u/!\underline{t}_2]$: Note that the reduction step cannot be at the root of $!\underline{t}_2$, since there are no rules in $\lambda^{!\bullet}$ that have a $!$ in the root of their left-hand sides. We consider four cases, depending on whether there is a $\bullet\mathsf{ls}$ step at the root, or a $\bullet\mathsf{gc}$ step at the root, or whether the step is internal to $\underline{t}_1$ or $\underline{t}_2$:
   5.1 If there is a $\bullet\mathsf{ls}$ step at the root, then $\underline{t}_2$ is of the form $\underline{t}_2 = (\bullet t_2'')\underline{\mathsf{L}}$. By Rem. E.11, this implies that t_2'' must actually be of the form $\lambda a.\underline{t}_2'[x/a]$ where $\underline{t}_2' \in \mathcal{T}^{\mathsf{S}}_\bullet$ and $a \notin \mathsf{fv}(\underline{t}_2')$. Moreover, $\underline{t}_1$ must be of the form $\underline{t}_1 = \mathsf{C}\langle\!\langle u\rangle\!\rangle$. The step is of the form:

$$\begin{array}{rl} \underline{t} = & \mathsf{C}\langle\!\langle u\rangle\!\rangle[u/!(\bullet(\lambda a.\underline{t}_2'[x/a])\underline{\mathsf{L}})] \\ \mapsto_{\bullet\mathsf{ls}} & \mathsf{C}\langle\!\langle(\bullet\lambda a.\underline{t}_2'[x/a])\underline{\mathsf{L}}\rangle\!\rangle[u/!(\bullet(\lambda a.\underline{t}_2'[x/a]))\underline{\mathsf{L}}] \\ = & s \end{array}$$

   Note that $\underline{t}_1 = \mathsf{C}\langle\!\langle u\rangle\!\rangle \in \mathcal{T}^{\mathsf{S}}_\bullet$ so, by Lem. E.6, we have that $\mathsf{C} \in \mathsf{Ctxs}^{\mathsf{S}}_\bullet$ and, again by Lem. E.6, $\mathsf{C}\langle\!\langle(\bullet\lambda a.\underline{t}_2'[x/a])\underline{\mathsf{L}}\rangle\!\rangle \in \mathcal{T}^{\mathsf{S}}_\bullet$. This in turn implies that $s \in \mathcal{T}^{\mathsf{S}}_\bullet$. Moreover, using Lem. E.6, we have that:

$$\begin{array}{rl} \underline{t}^{-\mathsf{S}} = & (\mathsf{C}\langle\!\langle u\rangle\!\rangle[u/!(\bullet(\lambda a.\underline{t}_2'[x/a])\underline{\mathsf{L}})])^{-\mathsf{S}} \\ = & \mathsf{C}^{-\mathsf{S}}\langle\!\langle u\rangle\!\rangle[u/(\lambda x.\underline{t}_2'^{-\mathsf{S}})\underline{\mathsf{L}}^{-\mathsf{S}}] \\ \mapsto_{\mathsf{lsw}} & \mathsf{C}^{-\mathsf{S}}\langle\!\langle(\lambda x.\underline{t}_2'^{-\mathsf{S}})\underline{\mathsf{L}}^{-\mathsf{S}}\rangle\!\rangle[u/(\lambda x.\underline{t}_2'^{-\mathsf{S}})\underline{\mathsf{L}}^{-\mathsf{S}}] \\ = & (\mathsf{C}\langle\!\langle(\bullet\lambda a.\underline{t}_2'[x/a])\underline{\mathsf{L}}\rangle\!\rangle[u/!(\bullet\lambda a.\underline{t}_2'[x/a])\underline{\mathsf{L}}])^{-\mathsf{S}} \\ = & s^{-\mathsf{S}} \end{array}$$

   5.2 If there is a $\bullet\mathsf{gc}$ step at the root, we have that $u \notin \mathsf{fv}(\underline{t}_1)$, and the step is of the form:

$$\begin{array}{rl} \underline{t} = & \underline{t}_1[u/!\underline{t}_2] \\ \mapsto_{\bullet\mathsf{gc}} & \underline{t}_1 \\ = & s \end{array}$$

   So $s \in \mathcal{T}^{\mathsf{S}}_\bullet$ and:

$$\begin{array}{rl} \underline{t}^{-\mathsf{S}} = & (\underline{t}_1[u/!\underline{t}_2])^{-\mathsf{S}} \\ = & \underline{t}_1^{-\mathsf{S}}[u/\underline{t}_2^{-\mathsf{S}}] \\ \mapsto_{\mathsf{gc}} & \underline{t}_1^{-\mathsf{S}} \\ = & s^{-\mathsf{S}} \end{array}$$

   To be able to perform the gc step, note that $u \notin \mathsf{fv}(\underline{t}_1^{-\mathsf{S}})$ because we know that $u \notin \mathsf{fv}(\underline{t}_1) = \mathsf{fv}(\underline{t}_1^{-\mathsf{S}})$ by Rem. E.10.
   5.3 If the step is internal to $\underline{t}_1$, then $\underline{t} = \underline{t}_1[u/!\underline{t}_2] \to_\bullet s_1[u/!\underline{t}_2] = s$ with $\underline{t}_1 \to_\bullet s_1$. By IH, $s_1 \in \mathcal{T}^{\mathsf{S}}_\bullet$, so $s = s_1[u/!\underline{t}_2] \in \mathcal{T}^{\mathsf{S}}_\bullet$ and $\underline{t}^{-\mathsf{S}} = \underline{t}_1^{-\mathsf{S}}[u/\underline{t}_2^{-\mathsf{S}}] \to^{=}_{\mathsf{Nd}} s_1^{-\mathsf{S}}[u/\underline{t}_2^{-\mathsf{S}}] = s^{-\mathsf{S}}$.
   5.4 If the step is internal to $\underline{t}_2$, then $\underline{t} = \underline{t}_1[u/!\underline{t}_2] \to_\bullet \underline{t}_1[u/s_2] = s$ with $\underline{t}_2 \to_\bullet s_2$. By IH, $s_2 \in \mathcal{T}^{\mathsf{S}}_\bullet$, so $s = \underline{t}_1[u/!s_2] \in \mathcal{T}^{\mathsf{S}}_\bullet$ and $\underline{t}^{-\mathsf{S}} = \underline{t}_1^{-\mathsf{S}}[u/\underline{t}_2^{-\mathsf{S}}] \to^{=}_{\mathsf{Nd}} s_1^{-\mathsf{S}}[u/s_2^{-\mathsf{S}}] = s^{-\mathsf{S}}$.

6. $\underline{t} = \underline{t}_1 \,!\underline{t}_2$: We consider three subcases, depending on whether the step is at the root, internal to $\underline{t}_1$, or internal to $\underline{t}_2$:
   6.1 If the step is at the root, it must be a $\bullet\mathsf{db}$ step. Hence $\underline{t}_1$ is of the form $(\lambda a.\, r)\mathsf{L}$. Then by Rem. E.11 $\mathsf{L} \in \mathsf{SCtxs}^{\mathsf{S}}_{\bullet}$ and $r = \underline{r}_1[u/a]$ for some $\underline{r}_1 \in \mathcal{T}^{\mathsf{S}}_{\bullet}$ such that $a \notin \mathsf{fv}(\underline{r}_1)$. The step is of the form $\underline{t} = (\lambda a.\, \underline{r}_1[u/a])\mathsf{L} \,!\underline{t}_2 \to_{\bullet} \underline{r}_1[u/!\underline{t}_2]\mathsf{L} = s$. Note that $s \in \mathcal{T}^{\mathsf{S}}_{\bullet}$ and $\underline{t}^{-\mathsf{V}} = (\lambda u.\, \underline{r}_1^{-\mathsf{V}})\mathsf{L}^{-\mathsf{V}}\, \underline{t}_2^{-\mathsf{V}} \to_{\mathsf{Nd}} \underline{r}_1^{-\mathsf{V}}[u/\underline{t}_2^{-\mathsf{V}}]\mathsf{L}^{-\mathsf{V}} = s^{-\mathsf{V}}$.
   6.2 If the step is internal to $\underline{t}_1$, the step is of the form $\underline{t} = \underline{t}_1 \,!\underline{t}_2 \to_{\bullet} s_1 \,!\underline{t}_2 = s$, with $\underline{t}_1 \to_{\bullet} s_1$. By IH we have that $s_1 \in \mathcal{T}^{\mathsf{S}}_{\bullet}$ and $\underline{t}_1^{-\mathsf{S}} \to^{=}_{\mathsf{Nd}} s_1^{-\mathsf{S}}$, so $s = s_1 \,!\underline{t}_2 \in \mathcal{T}^{\mathsf{S}}_{\bullet}$ and $\underline{t}^{-\mathsf{S}} = \underline{t}_1^{-\mathsf{S}}\, \underline{t}_2^{-\mathsf{S}} \to^{=}_{\mathsf{Nd}} s_1^{-\mathsf{S}}\, \underline{t}_2^{-\mathsf{S}} = s^{-\mathsf{S}}$.
   6.3 If the step is internal to $\underline{t}_2$, the step is of the form $\underline{t} = \underline{t}_1 \,!\underline{t}_2 \to_{\bullet} \underline{t}_1 \,!s_2 = s$, with $\underline{t}_2 \to_{\bullet} s_2$. By IH we have that $s_2 \in \mathcal{T}^{\mathsf{S}}_{\bullet}$ and $\underline{t}_2^{-\mathsf{S}} \to^{=}_{\mathsf{Nd}} s_2^{-\mathsf{S}}$, so $s = \underline{t}_1 \,!s_2 \in \mathcal{T}^{\mathsf{S}}_{\bullet}$ and $\underline{t}^{-\mathsf{S}} = \underline{t}_1^{-\mathsf{S}}\, \underline{t}_2^{-\mathsf{S}} \to^{=}_{\mathsf{Nd}} \underline{t}_1^{-\mathsf{S}}\, s_2^{-\mathsf{S}} = s^{-\mathsf{S}}$.

F Appendix: Simulating Weak Evaluation

Definition F.1. *We write $t \rightsquigarrow^{*}_{\langle \rho_1,\dots,\rho_n\rangle} t'$ if $t = t_0 \rightsquigarrow_{\rho_1} t_1 \rightsquigarrow_{\rho_2} \dots \rightsquigarrow_{\rho_n} t_n = t'$ holds for terms $t_0, \dots, t_n$.*

We write $\mathcal{R}_{\bullet}$ for the set of all possible rulenames:

$$\mathcal{R}_{\bullet} := \{\bullet\mathsf{db}, \bullet\mathsf{ls}, \bullet\mathsf{gc}\} \cup \{\varsigma(u, (\bullet t)\mathsf{L}) \mid \text{varying } u, t, \mathsf{L}\} \cup \{\iota(u) \mid \text{varying } u\}$$

Definition F.2 (CBN **Rulenames).** *We define a subset $\mathcal{R}^{\mathsf{N}}_{\bullet} \subseteq \mathcal{R}_{\bullet}$ by the following grammar:*

$$\underline{\rho} ::= \bullet\mathsf{db} \mid \varsigma(x, \bullet\underline{t}) \mid \bullet\mathsf{ls} \mid \bullet\mathsf{gc} \mid \bullet\mathsf{req}$$

where $\underline{t}$ stands for a term in $\mathcal{T}^{\mathsf{N}}_{\bullet}$. The translation of a CBN*-rulename is a sequence of $\mathcal{R}^{\mathsf{N}}_{\bullet}$ rulenames, given by:*

$$\begin{aligned}
\mathsf{db}^{\mathsf{N}} &:= \langle \bullet\mathsf{db} \rangle \\
\varsigma(x,t)^{\mathsf{N}} &:= \langle \varsigma(x, \bullet t^{\mathsf{N}}), \bullet\mathsf{req} \rangle \\
\mathsf{ls}^{\mathsf{N}} &:= \langle \bullet\mathsf{ls}, \bullet\mathsf{req} \rangle \\
\mathsf{gc}^{\mathsf{N}} &:= \langle \bullet\mathsf{gc} \rangle
\end{aligned}$$

The inverse translation of a $\mathcal{R}^{\mathsf{N}}_{\bullet}$ rulename is a sequence of CBN*-rulenames, given by:*

$$\begin{aligned}
\bullet\mathsf{db}^{-\mathsf{N}} &:= \langle \mathsf{db} \rangle \\
\varsigma(x, \bullet\underline{t})^{-\mathsf{N}} &:= \langle \varsigma(x, \underline{t}^{-\mathsf{N}}) \rangle \\
\bullet\mathsf{ls}^{-\mathsf{N}} &:= \langle \mathsf{ls} \rangle \\
\bullet\mathsf{gc}^{-\mathsf{N}} &:= \langle \mathsf{gc} \rangle \\
\bullet\mathsf{req}^{-\mathsf{N}} &:= \langle \rangle
\end{aligned}$$

Remark F.1. $\mathsf{fv}(\underline{\rho}^{-\mathsf{N}}) = \mathsf{fv}(\underline{\rho})$ is an immediate consequence of Rem. E.3.

Lemma F.1 (CBN **Evaluation – Simulation).** *If $t \rightsquigarrow^{\mathsf{N}}_{\rho} t'$ then $t^{\mathsf{N}} \rightsquigarrow^{*}_{\rho^{\mathsf{N}}} t'^{\mathsf{N}}$.*

Proof. By induction on the derivation of the step $t \rightsquigarrow^{\mathsf{N}}_{\rho} t'$:

1. $\mathtt{E}^{\mathsf{N}}$-db: Let $(\lambda x.\,t)\mathtt{L}\ s \rightsquigarrow^{\mathsf{N}}_{\mathsf{db}} t[x/s]\mathtt{L}$. Then:

$$\begin{array}{lll} ((\lambda x.\,t)\mathtt{L}\ s)^{\mathsf{N}} = & (\lambda a.\,t^{\mathsf{N}}[x/a])\mathtt{L}^{\mathsf{N}}\ !\bullet s^{\mathsf{N}} & \\ \rightsquigarrow_{\bullet\mathsf{db}} & t^{\mathsf{N}}[x/!\bullet s^{\mathsf{N}}]\mathtt{L}^{\mathsf{N}} & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{db} \\ = & (t[x/s]\mathtt{L})^{\mathsf{N}} & \end{array}$$

2. $\mathtt{E}^{\mathsf{N}}$-ς: Let $x \rightsquigarrow^{\mathsf{N}}_{\varsigma(x,t)} t$. Then:

$$\begin{array}{lll} x^{\mathsf{N}} = & \mathtt{req}(x) & \\ \rightsquigarrow_{\varsigma(x,\bullet t^{\mathsf{N}})} & \mathtt{req}(\bullet t^{\mathsf{N}}) & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{req}, \mathtt{E}^{\bullet}\text{-}\varsigma \\ \rightsquigarrow_{\bullet\mathsf{req}} & t^{\mathsf{N}} & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{req}\bullet \end{array}$$

3. $\mathtt{E}^{\mathsf{N}}$-$\mathtt{ls}$: Let $t[x/s] \rightsquigarrow^{\mathsf{N}}_{\mathsf{ls}} t'[x/s]$ be derived from $t \rightsquigarrow^{\mathsf{N}}_{\varsigma(x,s)} t'$. By IH we have that $t^{\mathsf{N}} \rightsquigarrow^{*}_{\varsigma(x,s)^{\mathsf{N}}} t'^{\mathsf{N}}$. Since $\varsigma(x,s)^{\mathsf{N}} = \langle\varsigma(x,\bullet s^{\mathsf{N}}), \bullet\mathsf{req}\rangle$, this means there exists a term $r \in \mathcal{T}_{\bullet}$ such that $t^{\mathsf{N}} \rightsquigarrow_{\varsigma(x,\bullet s^{\mathsf{N}})} r \rightsquigarrow_{\bullet\mathsf{req}} t'^{\mathsf{N}}$. Hence:

$$\begin{array}{lll} t[x/s]^{\mathsf{N}} = & t^{\mathsf{N}}[x/!\bullet s^{\mathsf{N}}] & \\ \rightsquigarrow_{\bullet\mathsf{ls}} & r[x/!\bullet s^{\mathsf{N}}] & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{ls} \text{ since } t^{\mathsf{N}} \rightsquigarrow_{\varsigma(x,\bullet s^{\mathsf{N}})} r \\ \rightsquigarrow_{\bullet\mathsf{req}} & t'^{\mathsf{N}}[x/!\bullet s^{\mathsf{N}}] & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{esL} \text{ since } r \rightsquigarrow_{\bullet\mathsf{req}} t'^{\mathsf{N}} \\ = & (t'[x/s])^{\mathsf{N}} & \end{array}$$

4. $\mathtt{E}^{\mathsf{N}}$-$\mathtt{gc}$: Let $t[x/s] \rightsquigarrow^{\mathsf{N}}_{\mathsf{gc}} t$ where $x \notin \mathsf{fv}(t)$. Recall that the translation does not create free variables (Rem. E.1), so $x \notin \mathsf{fv}(t^{\mathsf{N}})$. Hence:

$$\begin{array}{ll} (t[x/s])^{\mathsf{N}} = & t^{\mathsf{N}}[x/!\bullet s^{\mathsf{N}}] \\ \rightsquigarrow_{\bullet\mathsf{gc}} & t^{\mathsf{N}} \end{array}$$

5. $\mathtt{E}^{\mathsf{N}}$-$\mathtt{app}$: Let $t\ s \rightsquigarrow^{\mathsf{N}}_{\rho} t'\ s$ be derived from $t \rightsquigarrow^{\mathsf{N}}_{\rho} t'$. By IH $t^{\mathsf{N}} \rightsquigarrow^{*}_{\rho^{\mathsf{N}}} t'^{\mathsf{N}}$. Hence:

$$\begin{array}{lll} (t\ s)^{\mathsf{N}} = & t^{\mathsf{N}}\ !\bullet s^{\mathsf{N}} & \\ \rightsquigarrow^{*}_{\rho^{\mathsf{N}}} & t'^{\mathsf{N}}\ !\bullet s^{\mathsf{N}} & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{app} \text{ (many times)} \\ = & (t'\ s)^{\mathsf{N}} & \end{array}$$

6. $\mathtt{E}^{\mathsf{N}}$-$\mathtt{subL}$: Let $t[x/s] \rightsquigarrow^{\mathsf{N}}_{\rho} t'[x/s]$ be derived from $t \rightsquigarrow^{\mathsf{N}}_{\rho} t'$, where $x \notin \mathsf{fv}(\rho)$. By IH, $t^{\mathsf{N}} \rightsquigarrow^{*}_{\rho^{\mathsf{N}}} t'^{\mathsf{N}}$. Recall that the translation does not create free variables (Rem. E.1), so $x \notin \mathsf{fv}(\rho)$ implies $x \notin \mathsf{fv}(\rho^{\mathsf{N}})$. Hence:

$$\begin{array}{lll} (t[x/s])^{\mathsf{N}} = & t^{\mathsf{N}}[x/!\bullet s^{\mathsf{N}}] & \\ \rightsquigarrow^{*}_{\rho^{\mathsf{N}}} & t'^{\mathsf{N}}[x/!\bullet s^{\mathsf{N}}] & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{esL} \text{ (many times), since } x \notin \mathsf{fv}(\rho^{\mathsf{N}}) \\ = & (t'[x/s])^{\mathsf{N}} & \end{array}$$

Lemma F.2 (CBN **Evaluation – Inverse Simulation).** *Let* $\underline{t} \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, $\underline{\rho} \in \mathcal{R}^{\mathsf{N}}_{\bullet}$ *and* $s \in \mathcal{T}_{\bullet}$ *such that* $\underline{t} \rightsquigarrow_{\underline{\rho}} s$. *Then* $s \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ *and* $\underline{t}^{-\mathsf{N}}\ (\rightsquigarrow^{\mathsf{N}})^{*}_{\underline{\rho}^{-\mathsf{N}}}\ s^{-\mathsf{N}}$.

Proof. By induction on $\underline{t}$:

1. Translation of a variable before substitution, $\underline{t} = \mathtt{req}(u)$: The step is of the form $\underline{t} = \mathtt{req}(u) \rightsquigarrow_{\underline{\rho}} s$ and must be derived using the $\mathtt{E}^{\bullet}\text{-}\mathtt{req}$ rule from an internal step $u \rightsquigarrow_{\underline{\rho}} s'$. The internal step cannot be derived using the $\mathtt{E}^{\bullet}\text{-}\iota$ rule because $\underline{\rho} \in \mathcal{R}_{\bullet}^{\mathsf{N}}$ so $\underline{\rho}$ cannot be of the form $\iota(u)$. This means that the internal step can only be derived using and $\mathtt{E}^{\bullet}\text{-}\varsigma$, where, again, we know that $\underline{\rho} = \varsigma(u, \bullet\underline{r})$ because $\underline{\rho} \in \mathcal{R}_{\bullet}^{\mathsf{N}}$. Hence the step must be of the form $\mathtt{req}(u) \rightsquigarrow_{\varsigma(u,\bullet\underline{r})} \mathtt{req}(\bullet\underline{r}) = s$. Note that $s = \mathtt{req}(\bullet\underline{r}) \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and:
$$\underline{t}^{-\mathsf{N}} = \mathtt{req}(u)^{-\mathsf{N}} = u \rightsquigarrow^{\mathsf{N}}_{\varsigma(u,\underline{r}^{-\mathsf{N}})} = \underline{r}^{-\mathsf{N}} = \mathtt{req}(\bullet\underline{r})^{-\mathsf{N}} = s^{-\mathsf{N}}$$
2. Translation of a variable after substitution, $\underline{t} = \mathtt{req}(\bullet\underline{r})$: Note that the step can not be derived using the $\mathtt{E}^{\bullet}\text{-}\mathtt{req}$ rule, because there are no rules that allow deriving a step of the form $\bullet\underline{r} \rightsquigarrow_{\underline{\rho}} s'$. Hence the step is derived from the $\mathtt{E}^{\bullet}\text{-}\mathtt{req}\bullet$ rule and of the form $\mathtt{req}(\bullet\underline{r}) \rightsquigarrow_{\bullet\mathsf{req}} \underline{r} = s$. Note that $s = \underline{r} \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and:
$$\underline{t}^{-\mathsf{N}} = \mathtt{req}(\bullet\underline{r})^{-\mathsf{N}} = \underline{r}^{-\mathsf{N}} = s^{-\mathsf{N}}$$
3. Translation of an abstraction, $\underline{t} = \lambda x.\, \underline{t}_1[u/a]$: This case is impossible, as there are no rules that allow deriving a step of the form $\lambda x.\, \underline{t}_1[u/a] \rightsquigarrow_{\underline{\rho}} s$.
4. Translation of an application, $\underline{t} = \underline{t}_1\ !\bullet\underline{t}_2$: We consider two subcases, depending on the inference rule applied to conclude $\underline{t} \rightsquigarrow_{\underline{\rho}} s$:
   4.1 $\mathtt{E}^{\bullet}\text{-}\mathtt{db}$: Then $\underline{t} = (\lambda a.\, \underline{t}_{11}[u/a])\underline{\mathsf{L}}\ !\bullet\underline{t}_2$ where $\underline{t}_{11}, \underline{t}_2 \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and $\underline{\mathsf{L}} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{N}}$ and the step is of the form
$$\underline{t} = (\lambda a.\, \underline{t}_{11}[u/a])\underline{\mathsf{L}}\ !\bullet\underline{t}_2 \rightsquigarrow_{\bullet\mathsf{db}} \underline{t}_{11}[u/!\bullet\underline{t}_2]\underline{\mathsf{L}} = s$$
   Note that $s \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and:
$$\underline{t}^{-\mathsf{N}} = (\lambda u.\, \underline{t}_{11}^{-\mathsf{N}})\underline{\mathsf{L}}^{-\mathsf{N}}\, \underline{t}_2^{-\mathsf{N}} \rightsquigarrow^{\mathsf{N}}_{\mathsf{db}} \underline{t}_{11}^{-\mathsf{N}}[u/\underline{t}_2^{-\mathsf{N}}]\underline{\mathsf{L}}^{-\mathsf{N}} = s^{-\mathsf{N}}$$
   4.2 $\mathtt{E}^{\bullet}\text{-}\mathtt{app}$: Then the step is of the form $\underline{t} = \underline{t}_1\ !\bullet\underline{t}_2 \rightsquigarrow_{\underline{\rho}} s_1\ !\bullet\underline{t}_2 = s$ where $\underline{t}_1 \rightsquigarrow_{\underline{\rho}} s_1$. By IH we have that $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and $\underline{t}_1^{-\mathsf{N}} \rightsquigarrow_{\underline{\rho}^{-\mathsf{N}}} s_1^{-\mathsf{N}}$. So $s = s_1\ !\bullet\underline{t}_2 \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and, applying $\mathtt{E}^{\mathsf{N}}\text{-}\mathtt{app}$, we conclude:
$$\underline{t}^{-\mathsf{N}} = \underline{t}_1^{-\mathsf{N}}\, \underline{t}_2^{-\mathsf{N}} \rightsquigarrow^{\mathsf{N}}_{\underline{\rho}^{-\mathsf{N}}} s_1^{-\mathsf{N}}\, \underline{t}_2^{-\mathsf{N}}$$
5. Translation of an explicit substitution, $\underline{t} = \underline{t}_1[u/!\bullet\underline{t}_2]$: We consider five subcases, depending on the inference rule applied to conclude that there is a step $\underline{t} \rightsquigarrow_{\underline{\rho}} s$.
   5.1 $\mathtt{E}^{\bullet}\text{-}\mathtt{ls}$: The step is of the form $\underline{t} = \underline{t}_1[u/!\bullet\underline{t}_2] \rightsquigarrow_{\bullet\mathsf{ls}} s_1[u/!\bullet\underline{t}_2] = s$ where $\underline{t}_1 \rightsquigarrow_{\varsigma(u,\bullet\underline{t}_2)} s_1$. Note that $\varsigma(u, \bullet\underline{t}_2) \in \mathcal{R}_{\bullet}^{\mathsf{N}}$, so by IH we have that $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and $\underline{t}_1^{-\mathsf{N}} \rightsquigarrow^{\mathsf{N}}_{\varsigma(u,\underline{t}_2^{-\mathsf{N}})} s_1^{-\mathsf{N}}$. So $s_1[u/\underline{t}_2] \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and, applying $\mathtt{E}^{\mathsf{N}}\text{-}\mathtt{ls}$, we conclude:
$$\underline{t}^{-\mathsf{N}} = \underline{t}_1^{-\mathsf{N}}[u/\underline{t}_2^{-\mathsf{N}}] \rightsquigarrow^{\mathsf{N}}_{\mathsf{ls}} s_1^{-\mathsf{N}}[u/\underline{t}_2^{-\mathsf{N}}] = s^{-\mathsf{N}}$$
   5.2 $\mathtt{E}^{\bullet}\text{-}\mathtt{gc}$: The step is of the form $\underline{t} = \underline{t}_1[u/!\bullet\underline{t}_2] \rightsquigarrow_{\bullet\mathsf{gc}} \underline{t}_1 = s$ where $u \notin \mathsf{fv}(\underline{t}_1)$. Note that $s = \underline{t}_1 \in \mathcal{T}_{\bullet}^{\mathsf{N}}$. Note also that $u \notin \mathsf{fv}(\underline{t}_1^{-\mathsf{N}})$ by Rem. E.3. So:
$$\underline{t}^{-\mathsf{N}} = \underline{t}_1^{-\mathsf{N}}[u/\underline{t}_2^{-\mathsf{N}}] \rightsquigarrow^{\mathsf{N}}_{\mathsf{gc}} \underline{t}_1^{-\mathsf{N}} = s^{-\mathsf{N}}$$

5.3 $\mathtt{E}^{\bullet}\mathtt{\text{-}esL}$: The step is of the form $\underline{t} = \underline{t}_1[u/!\bullet\underline{t}_2] \rightsquigarrow_{\underline{\rho}} s_1[u/!\bullet\underline{t}_2] = s$ where $\underline{t}_1 \rightsquigarrow_{\underline{\rho}} s_1$ and $u \notin \mathsf{fv}(\underline{\rho})$. By IH we have that $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{N}}$ and $\underline{t}_1^{-\mathsf{N}} \rightsquigarrow_{\underline{\rho}^{-\mathsf{N}}} s_1^{-\mathsf{N}}$. Note that $s = s_1[u/!\bullet\underline{t}_2] \in \mathcal{T}_{\bullet}^{\mathsf{N}}$. Note also that $u \notin \mathsf{fv}(\underline{\rho}^{-\mathsf{N}})$ by Rem. F.1. So, applying $\mathtt{E}^{\mathsf{N}}\mathtt{\text{-}subL}$, we conclude:

$$\underline{t}^{-\mathsf{N}} = \underline{t}_1^{-\mathsf{N}}[u/\underline{t}_2^{-\mathsf{N}}] \rightsquigarrow_{\underline{\rho}^{-\mathsf{N}}} s_1^{-\mathsf{N}}[u/\underline{t}_2^{-\mathsf{N}}] = s^{-\mathsf{N}}$$

5.4 $\mathtt{E}^{\bullet}\mathtt{\text{-}esR}$: We argue that this case is impossible. Indeed, the step must be of the form $\underline{t} = \underline{t}_1[u/!\bullet\underline{t}_2] \rightsquigarrow_{\underline{\rho}} \underline{t}_1[u/s_2] = s$ where $!\bullet\underline{t}_2 \rightsquigarrow_{\underline{\rho}} s_2$. This is impossible because there are no rules that allow deriving such a step.

5.5 $\mathtt{E}^{\bullet}\mathtt{\text{-}es!}$: We argue that this case is impossible. Indeed, the step must be of the form $\underline{t} = \underline{t}_1[u/!\bullet\underline{t}_2] \rightsquigarrow_{\underline{\rho}} \underline{t}_1[u/!s_2] = s$ where $\underline{t}_1 \rightsquigarrow_{\iota(u)} \underline{t}_1$ and $\bullet\underline{t}_2 \rightsquigarrow_{\underline{\rho}} s_2$. This is impossible because there are no rules that allow deriving the step $\bullet\underline{t}_2 \rightsquigarrow_{\underline{\rho}} s_2$.

Definition F.3 (CBV **Rulenames).** *We define a subset $\mathcal{R}_{\bullet}^{\mathsf{V}} \subseteq \mathcal{R}_{\bullet}$ by the following grammar:*

$$\underline{\rho} ::= \bullet\mathsf{db} \mid \varsigma(x, \bullet\lambda a.\,\underline{t}[u/a]) \mid \bullet\mathsf{ls} \mid \bullet\mathsf{gc} \mid \bullet\mathsf{req}$$

where $\underline{t}$ stands for a term in $\mathcal{T}_{\bullet}^{\mathsf{V}}$. The translation of a CBV*-rulename is a sequence of $\mathcal{R}_{\bullet}^{\mathsf{V}}$ rulenames, given by:*

$$\begin{aligned}
\mathsf{db}^{\mathsf{V}} &:= \langle \bullet\mathsf{ls}, \bullet\mathsf{req}, \bullet\mathsf{db}, \bullet\mathsf{gc}\rangle \\
\varsigma(x, \mathsf{v})^{\mathsf{V}} &:= \langle \varsigma(x, \bullet\mathsf{v}^{(\mathsf{V})})\rangle \\
\mathsf{lsv}^{\mathsf{V}} &:= \langle \bullet\mathsf{ls}\rangle \\
\mathsf{gcv+}^{\mathsf{V}} &:= \langle \bullet\mathsf{gc}\rangle
\end{aligned}$$

where we define an auxiliary operation to translate values as follows:

$$(\lambda x.\,t)^{(\mathsf{V})} := \lambda a.\,t^{\mathsf{V}}[x/a]$$

We define an extended CBV *evaluation relation $t \blacktriangleright_{\rho}^{\mathsf{V}} s$, extending the system defining $\rightsquigarrow_{\rho}^{\mathsf{V}}$ with a further rulename $\mathsf{gcv+}^{-1}$ and the following rule:*

$$\frac{x \notin \mathsf{fv}(t)}{t \blacktriangleright_{\mathsf{gcv+}^{-1}}^{\mathsf{V}} t[x/\mathsf{v}^{+}]}\ \mathtt{E}^{\mathsf{V}}\mathtt{\text{-}gcv+}$$

The translation of a $\mathcal{R}_{\bullet}^{\mathsf{V}}$ rulename is a set of sequences of CBV*-rulenames, given by:*

$$\begin{aligned}
\bullet\mathsf{db}^{-\mathsf{V}} &:= \{\langle \mathsf{db}\rangle\} \\
\varsigma(x, \bullet\lambda a.\,\underline{t}[u/a])^{-\mathsf{V}} &:= \{\langle \varsigma(x, \lambda u.\,\underline{t}^{-\mathsf{V}})\rangle\} \\
\bullet\mathsf{ls}^{-\mathsf{V}} &:= \{\langle \mathsf{lsv}\rangle, \langle \mathsf{gcv+}^{-1}\rangle\} \\
\bullet\mathsf{gc}^{-\mathsf{V}} &:= \{\langle \mathsf{gcv+}\rangle\} \\
\bullet\mathsf{req}^{-\mathsf{V}} &:= \{\langle\rangle\}
\end{aligned}$$

Note that the result is a set of sequences, which should be understood as the fact that the translation is "non-deterministic".

Lemma F.3 (CBV **Evaluation – Simulation).** *If $t \rightsquigarrow_{\rho}^{\mathsf{V}} t'$ then $t^{\mathsf{V}} \rightsquigarrow_{\rho^{\mathsf{V}}}^{*} t'^{\mathsf{V}}$.*

Proof. By induction on the derivation of $t \rightsquigarrow_{\rho}^{\mathsf{V}} t'$:

1. $\mathtt{E}^{\mathsf{V}}\texttt{-db}$: Let $(\lambda x.\, t)\mathtt{L}\; s \rightsquigarrow_{\mathsf{db}}^{\mathsf{V}} t[x/s]\mathtt{L}$. Then:

$$\begin{array}{rll}
((\lambda x.\, t)\mathtt{L}\; s)^{\mathsf{V}} & = & \mathtt{req}(u[u/(!\bullet\lambda a.\, t^{\mathsf{V}}[x/a])\mathtt{L}^{\mathsf{V}}])\; s^{\mathsf{V}} \\
& \rightsquigarrow_{\bullet\mathsf{ls}} & \mathtt{req}((\bullet\lambda a.\, t^{\mathsf{V}}[x/a])[u/!\bullet\lambda a.\, t^{\mathsf{V}}[x/a]]\mathtt{L}^{\mathsf{V}})\; s^{\mathsf{V}} \\
& & \quad \text{by } \mathtt{E}^{\bullet}\texttt{-app}, \mathtt{E}^{\bullet}\texttt{-req}, \mathtt{E}^{\bullet}\texttt{-ls}, \mathtt{E}^{\bullet}\texttt{-}\varsigma \\
& \rightsquigarrow_{\bullet\mathsf{req}} & (\lambda a.\, t^{\mathsf{V}}[x/a])[u/!\bullet\lambda a.\, t^{\mathsf{V}}[x/a]]\mathtt{L}^{\mathsf{V}}\; s^{\mathsf{V}} \\
& & \quad \text{by } \mathtt{E}^{\bullet}\texttt{-app}, \mathtt{E}^{\bullet}\texttt{-req}\bullet \\
& \rightsquigarrow_{\bullet\mathsf{db}} & t^{\mathsf{V}}[x/s^{\mathsf{V}}][u/!\bullet\lambda a.\, t^{\mathsf{V}}[x/a]]\mathtt{L}^{\mathsf{V}} \\
& & \quad \text{by } \mathtt{E}^{\bullet}\texttt{-db} \\
& \rightsquigarrow_{\bullet\mathsf{gc}} & t^{\mathsf{V}}[x/s^{\mathsf{V}}]\mathtt{L}^{\mathsf{V}} \\
& & \quad \text{by } \mathtt{E}^{\bullet}\texttt{-gc} \\
& = & (t[x/s]\mathtt{L})^{\mathsf{V}}
\end{array}$$

   To justify the application of $\mathtt{E}^{\bullet}\texttt{-gc}$, note that by α-conversion we may assume that $u \notin \mathsf{fv}(t[x/s^{\mathsf{V}}]^{\mathsf{V}})$.
2. $\mathtt{E}^{\mathsf{V}}\texttt{-}\varsigma$: Let $x \rightsquigarrow_{\varsigma(x,\mathtt{v})}^{\mathsf{V}} \mathtt{v}$. Then:

$$\begin{array}{rll}
x^{\mathsf{V}} & = & !x^{\mathsf{V}} \\
& \rightsquigarrow_{\varsigma(x,\bullet\mathtt{v}^{(\mathsf{V})})} & !\bullet\mathtt{v}^{(\mathsf{V})} \text{ by } \mathtt{E}^{\bullet}\texttt{-}!\varsigma \\
& = & \mathtt{v}^{\mathsf{V}} \quad \text{as } !\bullet\mathtt{v}^{(\mathsf{V})} = \mathtt{v}^{\mathsf{V}} \text{ holds by definition}
\end{array}$$

3. $\mathtt{E}^{\mathsf{V}}\texttt{-lsv}$: Let $t[x/\mathtt{vL}] \rightsquigarrow_{\mathsf{lsv}}^{\mathsf{V}} t'[x/\mathtt{v}]\mathtt{L}$ be derived from $t \rightsquigarrow_{\varsigma(x,\mathtt{v})}^{\mathsf{V}} t'$. By IH, $t^{\mathsf{V}} \rightsquigarrow_{\varsigma(x,\mathtt{v})^{\mathsf{V}}}^{*} t'^{\mathsf{V}}$, that is, $t^{\mathsf{V}} \rightsquigarrow_{\varsigma(x,\bullet\mathtt{v}^{(\mathsf{V})})} t'^{\mathsf{V}}$. Hence:

$$\begin{array}{rll}
(t[x/\mathtt{vL}])^{\mathsf{V}} & = & t^{\mathsf{V}}[x/\mathtt{v}^{\mathsf{V}}\mathtt{L}^{\mathsf{V}}] \\
& = & t^{\mathsf{V}}[x/(!\bullet\mathtt{v}^{(\mathsf{V})})\mathtt{L}^{\mathsf{V}}] \text{ since } \mathtt{v}^{\mathsf{V}} = !\bullet\mathtt{v}^{(\mathsf{V})} \text{ holds by definition} \\
& \rightsquigarrow_{\mathsf{ls}} & t'^{\mathsf{V}}[x/!\bullet\mathtt{v}^{(\mathsf{V})}]\mathtt{L}^{\mathsf{V}} \quad \text{by } \mathtt{E}^{\bullet}\texttt{-ls}, \text{ since } t^{\mathsf{V}} \rightsquigarrow_{\varsigma(x,\mathtt{v})^{\mathsf{V}}} t'^{\mathsf{V}} \\
& = & t'^{\mathsf{V}}[x/\mathtt{v}^{\mathsf{V}}]\mathtt{L}^{\mathsf{V}} \\
& = & (t'[x/\mathtt{v}]\mathtt{L})^{\mathsf{V}}
\end{array}$$

4. $\mathtt{E}^{\mathsf{V}}\texttt{-gcv+}$: Let $t[x/\mathtt{v}^{+}\mathtt{L}] \rightsquigarrow_{\mathsf{gcv+}}^{\mathsf{V}} t\mathtt{L}$, where $x \notin \mathsf{fv}(t)$. Recall that the translation does not create free variables (Rem. E.4), so $x \notin \mathsf{fv}(t^{\mathsf{V}})$. Note that the translation of a lax value always starts with "!", *i.e.* $(\mathtt{v}^{+})^{\mathsf{V}}$ is of the form $!s$. Hence:

$$\begin{array}{rll}
(t[x/\mathtt{v}^{+}\mathtt{L}])^{\mathsf{V}} & = & t^{\mathsf{V}}[x/(\mathtt{v}^{+})^{\mathsf{V}}\mathtt{L}^{\mathsf{V}}] \\
& = & t^{\mathsf{V}}[x/(!s)\mathtt{L}^{\mathsf{V}}] \\
& \rightsquigarrow_{\bullet\mathsf{gc}} & t^{\mathsf{V}}\mathtt{L}^{\mathsf{V}} \qquad \text{by } \mathtt{E}^{\bullet}\texttt{-gc} \text{ since } x \notin \mathsf{fv}(t^{\mathsf{V}}) \\
& = & (t\mathtt{L})^{\mathsf{V}}
\end{array}$$

5. $\mathtt{E}^{\mathsf{V}}\texttt{-app}$: Let $t\; s \rightsquigarrow_{\rho}^{\mathsf{V}} t'\; s$ be derived from $t \rightsquigarrow_{\rho}^{\mathsf{V}} t'$. By IH we have that $t^{\mathsf{V}} \rightsquigarrow_{\rho^{\mathsf{V}}}^{*} t'^{\mathsf{V}}$. Hence:

$$\begin{array}{rll}
(t\; s)^{\mathsf{V}} & = & \mathtt{req}(u)[u/t^{\mathsf{V}}]\; s^{\mathsf{V}} \\
& \rightsquigarrow_{\rho^{\mathsf{V}}}^{*} & \mathtt{req}(u)[u/t'^{\mathsf{V}}]\; s^{\mathsf{V}} \text{ by } \mathtt{E}^{\bullet}\texttt{-app}, \mathtt{E}^{\bullet}\texttt{-esR} \text{ (many times)} \\
& = & (t'\; s)^{\mathsf{V}}
\end{array}$$

6. $\mathtt{E}^{\mathsf{V}}$-$\mathtt{subL}$: Let $t[x/s] \rightsquigarrow_{\rho}^{\mathsf{V}} t'[x/s]$ be derived from $t \rightsquigarrow_{\rho}^{\mathsf{V}} t'$, where $x \notin \mathsf{fv}(\rho)$. By IH we have that $t^{\mathsf{V}} \rightsquigarrow_{\rho^{\mathsf{V}}}^{*} t'^{\mathsf{V}}$. Recall that the translation does not create free variables (Rem. E.4), so $x \notin \mathsf{fv}(t^{\mathsf{V}})$, which means that $x \notin \mathsf{fv}(\rho^{\mathsf{V}})$. Hence:

$$\begin{array}{rll} (t[x/s])^{\mathsf{V}} = & t^{\mathsf{V}}[x/s^{\mathsf{V}}] & \\ \rightsquigarrow_{\rho^{\mathsf{V}}}^{*} & t'^{\mathsf{V}}[x/s^{\mathsf{V}}] & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{esL} \text{ (many times), since } x \notin \mathsf{fv}(\rho^{\mathsf{V}}) \\ = & (t'[x/s])^{\mathsf{V}} & \end{array}$$

7. $\mathtt{E}^{\mathsf{V}}$-$\mathtt{subR}$: Let $t[x/s] \rightsquigarrow_{\rho}^{\mathsf{V}} t[x/s']$ be derived from $s \rightsquigarrow_{\rho}^{\mathsf{V}} s'$. By IH we have that $s^{\mathsf{V}} \rightsquigarrow_{\rho^{\mathsf{V}}}^{*} s'^{\mathsf{V}}$. Hence:

$$\begin{array}{rll} (t[x/s])^{\mathsf{V}} = & t^{\mathsf{V}}[x/s^{\mathsf{V}}] & \\ \rightsquigarrow_{\rho^{\mathsf{V}}}^{*} & t^{\mathsf{V}}[x/s'^{\mathsf{V}}] & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{esR} \text{ (many times)} \\ = & (t[x/s'])^{\mathsf{V}} & \end{array}$$

Lemma F.4 (CBV **Evaluation – Inverse Simulation).** *Let $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, $\underline{\rho} \in \mathcal{R}_{\bullet}^{\mathsf{V}}$ and $s \in \mathcal{T}_{\bullet}$ such that $\underline{t} \rightsquigarrow_{\underline{\rho}}^{*} s$. Then $s \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and there is a sequence of rulenames $\boldsymbol{\rho} \in \underline{\rho}^{-\mathsf{V}}$ such that $\underline{t}^{-\mathsf{V}} (\blacktriangleright^{\mathsf{V}})_{\boldsymbol{\rho}}^{*} s^{-\mathsf{V}}$.*

Proof. By induction on $\underline{t}$:

1. Translation of a variable, $\underline{t} = !u$: The step must be derived using the $\mathtt{E}^{\bullet}\text{-}!\varsigma$ rule and thus of the form $\underline{t} = !u \rightsquigarrow_{\varsigma(u,(\bullet r)\mathtt{L})} !(\bullet r)\mathtt{L} = s$. Since $\underline{\rho} = \varsigma(u,(\bullet r)\mathtt{L}) \in \mathcal{R}_{\bullet}^{\mathsf{V}}$, we have that $\mathtt{L}$ is empty and $r = \lambda a.\,\underline{r}_1[v/a]$. Then $s = !\bullet\lambda a.\,\underline{r}_1[v/a] \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and:

$$\underline{t}^{-\mathsf{V}} = u \blacktriangleright_{\varsigma(u,\lambda v.\,\underline{r}_1^{-\mathsf{V}})}^{\mathsf{V}} = \lambda v.\,\underline{r}_1^{-\mathsf{V}} = s^{-\mathsf{V}}$$

2. Translation of an abstraction, $\underline{t} = !\bullet\lambda a.\,\underline{t}_1[u/a]$: This case is impossible, as there are no rules that allow deriving a step of the form $!\bullet\lambda a.\,\underline{t}_1[u/a] \rightsquigarrow_{\underline{\rho}} s$.
3. Request (1), $\underline{t} = \mathtt{req}(u)[u/\underline{t}_1]$: Then the step must be of the form $\mathtt{req}(u)[u/\underline{t}_1] \rightsquigarrow_{\underline{\rho}} s$. Note that the step cannot be an instance of the $\mathtt{E}^{\bullet}\text{-}\mathtt{gc}$ rule because $u \in \mathsf{fv}(\mathtt{req}(u))$. We consider four subcases, depending on whether the step is derived using $\mathtt{E}^{\bullet}\text{-}\mathtt{ls}$, $\mathtt{E}^{\bullet}\text{-}\mathtt{esL}$, $\mathtt{E}^{\bullet}\text{-}\mathtt{esR}$, or $\mathtt{E}^{\bullet}\text{-}\mathtt{es!}$:
   3.1 $\mathtt{E}^{\bullet}\text{-}\mathtt{esL}$: Then $\underline{t}_1 = (!(\bullet r)\mathtt{L}_1)\mathtt{L}_2$ and the step is of the form

$$\underline{t} = \mathtt{req}(u)[u/(!(\bullet r)\mathtt{L}_1)\mathtt{L}_2] \rightsquigarrow_{\bullet\mathsf{ls}} \mathtt{req}((\bullet r)\mathtt{L}_1)[u/!(\bullet r)\mathtt{L}_1]\mathtt{L}_2 = s$$

Since $\underline{t}_1 = (!(\bullet r)\mathtt{L}_1)\mathtt{L}_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ then by Rem. E.7 we have that $\underline{t}_1 = (!\bullet\lambda a.\,\underline{t}_{11}[v/a])\mathtt{L}_2$ where $\mathtt{L}_2 \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$. Hence $s = \mathtt{req}(\bullet\lambda a.\,\underline{t}_{11}[v/a])[u/!\bullet\lambda a.\,\underline{t}_{11}[v/a]]\underline{\mathtt{L}}_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and:

$$\begin{array}{rl} \underline{t}^{-\mathsf{V}} & = (\mathtt{req}(u)[u/(!\bullet\lambda a.\,\underline{t}_{11}[v/a])\mathtt{L}_2])^{-\mathsf{V}} \\ & = (\lambda v.\,\underline{t}_{11}^{-\mathsf{V}})\mathtt{L}_2^{-\mathsf{V}} \\ \blacktriangleright_{\mathsf{gcv+}^{-1}}^{\mathsf{V}} & (\lambda v.\,\underline{t}_{11}^{-\mathsf{V}})[u/\lambda v.\,\underline{t}_{11}^{-\mathsf{V}}]\underline{\mathtt{L}}_2^{-\mathsf{V}} \\ & = (\mathtt{req}(\bullet\lambda a.\,\underline{t}_{11}[v/a])[u/!\bullet\lambda a.\,\underline{t}_{11}[v/a]]\underline{\mathtt{L}}_2)^{-\mathsf{V}} \\ & = s^{-\mathsf{V}} \end{array}$$

3.2 $\mathtt{E}^{\bullet}\text{-}\mathtt{esL}$: We argue that this case is impossible. Indeed, the only way to reduce $\mathtt{req}(u)$ is by substituting u, *i.e.* with a step of the form $\mathtt{req}(u) \rightsquigarrow_{\varsigma(u,(\bullet r)\mathtt{L})} \mathtt{req}((\bullet r)\mathtt{L})$, but $\mathtt{E}^{\bullet}\text{-}\mathtt{esL}$ cannot be applied because $u \in \mathsf{fv}(\varsigma(u,(\bullet r)\mathtt{L}))$.

3.3 $\mathtt{E}^{\bullet}\text{-}\mathtt{esR}$: The step is of the form $\mathtt{req}(u)[u/\underline{t}_1] \rightsquigarrow_{\underline{\rho}} \mathtt{req}(u)[u/s_1] = s$. where $\underline{t}_1 \rightsquigarrow_{\underline{\rho}} s_1$. By IH we have that $s_1 \in \mathcal{T}_{\bullet}^{\vee}$ and $\underline{t}_1^{-\vee} (\blacktriangleright^{\vee})^*_{\boldsymbol{\rho}} s_1^{-\vee}$ for some $\boldsymbol{\rho} \in \underline{\rho}^{-\vee}$. This means that $s = \mathtt{req}(u)[u/s_1] \in \mathcal{T}_{\bullet}^{\vee}$ and:

$$\begin{aligned} \underline{t}^{-\vee} &= (\mathtt{req}(u)[u/\underline{t}_1])^{-\vee} \\ &= \underline{t}_1^{-\vee} \\ (\blacktriangleright^{\vee})^*_{\boldsymbol{\rho}} &\; s_1^{-\vee} \\ &= (\mathtt{req}(u)[u/s_1])^{-\vee} \\ &= s^{-\vee} \end{aligned}$$

3.4 $\mathtt{E}^{\bullet}\text{-}\mathtt{es!}$: Then $\underline{t}_1 = !r$ and the step is of the form $\mathtt{req}(u)[u/!r] \rightsquigarrow_{\underline{\rho}} \mathtt{req}(u)[u/!r'] = s$ where $\mathtt{req}(u) \rightsquigarrow_{\iota(u)} \mathtt{req}(u)$ and $r \rightsquigarrow_{\underline{\rho}} r'$. Since $\underline{t}_1 \in \mathcal{T}_{\bullet}^{\vee}$ we have that $\underline{t}_1$ must be either the translation of a variable ($\underline{t}_1 = !v$) or the translation of an abstraction ($\underline{t}_1 = !\bullet\lambda a.\, \underline{p}[x/a]$). We consider these two as subcases:

3.4.1 If $\underline{t}_1 = !v$, then $r = v$. The step $r = v \rightsquigarrow_{\underline{\rho}} r'$ can only be a substitution step, *i.e.* it must be of the form $r = v \rightsquigarrow_{\varsigma(v,(\bullet p)\mathtt{L})} (\bullet p)\mathtt{L} = r'$. Since by hypothesis $\underline{\rho} = \varsigma(v,(\bullet p)\mathtt{L}) \in \mathcal{R}_{\bullet}^{\vee}$, we have that $\mathtt{L}$ is empty and p is of the form $p = \lambda b.\, \underline{p}_1[w/b]$. The step is then of the form $\underline{t} = \mathtt{req}(u)[u/!v] \rightsquigarrow_{\varsigma(v,\bullet p)} \mathtt{req}(u)[u/!\bullet p] = s$. To conclude, note that $s = \mathtt{req}(u)[u/!\bullet p] \in \mathcal{T}_{\bullet}^{\vee}$ because $!\bullet p = !\bullet\lambda b.\, \underline{p}_1[w/b] \in \mathcal{T}_{\bullet}^{\vee}$ and note that:

$$\begin{aligned} \underline{t}^{-\vee} &= (\mathtt{req}(u)[u/!v])^{-\vee} \\ &= v \\ \blacktriangleright^{\vee}_{\varsigma(v,\lambda w.\, \underline{p}_1^{-\vee})} &\; \lambda w.\, \underline{p}_1^{-\vee} \\ &= (!\bullet p)^{-\vee} \\ &= (\mathtt{req}(u)[u/!\bullet p])^{-\vee} \\ &= s^{-\vee} \end{aligned}$$

3.4.2 If $\underline{t}_1 = !\bullet\lambda a.\, \underline{p}[x/a]$, then $r = \bullet\lambda a.\, \underline{p}[x/a]$. This case is impossible, as there are no rules that allow to derive a step of the form $r = \bullet\lambda a.\, \underline{p}[x/a] \rightsquigarrow_{\underline{\rho}} r'$.

4. Request (2), $\underline{t} = \mathtt{req}(\bullet\lambda a.\, \underline{t}_1[u/a])$: Then the step is of the form $\underline{t} = \mathtt{req}(\bullet\lambda a.\, \underline{t}_1[u/a]) \rightsquigarrow_{\underline{\rho}} s$. Note that the step cannot be internal to $\lambda a.\, \underline{t}_1[u/a]$ because there are no rules that allow to reduce inside a "$\bullet$". Hence the step must be an instance of the $\mathtt{E}^{\bullet}\text{-}\mathtt{req}\bullet$ rule, *i.e.* of the form: $\underline{t} = \mathtt{req}(\bullet\lambda a.\, \underline{t}_1[u/a]) \rightsquigarrow_{\bullet\mathsf{req}} \lambda a.\, \underline{t}_1[u/a] = s$: then note that $s \in \mathcal{T}_{\bullet}^{\vee}$ and:

$$\begin{aligned} \underline{t}^{-\vee} &= (\mathtt{req}(\bullet\lambda a.\, \underline{t}_1[u/a]))^{-\vee} \\ &= \lambda u.\, \underline{t}_1^{-\vee}) \\ &= (\lambda a.\, \underline{t}_1[u/a])^{-\vee} \\ &= s \end{aligned}$$

5. Abstraction, $\underline{t} = \lambda a.\, \underline{t}_1[u/a]$: This case is impossible, as there are no rules that allow deriving a step of the form $\lambda a.\, \underline{t}_1[u/a] \rightsquigarrow_{\rho} s$.
6. Application, $\underline{t} = \underline{t}_1\, \underline{t}_2$: We consider two subcases, depending on whether the step is derived using the $\mathtt{E}^{\bullet}\text{-}\mathtt{db}$ rule or the $\mathtt{E}^{\bullet}\text{-}\mathtt{app}$ rule:

6.1 $\mathtt{E}^{\bullet}\texttt{-db}$: Then $\underline{t}_1$ must be of the form $(\lambda a.\, r)\mathtt{L}$. and by Rem. E.7 we know in turn that $r = \underline{r}_1[u/a]$, where $a \notin \mathsf{fv}(\underline{r}_1)$ and $\underline{r}_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and $\mathtt{L} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$. The step is of the form $\underline{t} = (\lambda a.\, \underline{r}_1[u/a])\mathtt{L}\, \underline{t}_2 \rightsquigarrow_{\bullet\mathsf{db}} \underline{r}_1[u/\underline{t}_2]\mathtt{L} = s$. Note that $s \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and

$$\begin{aligned} \underline{t}^{-\mathsf{V}} &= (\lambda a.\, \underline{r}_1[u/a])\mathtt{L}\, \underline{t}_2^{-\mathsf{V}} \\ &= (\lambda u.\, \underline{r}_1^{-\mathsf{V}})\mathtt{L}^{-\mathsf{V}}\, \underline{t}_2^{-\mathsf{V}} \\ \blacktriangleright_{\mathsf{db}}^{\mathsf{V}}\ & \underline{r}_1^{-\mathsf{V}}[u/\underline{t}_2^{-\mathsf{V}}]\mathtt{L}^{-\mathsf{V}} \\ &= (\underline{r}_1[u/\underline{t}_2]\mathtt{L})^{-\mathsf{V}} \\ &= s^{-\mathsf{V}} \end{aligned}$$

6.2 $\mathtt{E}^{\bullet}\texttt{-app}$: Then the step is of the form $\underline{t} = \underline{t}_1\, \underline{t}_2 \rightsquigarrow_{\underline{\rho}} s_1\, \underline{t}_2 = s$ where $\underline{t}_1 \rightsquigarrow_{\underline{\rho}} s_1$. By IH, $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and $\underline{t}_1^{-\mathsf{V}}\ (\blacktriangleright^{\mathsf{V}})_{\boldsymbol{\rho}}^{*}\ s_1^{-\mathsf{V}}$ for some $\boldsymbol{\rho} \in \underline{\rho}^{-\mathsf{V}}$. Then $s = s_1\, \underline{t}_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and, applying $\mathtt{E}^{\mathsf{V}}\texttt{-app}$ once per each step, we have:

$$\underline{t}^{-\mathsf{V}} = (\underline{t}_1\, \underline{t}_2)^{-\mathsf{V}} = \underline{t}_1^{-\mathsf{V}}\, \underline{t}_2^{-\mathsf{V}}\ (\blacktriangleright^{\mathsf{V}})_{\boldsymbol{\rho}}^{*}\ s_1^{-\mathsf{V}}\, \underline{t}_2^{-\mathsf{V}} = (s_1\, \underline{t}_2)^{-\mathsf{V}} = s^{-\mathsf{V}}$$

7. Explicit substitution, $\underline{t} = \underline{t}_1[u/\underline{t}_2]$: We consider five subcases, depending on the rule used to derive the step:

7.1 $\mathtt{E}^{\bullet}\texttt{-ls}$: For the $\mathtt{E}^{\bullet}\texttt{-ls}$ rule to be applicable, $\underline{t}_2$ must be of the form $\underline{t}_2 = (!(\bullet r)\mathtt{L}_1)\mathtt{L}_2$. Since $\underline{t}_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, by Rem. E.7, we have that actually $\underline{t}_2 = (!\bullet\lambda a.\, \underline{r}_1[u/a])\mathtt{L}_2$, where $\underline{r}_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, $a \notin \mathsf{fv}(\underline{r}_1)$, and $\mathtt{L}_2 \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$. Thus the step is of the form $\underline{t} = \underline{t}_1[u/(!\bullet\lambda a.\, \underline{r}_1[u/a])\mathtt{L}_2] \rightsquigarrow_{\bullet\mathsf{ls}} s_1[u/!\bullet\lambda a.\, \underline{r}_1[u/a]]\mathtt{L}_2 = s$, where $\underline{t}_1 \rightsquigarrow_{\varsigma(u,\bullet\lambda a.\, \underline{r}_1[u/a])} s_1$. By IH, we have that $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and $\underline{t}_1^{-\mathsf{V}} \blacktriangleright_{\varsigma(u,\lambda u.\, \underline{r}_1^{-\mathsf{V}})}^{\mathsf{V}} s_1^{-\mathsf{V}}$. Hence $s = s_1[u/!\bullet\lambda a.\, \underline{r}_1[u/a]]\mathtt{L}_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and:

$$\begin{aligned} \underline{t}^{-\mathsf{V}} &= (\underline{t}_1[u/(!\bullet\lambda a.\, \underline{r}_1[u/a])\mathtt{L}_2])^{-\mathsf{V}} \\ &= \underline{t}_1^{-\mathsf{V}}[u/(\lambda u.\, \underline{r}_1^{-\mathsf{V}})\mathtt{L}_2^{-\mathsf{V}}] \\ \blacktriangleright_{\mathsf{lsv}}^{\mathsf{V}}\ & s_1^{-\mathsf{V}}[u/\lambda u.\, \underline{r}_1^{-\mathsf{V}}]\mathtt{L}_2^{-\mathsf{V}} \\ &= (s_1[u/!\bullet\lambda a.\, \underline{r}_1[u/a]]\mathtt{L}_2)^{-\mathsf{V}} \\ &= s^{-\mathsf{V}} \end{aligned}$$

7.2 $\mathtt{E}^{\bullet}\texttt{-gc}$: For the $\mathtt{E}^{\bullet}\texttt{-gc}$ rule to be applicable, $\underline{t}_2$ must be of the form $\underline{t}_2 = (!r)\mathtt{L}$. Since $\underline{t}_2 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$, by Rem. E.7, we have that $\mathtt{L} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{V}}$ and that r is either a variable ($r = v$) or of the form $r = \bullet\lambda a.\, \underline{r}_1[v/a]$, where $\underline{r}_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and $a \notin \mathsf{fv}(\underline{r}_1)$. Note that in both cases $(!r)^{-\mathsf{V}}$ is a lax value, because it is either of the form v or of the form $\lambda v.\, \underline{r}_1^{-\mathsf{V}}$. Moreover, the step is of the form $\underline{t} = \underline{t}_1[u/(\bullet r)\mathtt{L}] \rightsquigarrow_{\bullet\mathsf{gc}} \underline{t}_1\mathtt{L} = s$, where $u \notin \mathsf{fv}(\underline{t}_1)$. Note that $s = \underline{t}_1\mathtt{L} \in \mathcal{T}_{\bullet}^{\mathsf{V}}$. Moreover, we have that $u \notin \mathsf{fv}(\underline{t}_1^{-\mathsf{V}})$ by Rem. E.6, so:

$$\begin{aligned} \underline{t}^{-\mathsf{V}} &= (\underline{t}_1[u/(!r)\mathtt{L}])^{-\mathsf{V}} \\ &= \underline{t}_1^{-\mathsf{V}}[u/!r^{-\mathsf{V}}\mathtt{L}^{-\mathsf{V}}] \\ \blacktriangleright_{\mathsf{gcv+}}^{\mathsf{V}}\ & \underline{t}_1^{-\mathsf{V}}\mathtt{L}^{-\mathsf{V}} \\ &= (\underline{t}_1\mathtt{L})^{-\mathsf{V}} \\ &= s^{-\mathsf{V}} \end{aligned}$$

7.3 $\mathtt{E}^{\bullet}\texttt{-esL}$: The step is of the form $\underline{t} = \underline{t}_1[u/\underline{t}_2] \rightsquigarrow_{\underline{\rho}} s_1[u/\underline{t}_2] = s$, where $u \notin \mathsf{fv}(\underline{\rho})$ and $\underline{t}_1 \rightsquigarrow_{\underline{\rho}} s_1$. By IH, we have that $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{V}}$ and $\underline{t}_1^{-\mathsf{V}} \blacktriangleright_{\boldsymbol{\rho}}^{\mathsf{V}} s_1^{-\mathsf{V}}$ for some $\boldsymbol{\rho} \in \underline{\rho}^{-\mathsf{V}}$.

Then $s = s_1[u/\underline{t}_2] \in \mathcal{T}_\bullet^{\mathsf{V}}$ and:

$$\begin{array}{lll} \underline{t}^{-\mathsf{V}} & = (\underline{t}_1[u/\underline{t}_2])^{-\mathsf{V}} & \\ & = \underline{t}_1^{-\mathsf{V}}[u/\underline{t}_2^{-\mathsf{V}}] & \\ & \blacktriangleright_{\rho}^{\mathsf{V}} s_1^{-\mathsf{V}}[u/\underline{t}_2^{-\mathsf{V}}] & \text{by } \mathtt{E}^{\mathsf{V}}\text{-}\mathtt{subL} \text{ (many times)} \\ & = (s_1[u/\underline{t}_2])^{-\mathsf{V}} & \\ & = s^{-\mathsf{V}} & \end{array}$$

7.4 $\mathtt{E}^\bullet\text{-}\mathtt{esR}$: The step is of the form $\underline{t} = \underline{t}_1[u/\underline{t}_2] \rightsquigarrow_{\underline{\rho}} \underline{t}_1[u/s_2] = s$, where $\underline{t}_2 \rightsquigarrow_{\underline{\rho}} s_2$. By IH, we have that $s_2 \in \mathcal{T}_\bullet^{\mathsf{V}}$ and $\underline{t}_2^{-\mathsf{V}} \blacktriangleright_{\boldsymbol{\rho}}^{\mathsf{V}} s_2^{-\mathsf{V}}$ for some $\boldsymbol{\rho} \in \underline{\rho}^{-\mathsf{V}}$. Then $s = \underline{t}_1[u/s_2] \in \mathcal{T}_\bullet^{\mathsf{V}}$ and:

$$\begin{array}{lll} \underline{t}^{-\mathsf{V}} & = (\underline{t}_1[u/\underline{t}_2])^{-\mathsf{V}} & \\ & = \underline{t}_1^{-\mathsf{V}}[u/\underline{t}_2^{-\mathsf{V}}] & \\ & \blacktriangleright_{\rho}^{\mathsf{V}} \underline{t}_1^{-\mathsf{V}}[u/s_2^{-\mathsf{V}}] & \text{by } \mathtt{E}^{\mathsf{V}}\text{-}\mathtt{subR} \text{ (many times)} \\ & = (\underline{t}_1[u/s_2])^{-\mathsf{V}} & \\ & = s^{-\mathsf{V}} & \end{array}$$

7.5 $\mathtt{E}^\bullet\text{-}\mathtt{es!}$: For the $\mathtt{E}^\bullet\text{-}\mathtt{es!}$ to be applicable, $\underline{t}_2$ must be of the form $\underline{t}_2 = !r$. Since $\underline{t}_2 \in \mathcal{T}_\bullet^{\mathsf{V}}$, by Rem. E.7, we have r is either a variable ($r = v$) or of the form $r = \bullet\lambda a.\, \underline{r}_1[v/a]$, where $\underline{r}_1 \in \mathcal{T}_\bullet^{\mathsf{V}}$ and $a \notin \mathsf{fv}(\underline{r}_1)$. Moreover, the step is of the form $\underline{t} = \underline{t}_1[u/!r] \rightsquigarrow_{\underline{\rho}} \underline{t}_1[u/s_2] = s$ where $\underline{t}_1 \rightsquigarrow_{\iota(u)} \underline{t}_1$ and $r \rightsquigarrow_{\underline{\rho}} s_2$. Note that r cannot be of the form $\bullet\lambda a.\, \underline{r}_1[v/a]$, because there are no rules that allow deriving a step $\bullet\lambda a.\, \underline{r}_1[v/a] \rightsquigarrow_{\underline{\rho}} s_2$. Hence r must be a variable, *i.e.* $r = v$. The step $r = v \rightsquigarrow_{\underline{\rho}} s_2$ can only be derived using the $\mathtt{E}^\bullet\text{-}\varsigma$ rule, so $\underline{\rho} = \varsigma(v, (\bullet p)\mathrm{L})$ and $s_2 = (\bullet p)\mathrm{L}$. Since $\underline{\rho} \in \mathcal{R}_\bullet^{\mathsf{V}}$, we know that $(\bullet p)\mathrm{L}$ must be of the form $(\bullet p)\mathrm{L} = \bullet\lambda b.\, \underline{p}_1[w/b]$. In summary, the step is of the form $\underline{t} = \underline{t}_1[u/!v] \rightsquigarrow_{\varsigma(v, \bullet\lambda b.\, \underline{p}_1[w/b])} \underline{t}_1[u/!\bullet\lambda b.\, \underline{p}_1[w/b]] = s$. Note that $s \in \mathcal{T}_\bullet^{\mathsf{V}}$ and:

$$\begin{array}{lll} \underline{t}^{-\mathsf{V}} & = (\underline{t}_1[u/!v])^{-\mathsf{V}} & \\ & = \underline{t}_1^{-\mathsf{V}}[u/v] & \\ \blacktriangleright_{\varsigma(v, \lambda w.\, \underline{p}_1^{-\mathsf{V}})}^{\mathsf{V}} & \underline{t}_1^{-\mathsf{V}}[u/\lambda w.\, \underline{p}_1^{-\mathsf{V}}] & \text{by } \mathtt{E}^{\mathsf{V}}\text{-}\varsigma, \mathtt{E}^{\mathsf{V}}\text{-}\mathtt{subR} \\ & = (\underline{t}_1[u/!\bullet\lambda b.\, \underline{p}_1[w/b]])^{-\mathsf{V}} & \\ & = s^{-\mathsf{V}} & \end{array}$$

Definition F.4 (CBS **Rulenames).** *We define a subset $\mathcal{R}_\bullet^{\mathsf{S}} \subseteq \mathcal{R}_\bullet$ by the following grammar:*

$$\underline{\rho} ::= \bullet\mathsf{db} \mid \varsigma(x, (\bullet\lambda a.\, \underline{t}[u/a])\underline{\mathrm{L}}) \mid \bullet\mathsf{ls} \mid \bullet\mathsf{gc} \mid \bullet\mathsf{req}$$

where $\underline{t}$ stands for a term in $\mathcal{T}_\bullet^{\mathsf{S}}$. The translation of a CBS*-rulename is a sequence of $\mathcal{R}_\bullet^{\mathsf{S}}$ rulenames, given by:*

$$\begin{array}{rl} \mathsf{db}^{\mathsf{S}} & := \langle \bullet\mathsf{req}, \bullet\mathsf{db} \rangle \\ \varsigma(x, \mathsf{vL})^{\mathsf{S}} & := \langle \varsigma(x, (\bullet\mathsf{v}^{(\mathsf{S})})\mathrm{L}^{\mathsf{S}}) \rangle \\ \mathsf{lsw}^{\mathsf{S}} & := \langle \bullet\mathsf{ls} \rangle \\ \mathsf{gc}^{\mathsf{S}} & := \langle \bullet\mathsf{gc} \rangle \\ \iota(x)^{\mathsf{S}} & := \langle \iota(x) \rangle \end{array}$$

where, moreover, we define an auxiliary operation to translate values as follows: $(\lambda x.\, t)^{(\mathsf{S})} := \lambda a.\, t^{\mathsf{S}}[x/a]$.

The translation of a $\mathcal{R}^{\mathsf{S}}_{\bullet}$ *rulename is a sequence of* CBS*-rulenames, given by:*

$$\begin{array}{rcl} \bullet\mathsf{db}^{-\mathsf{S}} & := & \langle \mathsf{db} \rangle \\ \varsigma(x, (\bullet\lambda a.\, \underline{t}[u/a])\underline{\mathtt{L}})^{-\mathsf{S}} & := & \langle \varsigma(x, (\lambda u.\, \underline{t}^{-\mathsf{S}})\underline{\mathtt{L}}^{-\mathsf{S}}) \rangle \\ \bullet\mathsf{ls}^{-\mathsf{S}} & := & \langle \mathsf{lsw} \rangle \\ \bullet\mathsf{gc}^{-\mathsf{S}} & := & \langle \mathsf{gc} \rangle \\ \bullet\mathsf{req}^{-\mathsf{S}} & := & \langle \rangle \\ \iota(x)^{-\mathsf{S}} & := & \langle \iota(x) \rangle \end{array}$$

Lemma F.5 (CBS **Evaluation – Simulation).** *If* $t \rightsquigarrow^{\mathsf{S}}_{\rho} t'$ *then* $t^{\mathsf{S}} \rightsquigarrow^{*}_{\rho^{\mathsf{S}}} t'^{\mathsf{S}}$.

Proof. By induction on the derivation of $t \rightsquigarrow^{\mathsf{S}}_{\rho} t'$:

1. $\mathtt{E}^{\mathsf{S}}$-$\mathtt{db}$: Let $(\lambda x.\, t)\mathtt{L}\; s \rightsquigarrow^{\mathsf{S}}_{\mathsf{db}} t[x/s]\mathtt{L}$. Then:

$$\begin{array}{rll} ((\lambda x.\, t)\mathtt{L}\; s)^{\mathsf{S}} = & \mathtt{req}(\bullet(\lambda a.\, t^{\mathsf{S}}[x/a])\mathtt{L}^{\mathsf{S}})\; !s^{\mathsf{S}} & \\ \rightsquigarrow_{\bullet\mathsf{req}} & (\lambda a.\, t^{\mathsf{S}}[x/a])\mathtt{L}^{\mathsf{S}}\; !s^{\mathsf{S}} & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{app},\, \mathtt{E}^{\bullet}\text{-}\mathtt{req}\bullet \\ \rightsquigarrow_{\bullet\mathsf{db}} & t^{\mathsf{S}}[x/!s^{\mathsf{S}}]\mathtt{L}^{\mathsf{S}} & \\ = & (t[x/s]\mathtt{L})^{\mathsf{S}} & \end{array}$$

2. $\mathtt{E}^{\mathsf{S}}$-ς: Let $x \rightsquigarrow^{\mathsf{S}}_{\varsigma(x,\mathtt{vL})} \mathtt{vL}$. Then by $\mathtt{E}^{\bullet}$-ς:

$$x^{\mathsf{S}} = x \rightsquigarrow_{\varsigma(x,(\bullet\mathtt{v}^{(\mathsf{S})})\mathtt{L}^{\mathsf{S}})} \bullet\mathtt{v}^{(\mathsf{S})}\mathtt{L}^{\mathsf{S}} = \mathtt{v}^{\mathsf{S}}\mathtt{L}^{\mathsf{S}} = (\mathtt{vL})^{\mathsf{S}}$$

   Note that $\mathtt{v}^{\mathsf{S}} = \bullet\mathtt{v}^{(\mathsf{S})}$ holds by definition.
3. $\mathtt{E}^{\mathsf{S}}$-ς_2: Let $t[u/x] \rightsquigarrow^{\mathsf{S}}_{\varsigma(x,\mathtt{vL})} t[u/\mathtt{vL}]$. Then by $\mathtt{E}^{\bullet}$-$\mathtt{esR}$ and $\mathtt{E}^{\bullet}$-$!\varsigma$:

$$t[u/x]^{\mathsf{S}} = t^{\mathsf{S}}[u/!x] \rightsquigarrow^{\mathsf{S}}_{\varsigma(x,(\bullet\mathtt{v}^{(\mathsf{S})})\mathtt{L}^{\mathsf{S}})} t^{\mathsf{S}}[u/!(\bullet\mathtt{v}^{(\mathsf{S})})\mathtt{L}^{\mathsf{S}}] = (t[u/\mathtt{vL}])^{\mathsf{S}}$$

4. $\mathtt{E}^{\mathsf{S}}$-$\mathtt{lsw}$: Let $t[x/\mathtt{vL}] \rightsquigarrow^{\mathsf{S}}_{\mathsf{lsw}} t'[x/\mathtt{vL}]$ be derived from $t \rightsquigarrow^{\mathsf{S}}_{\varsigma(x,\mathtt{vL})} t'$. By IH we have that $t^{\mathsf{S}} \rightsquigarrow^{*}_{\varsigma(x,\mathtt{vL})^{\mathsf{S}}} t'^{\mathsf{S}}$, that is, $t^{\mathsf{S}} \rightsquigarrow_{\varsigma(x,\bullet\mathtt{v}^{(\mathsf{S})}\mathtt{L}^{\mathsf{S}})} t'^{\mathsf{S}}$. Hence:

$$\begin{array}{rll} (t[x/\mathtt{vL}])^{\mathsf{S}} = & t^{\mathsf{S}}[x/!(\mathtt{vL})^{\mathsf{S}}] & \\ = & t^{\mathsf{S}}[x/!(\bullet\mathtt{v}^{(\mathsf{S})}\mathtt{L}^{\mathsf{S}})] & \text{as } \mathtt{v}^{\mathsf{S}} = \bullet\mathtt{v}^{(\mathsf{S})} \text{ by definition} \\ \rightsquigarrow_{\bullet\mathsf{ls}} & t'^{\mathsf{S}}[x/!(\bullet\mathtt{v}^{(\mathsf{S})}\mathtt{L}^{\mathsf{S}})] & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{ls} \text{ since } t^{\mathsf{S}} \rightsquigarrow_{\varsigma(x,\bullet\mathtt{v}^{(\mathsf{S})}\mathtt{L}^{\mathsf{S}})} t'^{\mathsf{S}} \\ = & t'^{\mathsf{S}}[x/!(\mathtt{vL}^{\mathsf{S}})] & \\ = & (t'[x/\mathtt{vL}])^{\mathsf{S}} & \end{array}$$

5. $\mathtt{E}^{\mathsf{S}}$-$\mathtt{gc}$: Let $t[x/s] \rightsquigarrow^{\mathsf{S}}_{\mathsf{gc}} t$, where $x \notin \mathsf{fv}(t)$. Recall that the translation does not create free variables (Rem. E.8), so $x \notin \mathsf{fv}(t^{\mathsf{S}})$. Then:

$$\begin{array}{rll} (t[x/s])^{\mathsf{S}} = & t^{\mathsf{S}}[x/!s^{\mathsf{S}}] & \\ \rightsquigarrow_{\bullet\mathsf{gc}} & t^{\mathsf{S}} & \text{by } \mathtt{E}^{\bullet}\text{-}\mathtt{gc} \text{ since } x \notin \mathsf{fv}(t^{\mathsf{S}}) \end{array}$$

6. $\mathtt{E}^{\mathsf{S}}$-ι: Let $x \rightsquigarrow^{\mathsf{S}}_{\iota(x)} x$. Then $x^{\mathsf{S}} = x \rightsquigarrow_{\iota(x)} x = x^{\mathsf{S}}$ by $\mathtt{E}^{\bullet}$-ι.

7. $\mathtt{E}^{\mathsf{S}}$-app: Let $t\ s \rightsquigarrow_{\rho}^{\mathsf{S}} t'\ s$ be derived from $t \rightsquigarrow_{\rho}^{\mathsf{S}} t'$. By IH $t^{\mathsf{S}} \rightsquigarrow_{\rho^{\mathsf{S}}}^{*} t'^{\mathsf{S}}$. Hence:

$$\begin{array}{rl} (t\ s)^{\mathsf{S}} = & \mathtt{req}(t^{\mathsf{S}})\ !s^{\mathsf{S}} \\ \rightsquigarrow_{\rho^{\mathsf{S}}}^{*} & \mathtt{req}(t'^{\mathsf{S}})\ !s^{\mathsf{S}} \text{ by } \mathtt{E}^{\bullet}\text{-app}, \mathtt{E}^{\bullet}\text{-req (many times)} \\ = & (t'\ s)^{\mathsf{S}} \end{array}$$

8. $\mathtt{E}^{\mathsf{S}}$-subL: Let $t[x/s] \rightsquigarrow_{\rho}^{\mathsf{S}} t'[x/s]$ be derived from $t \rightsquigarrow_{\rho}^{\mathsf{S}} t'$, where $x \notin \mathsf{fv}(\rho)$. By IH $t^{\mathsf{S}} \rightsquigarrow_{\rho^{\mathsf{S}}}^{*} t'^{\mathsf{S}}$. Recall that the translation does not create free variables (Rem. E.8), so $x \notin \mathsf{fv}(\rho^{\mathsf{S}})$. Hence:

$$\begin{array}{rl} (t[x/s])^{\mathsf{S}} = & t^{\mathsf{S}}[x/!s^{\mathsf{S}}] \\ \rightsquigarrow_{\rho^{\mathsf{S}}} & t'^{\mathsf{S}}[x/!s^{\mathsf{S}}] \text{ by } \mathtt{E}^{\bullet}\text{-esL (many times), since } x \notin \mathsf{fv}(\rho^{\mathsf{S}}) \\ = & (t'[x/s])^{\mathsf{S}} \end{array}$$

9. $\mathtt{E}^{\mathsf{S}}$-subR: Let $t[x/s] \rightsquigarrow_{\rho}^{\mathsf{S}} t[x/s']$ be derived from $t \rightsquigarrow_{\iota(x)}^{\mathsf{S}} t$ and $s \rightsquigarrow_{\rho}^{\mathsf{S}} s'$. By IH we have that $t^{\mathsf{S}} \rightsquigarrow_{\iota(x)} t^{\mathsf{S}}$ and $s^{\mathsf{S}} \rightsquigarrow_{\rho^{\mathsf{S}}}^{*} s'^{\mathsf{S}}$. Note that ρ^{S} must be a sequence of the form $\langle \rho_1, \ldots, \rho_n \rangle$, so there exist terms $s_0, \ldots, s_n \in \mathcal{T}_{\bullet}$ such that $s^{\mathsf{S}} = s_0 \rightsquigarrow_{\rho_1} s_1 \rightsquigarrow_{\rho_2} \ldots s_n = s'^{\mathsf{S}}$. Hence:

$$\begin{array}{rl} t[x/s]^{\mathsf{S}} = & t^{\mathsf{S}}[x/!s^{\mathsf{S}}] \\ = & t^{\mathsf{S}}[x/!s_0] \\ \rightsquigarrow_{\rho_1} & t^{\mathsf{S}}[x/!s_1] \ \text{ by } \mathtt{E}^{\bullet}\text{-es!} \\ & \vdots \\ \rightsquigarrow_{\rho_n} & t^{\mathsf{S}}[x/!s_n] \ \text{ by } \mathtt{E}^{\bullet}\text{-es!} \\ = & t^{\mathsf{S}}[x/!s'^{\mathsf{S}}] \\ = & t[x/s']^{\mathsf{S}} \end{array}$$

Lemma F.6 (CBS **Evaluation – Inverse Simulation).** *Let $\underline{t} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$, $\underline{\rho} \in \mathcal{R}_{\bullet}^{\mathsf{S}}$ and $s \in \mathcal{T}_{\bullet}$ such that $\underline{t} \rightsquigarrow_{\underline{\rho}} s$. Then $s \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\underline{t}^{-\mathsf{S}}\ (\rightsquigarrow^{\mathsf{S}})_{\rho^{-\mathsf{S}}}^{*}\ s^{-\mathsf{S}}$.*

Proof. By induction on $\underline{t}$:

1. Variable, $\underline{t} = u$: Then the step can only be derived using the $\mathtt{E}^{\bullet}$-ς rule, *i.e.* of the form $\underline{t} = u \rightsquigarrow_{\varsigma(u,(\bullet r)\mathtt{L})} (\bullet r)\mathtt{L} = s$. Note that $\underline{\rho} = \varsigma(u, (\bullet r)\mathtt{L}) \in \mathcal{R}_{\bullet}^{\mathsf{S}}$ so we know that r is of the form $\lambda a.\,\underline{r}[v/a]$ and $\mathtt{L} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{S}}$. Hence $s = (\bullet\lambda a.\,\underline{r}[v/a])\mathtt{L} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and:

$$\begin{array}{rl} \underline{t}^{-\mathsf{S}} = & u \\ \rightsquigarrow_{\varsigma(u,(\lambda v.\,\underline{r}^{-\mathsf{S}})\mathtt{L}^{-\mathsf{S}})}^{\mathsf{S}} & (\lambda v.\,\underline{r}^{-\mathsf{S}})\mathtt{L}^{-\mathsf{S}} \\ = & ((\bullet\lambda a.\,\underline{r}[v/a])\mathtt{L})^{-\mathsf{S}} \\ = & s^{-\mathsf{S}} \end{array}$$

2. Translation of an abstraction, $\underline{t} = \bullet\lambda a.\,\underline{t_1}[u/a]$ with $a \notin \mathsf{fv}(\underline{t_1})$: this case is impossible, as there are no reduction rules that allow deriving a step $\underline{t} = \bullet\lambda a.\,\underline{t_1}[u/a] \rightsquigarrow_{\underline{\rho}} s$.
3. Request, $\underline{t} = \mathtt{req}(\underline{t_1})$: we consider two subcases, depending on whether the step is derived using the $\mathtt{E}^{\bullet}$-$\mathtt{req}\bullet$ or the $\mathtt{E}^{\bullet}$-$\mathtt{req}$ rule:

3.1 $\mathtt{E}^{\bullet}\texttt{-req}\bullet$: Note that, for the $\mathtt{E}^{\bullet}\texttt{-req}\bullet$ rule to be applicable, $\underline{t}_1$ must be of the form $\underline{t}_1 = (\bullet r)\mathtt{L}$. Moreover, since $\underline{t}_1 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ we have that $\mathtt{L} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{S}}$ and that r is of the form $r = \lambda a.\,\underline{r}_1[u/a]$ with $\underline{r}_1 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $a \notin \mathsf{fv}(\underline{r}_1)$. Then the step is of the form $\underline{t} = \mathtt{req}((\bullet\lambda a.\,\underline{r}_1[u/a])\mathtt{L}) \rightsquigarrow_{\bullet\mathsf{req}} (\lambda a.\,\underline{r}_1[u/a])\mathtt{L} = s$. Note that $s = (\lambda a.\,\underline{r}_1[u/a])\mathtt{L} \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and:

$$\begin{aligned}
\underline{t}^{-\mathsf{S}} &= \mathtt{req}((\bullet\lambda a.\,\underline{r}_1[u/a])\mathtt{L})^{-\mathsf{S}}\\
&= (\lambda u.\,\underline{r}_1^{-\mathsf{S}})\mathtt{L}^{-\mathsf{S}}\\
&= (\lambda a.\,\underline{r}_1[u/a])\mathtt{L}^{-\mathsf{S}}\\
&= s^{-\mathsf{S}}
\end{aligned}$$

3.2 $\mathtt{E}^{\bullet}\texttt{-req}$: Then the step is of the form $\underline{t} = \mathtt{req}(\underline{t}_1) \rightsquigarrow_{\underline{\rho}} \mathtt{req}(s_1) = s$, where $\underline{t}_1 \rightsquigarrow_{\underline{\rho}} s_1$. By IH, we have that $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\underline{t}_1^{-\mathsf{S}} (\rightsquigarrow^{\mathsf{S}})^*_{\underline{\rho}^{-\mathsf{S}}} s_1^{-\mathsf{S}}$. Note that $s = \mathtt{req}(s_1) \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and:

$$\begin{aligned}
t^{-\mathsf{S}} &= \mathtt{req}(\underline{t}_1)^{-\mathsf{S}}\\
&= \underline{t}_1^{-\mathsf{S}}\\
&(\rightsquigarrow^{\mathsf{S}})^*_{\underline{\rho}^{-\mathsf{S}}}\ s_1^{-\mathsf{S}}\\
&= \mathtt{req}(s_1)^{-\mathsf{S}}\\
&= s^{-\mathsf{S}}
\end{aligned}$$

4. Translation of an explicit substitution, $\underline{t} = \underline{t}_1[u/!\underline{t}_2]$: We consider five subcases, depending on the rule used to derive the step:

4.1 $\mathtt{E}^{\bullet}\texttt{-ls}$: For the $\mathtt{E}^{\bullet}\texttt{-ls}$ rule to be applicable, $\underline{t}_2$ must be of the form $\underline{t}_2 = (\bullet r)\mathtt{L}$. Moreover, since $\underline{t}_2 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$, we know that $\mathtt{L} \in \mathsf{SCtxs}_{\bullet}^{\mathsf{S}}$ and that r is of the form $r = \lambda a.\,\underline{r}_1[v/a]$ where $\underline{r}_1 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $a \notin \mathsf{fv}(\underline{r}_1)$. The step is of the form $\underline{t} = \underline{t}_1[u/!\underline{t}_2] \rightsquigarrow_{\bullet\mathsf{ls}} s_1[u/!\underline{t}_2] = s$, where $\underline{t}_1 \rightsquigarrow_{\varsigma(u,(\bullet\lambda a.\,\underline{r}_1[v/a])\mathtt{L})} s_1$. Note that $\varsigma(u,(\bullet\lambda a.\,\underline{r}_1[v/a])\mathtt{L}) \in \mathcal{R}_{\bullet}^{\mathsf{S}}$ so we may apply the IH to obtain that $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\underline{t}_1^{-\mathsf{S}} \rightsquigarrow^{\mathsf{S}}_{\varsigma(u,(\lambda a.\,\underline{r}_1^{-\mathsf{S}})\mathtt{L}^{-\mathsf{S}})} s_1^{-\mathsf{S}}$. Hence $s = s_1[u/!\underline{t}_2] \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and:

$$\begin{aligned}
\underline{t}^{-\mathsf{S}} &= (\underline{t}_1[u/!(\bullet\lambda a.\,\underline{r}_1[v/a])\mathtt{L}])^{-\mathsf{S}}\\
&= \underline{t}_1^{-\mathsf{S}}[u/(\lambda v.\,\underline{r}_1^{-\mathsf{S}})\mathtt{L}^{-\mathsf{S}}]\\
&\rightsquigarrow^{\mathsf{S}}_{\mathsf{lsw}}\ s_1^{-\mathsf{S}}[u/(\lambda v.\,\underline{r}_1^{-\mathsf{S}})\mathtt{L}^{-\mathsf{S}}] && \text{by } \mathtt{E}^{\mathsf{S}}\texttt{-lsw}\\
&= (s_1[u/!\underline{t}_2])^{-\mathsf{S}}\\
&= s^{-\mathsf{S}}
\end{aligned}$$

4.2 $\mathtt{E}^{\bullet}\texttt{-gc}$: The step is of the form $\underline{t} = \underline{t}_1[u/!\underline{t}_2] \to_{\bullet\mathsf{gc}} \underline{t}_1 = s$, where $u \notin \mathsf{fv}(\underline{t}_1)$. Then $s = \underline{t}_1 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$. Moreover, since $\mathsf{fv}(\underline{t}_1) = \mathsf{fv}(\underline{t}_1^{-\mathsf{S}})$ by Rem. E.10, we have that $u \notin \mathsf{fv}(\underline{t}_1)$, so:

$$\begin{aligned}
\underline{t}^{-\mathsf{S}} &= (\underline{t}_1[u/!\underline{t}_2])^{-\mathsf{S}}\\
&\rightsquigarrow^{\mathsf{S}}_{\mathsf{gc}}\ \underline{t}_1^{-\mathsf{S}}\\
&= s^{-\mathsf{S}}
\end{aligned}$$

4.3 $\mathtt{E}^{\bullet}\texttt{-esL}$: The step is of the form $\underline{t} = \underline{t}_1[u/!\underline{t}_2] \rightsquigarrow_{\underline{\rho}} s_1[u/!\underline{t}_2] = s$, derived from an internal step $\underline{t}_1 \rightsquigarrow_{\underline{\rho}} s_1$ where $u \notin \mathsf{fv}(\underline{\rho})$. By IH, we have that $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and

$\underline{t}_1^{-\mathsf{S}} \ (\rightsquigarrow^{\mathsf{S}})^*_{\underline{\rho}^{-\mathsf{S}}} \ s_1^{-\mathsf{S}}$, so $s = s_1[u/!\underline{t}_2] \in \mathcal{T}_\bullet^{\mathsf{S}}$ and:

$$\begin{array}{rll} \underline{t}^{-\mathsf{S}} & = \underline{t}_1[u/!\underline{t}_2]^{-\mathsf{S}} & \\ & = \underline{t}_1^{-\mathsf{S}}[u/\underline{t}_2^{-\mathsf{S}}] & \\ (\rightsquigarrow^{\mathsf{S}})^*_{\underline{\rho}^{-\mathsf{S}}} & s_1^{-\mathsf{S}}[u/\underline{t}_2^{-\mathsf{S}}] & \text{by } \mathtt{E}^{\mathsf{S}}\text{-}\mathtt{subL} \text{ (many times)} \\ & = (s_1[u/!\underline{t}_2])^{-\mathsf{S}} & \\ & = s^{-\mathsf{S}} & \end{array}$$

4.4 $\mathtt{E}^\bullet$-$\mathtt{esR}$: The step is of the form $\underline{t} = \underline{t}_1[u/!\underline{t}_2] \rightsquigarrow_{\underline{\rho}} \underline{t}_1[u/s_2] = s$, and derived from an internal step $!\underline{t}_2 \rightsquigarrow_{\underline{\rho}} s_2$. The internal step can only be derived using the $\mathtt{E}^\bullet$-$!\varsigma$ rule, namely $\underline{t}_2$ must be a variable ($\underline{t}_2 = v$) and the internal step is of the form $!v \rightsquigarrow_{\varsigma(v,(\bullet\lambda a.\,\underline{r}[w/a])\mathtt{L})} !(\bullet\lambda a.\,\underline{r}[w/a])\mathtt{L} = s_2$, where $\underline{r} \in \mathcal{T}_\bullet^{\mathsf{S}}$, $a \notin \mathsf{fv}(\underline{r})$, and $\mathtt{L} \in \mathsf{SCtxs}_\bullet^{\mathsf{S}}$. Then $s = \underline{t}_1[u/!(\bullet\lambda a.\,\underline{r}[w/a])\mathtt{L}] \in \mathcal{T}_\bullet^{\mathsf{S}}$ and:

$$\begin{array}{rll} \underline{t}^{-\mathsf{S}} & = (\underline{t}_1[u/!v])^{-\mathsf{S}} & \\ & = \underline{t}_1^{-\mathsf{S}}[u/v] & \\ \rightsquigarrow^{\mathsf{S}}_{\varsigma(v,(\lambda w.\,\underline{r}^{-\mathsf{S}})\mathtt{L}^{-\mathsf{S}})} & \underline{t}_1^{-\mathsf{S}}[u/(\lambda w.\,\underline{r}^{-\mathsf{S}})\mathtt{L}^{-\mathsf{S}}] & \text{by } \mathtt{E}^{\mathsf{S}}\text{-}\varsigma_2 \\ & = (\underline{t}_1[u/!(\bullet\lambda a.\,\underline{r}[w/a])\mathtt{L}])^{-\mathsf{S}} & \\ & = s^{-\mathsf{S}} & \end{array}$$

4.5 $\mathtt{E}^\bullet$-$\mathtt{es!}$: The step is of the form $\underline{t} = \underline{t}_1[u/!\underline{t}_2] \rightsquigarrow_{\underline{\rho}} \underline{t}_1[u/!s_2] = s$, where $\underline{t}_1 \rightsquigarrow_{\iota(u)} \underline{t}_1$ and $\underline{t}_2 \rightsquigarrow_{\underline{\rho}} s_2$. By IH on the first premise, we have that $\underline{t}_1^{-\mathsf{S}} \rightsquigarrow^{\mathsf{S}}_{\iota(u)} \underline{t}_1^{-\mathsf{S}}$. By IH on the second premise, we have that $s_2 \in \mathcal{T}_\bullet^{\mathsf{S}}$ and $\underline{t}_2^{-\mathsf{S}} \ (\rightsquigarrow^{\mathsf{S}})^*_{\underline{\rho}^{-\mathsf{S}}} \ s_2^{-\mathsf{S}}$. Then $s = \underline{t}_1[u/!s_2] \in \mathcal{T}_\bullet^{\mathsf{S}}$ and:

$$\begin{array}{rll} \underline{t}^{-\mathsf{S}} & = (\underline{t}_1[u/!\underline{t}_2])^{-\mathsf{S}} & \\ & = \underline{t}_1^{-\mathsf{S}}[u/\underline{t}_2^{-\mathsf{S}}] & \\ (\rightsquigarrow^{\mathsf{S}})^*_{\underline{\rho}^{-\mathsf{S}}} & \underline{t}_1^{-\mathsf{S}}[u/s_2^{-\mathsf{S}}] & \text{by } \mathtt{E}^{\mathsf{S}}\text{-}\mathtt{subR} \text{ (many times)} \\ & = (\underline{t}_1[u/!s_2])^{-\mathsf{S}} & \\ & = s^{-\mathsf{S}} & \end{array}$$

5. Abstraction, $\underline{t} = \lambda a.\,\underline{t}_1[u/a]$ with $a \notin \mathsf{fv}(\underline{t}_1)$: this case is impossible, as there are no reduction rules that allow deriving a step $\underline{t} = \lambda a.\,\underline{t}_1[u/a] \rightsquigarrow_{\underline{\rho}} s$.
6. Translation of an application, $\underline{t} = \underline{t}_1 \, !\underline{t}_2$: We consider two subcases, depending on whether the step is derived from $\mathtt{E}^\bullet$-$\mathtt{db}$ or $\mathtt{E}^\bullet$-$\mathtt{app}$:

6.1 $\mathtt{E}^\bullet$-$\mathtt{db}$: For the $\mathtt{E}^\bullet$-$\mathtt{db}$ rule to be applicable, $\underline{t}_1$ must be of the form $(\lambda a.\,r)\mathtt{L}$. Moreover, since $\underline{t}_1 \in \mathcal{T}_\bullet^{\mathsf{S}}$, we have that $\mathtt{L} \in \mathsf{SCtxs}_\bullet^{\mathsf{S}}$ and that r is of the form $r = \underline{r}_1[u/a]$, where $\underline{r}_1 \in \mathcal{T}_\bullet^{\mathsf{S}}$ and $a \notin \mathsf{fv}(\underline{r}_1)$. The step is of the form $\underline{t} = (\lambda a.\,\underline{r}_1[u/a])\mathtt{L} \, !\underline{t}_2 \rightsquigarrow_{\bullet\mathsf{db}} \underline{r}_1[u/!\underline{t}_2]\mathtt{L} = s$. So $s = \underline{r}_1[u/!\underline{t}_2]\mathtt{L} \in \mathcal{T}_\bullet^{\mathsf{S}}$ and:

$$\begin{array}{rl} \underline{t}^{-\mathsf{S}} & = ((\lambda a.\,\underline{r}_1[u/a])\mathtt{L} \, !\underline{t}_2)^{-\mathsf{S}} \\ & = (\lambda u.\,\underline{r}_1^{-\mathsf{S}})\mathtt{L}^{-\mathsf{S}} \, \underline{t}_2^{-\mathsf{S}} \\ \rightsquigarrow^{\mathsf{S}}_{\mathsf{db}} & \underline{r}_1^{-\mathsf{S}}[u/\underline{t}_2^{-\mathsf{S}}]\mathtt{L}^{-\mathsf{S}} \\ & = ((\underline{r}_1[u/!\underline{t}_2]\mathtt{L})^{-\mathsf{S}} \\ & = s^{-\mathsf{S}} \end{array}$$

6.2 $\mathtt{E}^{\bullet}$-app: The step is of the form $\underline{t} = \underline{t}_1\, !\underline{t}_2 \rightsquigarrow_{\underline{\rho}} s_1\, !\underline{t}_2 = s$, and derived from an internal step $\underline{t}_1 \rightsquigarrow_{\underline{\rho}} s_1$. By IH, we have that $s_1 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and $\underline{t}_1^{-\mathsf{S}} (\rightsquigarrow^{\mathsf{S}})^{*}_{\underline{\rho}^{-\mathsf{S}}} s_1^{-\mathsf{S}}$. So $s = s_1\, !\underline{t}_2 \in \mathcal{T}_{\bullet}^{\mathsf{S}}$ and:

$$
\begin{array}{rll}
\underline{t}^{-\mathsf{S}} & = (\underline{t}_1\, !\underline{t}_2)^{-\mathsf{S}} & \\
& = \underline{t}_1^{-\mathsf{S}}\, \underline{t}_2^{-\mathsf{S}} & \\
(\rightsquigarrow^{\mathsf{S}})^{*}_{\underline{\rho}^{-\mathsf{S}}} & s_1^{-\mathsf{S}}\, \underline{t}_2^{-\mathsf{S}} & \text{by } \mathtt{E}^{\mathsf{S}}\texttt{-app (many times)} \\
& = (s_1\, \underline{t}_2)^{-\mathsf{S}} & \\
& = s^{-\mathsf{S}} &
\end{array}
$$

G Appendix: The Bang Calculus

G.1 A Simplified Presentation of the Bang Calculus

A binary relation $\ltimes \subseteq \mathcal{T}_{\mathsf{B}^{\mathsf{der}}} \times \mathcal{T}_{\mathsf{B}}$, called *dereliction unfolding* is defined inductively as:

$$
\frac{}{x \ltimes x} \qquad \frac{t \ltimes s}{\lambda x.\, t \ltimes \lambda x.\, s} \qquad \frac{t_1 \ltimes s_1 \quad t_2 \ltimes s_2}{t_1\, t_2 \ltimes s_1\, s_2} \qquad \frac{t_1 \ltimes s_1 \quad t_2 \ltimes s_2}{t_1[x/t_2] \ltimes s_1[x/s_2]}
$$

$$
\frac{t \ltimes s}{!t \ltimes !s} \qquad \frac{t \ltimes s}{\mathtt{der}(t) \ltimes x[x/s]} \qquad \frac{t \ltimes s \quad x \notin \mathsf{fv}(s)}{t \ltimes s[x/!r]}
$$

The last rule is required since a dereliction step $\mathtt{der}((!t)\mathtt{L}) \to_{\mathsf{d!}} t\mathtt{L}$ is simulated as a $\mathsf{ls!}$ step $x[x/(!t)\mathtt{L}] \to_{\mathsf{ls!}} t[x/(!t)]\mathtt{L}$.

Remark G.1. Note that $t \in \mathcal{T}_{\mathsf{B}}$ implies $t \ltimes t$.

Lemma G.1 (Simulation of dereliction [Proofs on Sec. G.2]).

1. *If $t \ltimes s$ and $t \to_{\mathsf{B}^{\mathsf{der}}} t'$, there exists s' such that $s \to_{\mathsf{B}}^{*} s'$ and $t' \ltimes s'$. Moreover, if the step $t \to_{\mathsf{B}^{\mathsf{der}}} t'$ is not a $\mathsf{gc!}$ step, the reduction $s \to_{\mathsf{B}}^{*} s'$ consists of exactly one step.*
2. *If $t \ltimes s$ and $s \to_{\mathsf{B}^{\mathsf{der}}} s'$, there exists t' such that $t \to_{\mathsf{B}}^{=} t'$ and $t' \ltimes s'$.*

In the second part of the lemma, the only situation where a step is simulated by an empty $\mathsf{B}^{\mathsf{der}}$-step is when an $\mathsf{ls!}$-step is made inside the body of a garbage substitution. For example, $x[z/!r] \ltimes x[y/!z][z/!r]$. The preceding lemma has the following consequence:

Proposition G.1 (Simplified Bang simulation [proof of Prop. 7.1]). $\to_{\mathsf{B}^{\mathsf{der}}}$ *and* $\to_{\mathsf{B}}$ *simulate each other.*

Proof. To show that $\to_{\mathsf{B}}$ simulates $\to_{\mathsf{B}^{\mathsf{der}}}$, consider a reduction sequence $t_1 \to_{\mathsf{B}^{\mathsf{der}}} t_2 \to_{\mathsf{B}^{\mathsf{der}}} \ldots t_{n-1} \to_{\mathsf{B}^{\mathsf{der}}} t_n$, let s_1 be the smallest term such that $t_1 \ltimes s_1$ and proceed by induction on n, resorting to Lem. G.1(1). To show that $\to_{\mathsf{B}^{\mathsf{der}}}$ simulates $\to_{\mathsf{B}}$, we resort to Rem. G.1 and Lem. G.1(2). The $\to_{\mathsf{B}^{\mathsf{der}}}$ step can be taken to never be empty since $\to_{\mathsf{B}} \subseteq \to_{\mathsf{B}^{\mathsf{der}}}$ and the derivation of Rem. G.1 does not erase substitutions.

G.2 Embedding the Bang Calculus

Remark G.2. Suppose $t \ltimes s$. Then:

1. $\mathsf{fv}(t) \subseteq \mathsf{fv}(s)$, and;
2. if the rule gc is not used in the derivation, then $\mathsf{fv}(t) = \mathsf{fv}(s)$.

Lemma G.2. *1. $\lambda x.\, t \ltimes s$ implies $s = (\lambda x.\, s_1)\mathsf{L}$ with $t \ltimes s_1$ and $\mathsf{dom}(\mathsf{L}) \cap \mathsf{fv}(\lambda x.\, s_1) = \emptyset$.*

2. *$(\lambda x.\, t)\mathsf{L} \ltimes s$ implies $s = (\lambda x.\, s_1)\mathsf{L}'$ with $t \ltimes s_1$ and $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{L}'$ and $\mathcal{X} \cap \mathsf{fv}(\lambda x.\, s_1) = \emptyset$.*
3. *$\mathsf{C}\langle\!\langle x \rangle\!\rangle \ltimes s$ implies $s = \mathsf{C}'\langle\!\langle x \rangle\!\rangle$ and $\mathsf{C} \ltimes_{\mathcal{X}} \mathsf{C}'$ and $x \notin \mathcal{X}$.*
4. *$(!t)\mathsf{L} \ltimes s$ implies $s = (!s_1)\mathsf{L}'$ and $t \ltimes s_1$ and $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{L}'$ and $\mathcal{X} \cap \mathsf{fv}(!s_1) = \emptyset$.*
5. *$t \ltimes s$ and $x \notin \mathsf{fv}(t)$ and $x \in \mathsf{fv}(s)$ implies $s = \mathsf{C}\langle\!\langle s_1[y/!s_2] \rangle\!\rangle$ and $y \notin \mathsf{fv}(s_1)$ and $x \in \mathsf{fv}(!s_2)$ and $t \ltimes \mathsf{C}\langle\!\langle s_1 \rangle\!\rangle$.*
6. *$t \ltimes s$ and $x \notin \mathsf{fv}(t)$ and $x \in \mathsf{fv}(s)$ implies there exists s' such that $s \to^*_{\mathsf{gc!}} s'$ and $t \ltimes s'$ and $x \notin \mathsf{fv}(s')$.*
7. *$t \ltimes s$ and $\mathsf{C} \ltimes_{\mathcal{X}} \mathsf{D}$ and $\mathcal{X} \cap \mathsf{fv}(s) = \emptyset$ implies $\mathsf{C}\langle\!\langle t \rangle\!\rangle \ltimes \mathsf{D}\langle\!\langle s \rangle\!\rangle$.*
8. *$t \ltimes s$ and $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{K}$ and $\mathcal{X} \cap \mathsf{fv}(s) = \emptyset$ implies $\mathsf{L}\langle t \rangle \ltimes \mathsf{K}\langle s \rangle$.*

Proof. Item 1. By induction on the derivation of $\lambda x.\, t \ltimes s$. Notice that only two rules apply.

1. $s = \lambda x.\, s_1$ and the derivation ends in:

$$\frac{t \ltimes s_1}{\lambda x.\, t \ltimes \lambda x.\, s_1}\,\mathsf{abs}$$

   We set $\mathsf{L}' := \Box$ and conclude.
2. Then $s = s_1[y/!s_2]$ and the derivation ends in:

$$\frac{\lambda x.\, t \ltimes s_1 \quad y \notin \mathsf{fv}(s_1)}{\lambda x.\, t \ltimes s_1[y/!s_2]}\,\mathsf{gc}$$

   From the IH, $s_1 = (\lambda x.\, s_{11})\mathsf{L}'$ with $t \ltimes s_{11}$ and $\mathsf{dom}(\mathsf{L}') \cap \mathsf{fv}(\lambda x.\, s_{11}) = \emptyset$. We set $\mathsf{L} := \mathsf{L}'[y/!s_2]$ and conclude.

Item 2. By induction on L using item 1 for the base case. For the inductive case, only two subcases arise (the derivation ends in esub or gc); in each we use the IH.

Item 3. By induction on the derivation of $\mathsf{C}\langle\!\langle x \rangle\!\rangle \ltimes s$.

Item 4. By induction on the derivation of $t \ltimes s$.

Item 5. By induction on $t \ltimes s$.

- $t = x \ltimes x = s$. The result holds vacuously since $x \notin \mathsf{fv}(t)$ and $x \in \mathsf{fv}(s)$ is not possible.
- $t = \lambda z.\, t_1 \ltimes \lambda z.\, s_1 = s$ follows from $t_1 \ltimes s_1$ and $x \notin \mathsf{fv}(\lambda z.\, t_1)$ and $x \in \mathsf{fv}(\lambda z.\, s_1)$ and, w.l.o.g., we assume $x \neq z$. Note that $x \notin \mathsf{fv}(t_1)$ and $x \in \mathsf{fv}(s_1)$. By the IH, $s_1 = \mathsf{C}_1\langle\!\langle s_{11}[y/!s_{12}] \rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t_1 \ltimes \mathsf{C}_1\langle\!\langle s_{11} \rangle\!\rangle$. We set $\mathsf{C} := \lambda z.\, \mathsf{C}_1$. Note that $s = \mathsf{C}\langle\!\langle s_{11}[y/!s_{12}] \rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t \ltimes \mathsf{C}\langle\!\langle s_{11} \rangle\!\rangle$. The latter follows from $t_1 \ltimes \mathsf{C}_1\langle\!\langle s_{11} \rangle\!\rangle$ and abs. Thus we conclude.
- $t = t_1\, t_2 \ltimes s_1\, s_2 = s$ follows from $t_1 \ltimes s_1$ and $t_2 \ltimes s_2$ and $x \notin \mathsf{fv}(t)$ and $x \in \mathsf{fv}(s)$. Note that $x \notin \mathsf{fv}(t_1)$ and $x \notin \mathsf{fv}(t_2)$. Also, either $x \in \mathsf{fv}(s_1)$ or $x \in \mathsf{fv}(s_2)$. We consider three cases:

1. $x \in \mathsf{fv}(s_1)$ and $x \notin \mathsf{fv}(s_2)$. By the IH, $s_1 = \mathsf{C}_1\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t_1 \ltimes \mathsf{C}_1\langle\!\langle s_{11}\rangle\!\rangle$. We set $\mathsf{C} := \mathsf{C}_1\, s_2$. Note that $s = \mathsf{C}\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t \ltimes \mathsf{C}\langle\!\langle s_{11}\rangle\!\rangle$. The latter follows from $t_1 \ltimes \mathsf{C}_1\langle\!\langle s_{11}\rangle\!\rangle$ and $t_2 \ltimes s_2$ and $\mathtt{app}$. Thus we conclude
2. $x \notin \mathsf{fv}(s_1)$ and $x \in \mathsf{fv}(s_2)$. Similar to the previous case.
3. $x \in \mathsf{fv}(s_1)$ and $x \in \mathsf{fv}(s_2)$. Similar to the previous case but using the IH twice.

- $t = t_1[z/t_2] \ltimes s_1[z/s_2]$ follows from $t_1 \ltimes s_1$ and $t_2 \ltimes s_2$ and $x \notin \mathsf{fv}(t)$ and $x \in \mathsf{fv}(s)$ and, w.l.o.g., we assume $x \neq z$. Note that $x \notin \mathsf{fv}(t_1)$ and $x \notin \mathsf{fv}(t_2)$. Also, either $x \in \mathsf{fv}(s_1)$ or $x \in \mathsf{fv}(s_2)$. We consider two cases:
  1. $x \in \mathsf{fv}(s_1)$ and $x \notin \mathsf{fv}(s_2)$. By the IH, $s_1 = \mathsf{C}_1\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t_1 \ltimes \mathsf{C}_1\langle\!\langle s_{11}\rangle\!\rangle$. We set $\mathsf{C} := \mathsf{C}_1[z/s_2]$. Note that $s = \mathsf{C}\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t \ltimes \mathsf{C}\langle\!\langle s_1\rangle\!\rangle$. The latter follows from $t_1 \ltimes \mathsf{C}_1\langle\!\langle s_{11}\rangle\!\rangle$ and $t_2 \ltimes s_2$ and $\mathtt{app}$. Thus we conclude
  2. $x \notin \mathsf{fv}(s_1)$ and $x \in \mathsf{fv}(s_2)$. Similar to the previous case.
  3. $x \in \mathsf{fv}(s_1)$ and $x \in \mathsf{fv}(s_2)$. Similar to the previous case, but using the IH twice.
- $t = !t_1 \ltimes !s_1 = s$ follows from $t_1 \ltimes s_1$ and $x \notin \mathsf{fv}(t)$ and $x \in \mathsf{fv}(s)$. Note that $x \notin \mathsf{fv}(t_1)$ and $x \in \mathsf{fv}(s_1)$. By the IH, $s_1 = \mathsf{C}_1\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t_1 \ltimes \mathsf{C}_1\langle\!\langle s_{11}\rangle\!\rangle$. We set $\mathsf{C} := !\mathsf{C}_1$. Note that $s = \mathsf{C}\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t \ltimes \mathsf{C}\langle\!\langle s_1\rangle\!\rangle$. The latter follows from $t_1 \ltimes \mathsf{C}'_1\langle\!\langle s_{11}\rangle\!\rangle$ and $\mathtt{ofc}$. Thus we conclude.
- $t = \mathtt{der}(t_1) \ltimes z[z/s_1] = s$ follows from $t_1 \ltimes s_1$ and $x \notin \mathsf{fv}(t)$ and $x \in \mathsf{fv}(s)$. Note that $x \notin \mathsf{fv}(t_1)$ and $x \in \mathsf{fv}(s_1)$. By the IH, $s_1 = \mathsf{C}'_1\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t_1 \ltimes \mathsf{C}'_1\langle\!\langle s_{11}\rangle\!\rangle$. We set $\mathsf{C}' := z[z/\mathsf{C}'_1]$. Note that $s = \mathsf{C}'\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t \ltimes \mathsf{C}'\langle\!\langle s_1\rangle\!\rangle$. The latter follows from $t_1 \ltimes \mathsf{C}'_1\langle\!\langle s_{11}\rangle\!\rangle$ and $\mathtt{der}$. Thus we conclude.
- $t \ltimes s_1[z/!s_2] = s$ follows from $t \ltimes s_1$ and $z \notin \mathsf{fv}(s_1)$ and $x \notin \mathsf{fv}(t)$ and $x \in \mathsf{fv}(s)$. Note that either $x \in \mathsf{fv}(s_1)$ or $x \in \mathsf{fv}(!s_2)$. We consider two cases:
  1. $x \in \mathsf{fv}(s_1)$. By the IH, $s_1 = \mathsf{C}_1\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t_1 \ltimes \mathsf{C}_1\langle\!\langle s_{11}\rangle\!\rangle$. We set $\mathsf{C} := \mathsf{C}_1[z/!s_2]$. Note that $s = \mathsf{C}\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t \ltimes \mathsf{C}\langle\!\langle s_1\rangle\!\rangle$. The latter follows from $t_1 \ltimes \mathsf{C}_1\langle\!\langle s_{11}\rangle\!\rangle$ and $\mathtt{gc}$. Thus we conclude
  2. $x \in \mathsf{fv}(s_2)$. We set $\mathsf{C} := \Box$ and conclude.

Item 6. By induction on the size n of s. If $n = 1$, then the result holds trivially since by the derivation must end in $\mathtt{var}$ and hence $x \notin \mathsf{fv}(t)$ and $x \in \mathsf{fv}(s)$ is not possible. Suppose $n > 0$. Then from item 5, $s = \mathsf{C}\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle$ and $y \notin \mathsf{fv}(s_{11})$ and $x \in \mathsf{fv}(!s_{12})$ and $t \ltimes \mathsf{C}\langle\!\langle s_{11}\rangle\!\rangle = s_1$. Note that $s = \mathsf{C}\langle\!\langle s_{11}[y/!s_{12}]\rangle\!\rangle \to_{\mathsf{gc!}} \mathsf{C}\langle\!\langle s_{11}\rangle\!\rangle$. If $x \notin \mathsf{fv}(s_1)$, then we conclude. Otherwise, we can apply the IH on $t \ltimes \mathsf{C}\langle\!\langle s_{11}\rangle\!\rangle$ and conclude from that.

Item 7. By induction on the derivation of $\mathsf{C} \ltimes_{\mathcal{X}} \mathsf{D}$.

Item 8. By induction on the derivation of $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{K}$.

Lemma G.3 (Forwards simulation of dereliction).

$$\begin{tikzcd} t \arrow[d, "\mathfrak{r}"'] & \ltimes & s \arrow[d, dashed, "\mathsf{B}"] \\ t' & \ltimes & s' \end{tikzcd} \qquad \begin{tikzcd} t \arrow[d, "\mathsf{gc!}"'] & \ltimes & s \arrow[d, dashed, two heads, "\mathsf{gc!}"] \\ t' & \ltimes & s' \end{tikzcd} \tag{2}$$

where $\mathfrak{r} \in \{\mathsf{db}, \mathsf{ls!}, \mathsf{d!}\}$.

Proof. We prove both items by induction on the derivation of $t \ltimes s$.

1. $t \ltimes s$ is $x \ltimes x$. Both items are immediate since there are no $\to_{\mathsf{B}^{\mathsf{der}}}$ steps from x.
2. $t \ltimes s$ is $\lambda x.\, t_1 \ltimes \lambda x.\, s_1$ and follows from $t_1 \ltimes s_1$.
   - Item 1. The reduction must be internal: $\lambda x.\, t_1 \to_{\neg\mathsf{gc}!} \lambda x.\, t_1' = t'$ follows from $t_1 \to_{\neg\mathsf{gc}!} t_1'$. From the IH there exists s_1' such that $s_1 \to_{\mathsf{B}} s_1'$ and $t_1' \ltimes s_1'$. Then $\lambda x.\, s_1 \to_{\mathsf{B}} \lambda x.\, s_1'$ and moreover $\lambda x.\, t_1' \ltimes \lambda x.\, s_1'$. Thus we conclude by setting s' to be $\lambda x.\, s_1'$.
   - Item 2. Similar to Item 1.
3. $t \ltimes s$ is $t_1\, t_2 \ltimes s_1\, s_2$ and follows from $t_1 \ltimes s_1$ and $t_2 \ltimes s_2$
   - Item 1. There are three cases.
     3.1 The reduction is internal to t_1: $t = t_1\, t_2 \to_{\neg\mathsf{gc}!} t_1' t_2 = t'$ follows from $t_1 \to_{\neg\mathsf{gc}!} t_1'$. From the IH there exists s_1' such that $s_1 \to_{\mathsf{B}} s_1'$ and $t_1' \ltimes s_1'$. Then $s_1\, s_2 \to_{\mathsf{B}} s_1'\, s_2$ and moreover $t_1'\, t_2 \ltimes s_1'\, s_2$. Thus we conclude by setting s' to be $s_1'\, s_2$.
     3.2 The reduction is internal to t_2: $t = t_1\, t_2 \to_{\neg\mathsf{gc}!} t_1 t_2' = t'$ follows from $t_2 \to_{\neg\mathsf{gc}!} t_2'$. Similar to the previous case.
     3.3 The reduction is at the root of t. Then the step is a db step: $t = (\lambda x.\, t_{11})\mathsf{L}\, t_2 \to_{\mathsf{db}} t_{11}[x/t_2]\mathsf{L} = t'$. By Lem. G.2(2) there exists s_{11}, L' such that $s_1 = (\lambda x.\, s_{11})\mathsf{L}'$ with $t_{11} \ltimes s_{11}$ and $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{L}'$ and $\mathcal{X} \cap \mathsf{fv}(\lambda x.\, s_{11}) = \varnothing$.
     We set s' to be $s_{11}[x/s_2]\mathsf{L}'$. Note that $s = (\lambda x.\, s_{11})\mathsf{L}'\, s_2 \to_{\mathsf{db}} s_{11}[x/s_2]\mathsf{L}'$. From $t_{11} \ltimes s_{11}$ and $t_2 \ltimes s_2$, $t_{11}[x/t_2] \ltimes s_{11}[x/s_2]$ holds. Finally, since $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{L}'$ and $t_2 \ltimes s_2$, also $[x/t_2]\mathsf{L} \ltimes_{\mathcal{X}} [x/s_2]\mathsf{L}'$. We obtain $t_{11}[x/t_2]\mathsf{L} \ltimes s_{11}[x/s_2]\mathsf{L}'$ from Lem. G.2(8).
   - Item 2. There are two cases.
     3.1 The reduction is internal to t_1: $t = t_1\, t_2 \to_{\mathsf{gc}!} t_1' t_2 = t'$ follows from $t_1 \to_{\mathsf{gc}!} t_1'$. From the IH there exists s_1' such that $s_1 \to_{\mathsf{gc}!}^{*} s_1'$ and $t_1' \ltimes s_1'$. Then $s_1\, s_2 \to_{\mathsf{gc}!}^{*} s_1'\, s_2$ and moreover $t_1'\, t_2 \ltimes s_1'\, s_2$. Thus we conclude by setting s' to be $s_1'\, s_2$.
     3.2 The reduction is internal to t_2: $t = t_1\, t_2 \to_{\mathsf{gc}!} t_1 t_2' = t'$ follows from $t_2 \to_{\mathsf{gc}!} t_2'$. Similar to the previous case.
4. $t \ltimes s$ is $t_1[x/t_2] \ltimes s_1[x/s_2]$ and follows from $t_1 \ltimes s_1$ and $t_2 \ltimes s_2$.
   - Item 1. There are three cases.
     4.1 The reduction is internal to t_1. Then $t_1[x/t_2] \to_{\neg\mathsf{gc}!} t_1'[x/t_2] = t'$ follows from $t_1 \to_{\neg\mathsf{gc}!} t_1'$. From the hypothesis $t_1 \ltimes s_1$ and the IH, there exists s_1' such that $s_1 \to_{\mathsf{B}} s_1'$ and $t_1' \ltimes s_1'$. Then we set s' to be $s_1'[x/s_2]$. Note that $s_1[x/s_2] \to_{\mathsf{B}} s_1'[x/s_2]$ and $t_1'[x/t_2] \ltimes s_1'[x/s_2]$, and we conclude.
     4.2 The reduction is internal to t_2. Similar to the previous case.
     4.3 The reduction is at the root of t. Then the reduction step must be a $\mathsf{ls}!$-step: $t = \mathsf{C}\langle\!\langle x\rangle\!\rangle[x/(!t_{21})\mathsf{L}] \mapsto_{\mathsf{ls}!} \mathsf{C}\langle\!\langle t_{21}\rangle\!\rangle[x/!t_{21}]\mathsf{L} = t'$ and $x \notin \mathsf{fv}(t_{21})$ and $\mathsf{fv}(\mathsf{C}) \cap \mathsf{dom}(\mathsf{L}) = \varnothing$. From $\mathsf{C}\langle\!\langle x\rangle\!\rangle \ltimes s_1$ and Lem. G.2(3), $s_1 = \mathsf{C}'\langle\!\langle x\rangle\!\rangle$ and $\mathsf{C} \ltimes_{\mathcal{X}} \mathsf{C}'$ and $x \notin \mathcal{X}$. Similarly, from $(!t_{21})\mathsf{L} \ltimes s_2$ and Lem. G.2(4), $s_2 = (!s_{21})\mathsf{L}'$ and $t_{21} \ltimes s_{21}$ and $\mathsf{L} \ltimes_{\mathcal{Y}} \mathsf{L}'$ and $\mathcal{Y} \cap \mathsf{fv}(!s_{21}) = \varnothing$. Then $s = \mathsf{C}'\langle\!\langle x\rangle\!\rangle[x/(!s_{21})\mathsf{L}'] \to_{\mathsf{B}} \mathsf{C}'\langle\!\langle s_{21}\rangle\!\rangle[x/!s_{21}]\mathsf{L}'$. From $t_{21} \ltimes s_{21}$ and $\mathsf{C} \ltimes_{\mathcal{X}} \mathsf{C}'$, we deduce from Lem. G.2(7), that $\mathsf{C}\langle\!\langle t_{21}\rangle\!\rangle \ltimes \mathsf{C}'\langle\!\langle s_{21}\rangle\!\rangle$. We may assume that $\mathsf{dom}(\mathsf{L}') \cap \mathsf{fv}(\mathsf{C}'\langle\!\langle s_{21}\rangle\!\rangle) = \varnothing$. Finally, we set s' to be $\mathsf{C}'\langle\!\langle s_{21}\rangle\!\rangle[x/!s_{21}]\mathsf{L}'$ and conclude with $\mathsf{C}\langle\!\langle t_{21}\rangle\!\rangle[x/!t_{21}]\mathsf{L} \ltimes \mathsf{C}'\langle\!\langle s_{21}\rangle\!\rangle[x/!s_{21}]\mathsf{L}'$ by Lem. G.2(8).

- Item 2. There are three cases.
  - 4.1 The reduction is internal to t_1. Same as above.
  - 4.2 The reduction is internal to t_2. Same as above.
  - 4.3 The reduction is at the root of t. Then the reduction step must be a $\mathsf{gc!}$-step: $t = t_1[x/(!t_{21})\mathsf{L}] \mapsto_{\mathsf{gc!}} t_1\mathsf{L} = t'$ and $x \notin \mathsf{fv}(t_1)$. From $(!t_{21})\mathsf{L} \ltimes s_2$ and Lem. G.2(4), $s_2 = (!s_{21})\mathsf{L}'$ and $t_{21} \ltimes s_{21}$ and $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{L}'$ and $\mathcal{X} \cap \mathsf{fv}(!s_{21}) = \emptyset$. Moreover, we may assume that not only $\mathcal{X} \cap \mathsf{fv}(!s_{21}) = \emptyset$, but also $\mathcal{X} \cap \mathsf{fv}(s_1) = \emptyset$. Consider $s_1[x/(!s_{21})\mathsf{L}']$. There are two further cases:
    - 4.3.1 $x \notin \mathsf{fv}(s_1)$. Then $s = s_1[x/(!s_{21})\mathsf{L}'] \mapsto_{\mathsf{gc!}} s_1\mathsf{L}' = s'$. Note that from $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{L}'$, $\mathsf{dom}(\mathsf{L}') \cap \mathsf{fv}(s_1) = \emptyset$ and Lem. G.2(8), we have $t_1\mathsf{L} \ltimes s_1\mathsf{L}'$. We thus set s' to be $s_1\mathsf{L}'$ and conclude.
    - 4.3.2 $x \in \mathsf{fv}(s_1)$. Since $x \notin \mathsf{fv}(t_1)$ and $t_1 \ltimes s_1$, then by Lem. G.2(6), there exists s'_1 such that $x \notin \mathsf{fv}(s'_1)$ and $t_1 \ltimes s'_1$ and $s_1 \to^*_{\mathsf{gc!}} s'$. Then $s_1[x/(!s_{21})\mathsf{L}'] \to^*_{\mathsf{gc!}} s'_1[x/(!s_{21})\mathsf{L}'] \to_{\mathsf{gc!}} s'_1\mathsf{L}'$. We thus set s' to be $s'_1\mathsf{L}'$ and conclude.

5. $t \ltimes s$ is $!t_1 \ltimes !s_1$ and follows from $t_1 \ltimes s_1$.
   - Item 1.
     - 5.1 The reduction must be internal to t_1. Then $!t_1 \to_{\neg\mathsf{gc!}} !t'_1 = t'$ follows from $t_1 \to_{\neg\mathsf{gc!}} t'_1$. From the hypothesis $t_1 \ltimes s_1$ and the IH, there exists s'_1 such that $s_1 \to_{\mathsf{B}} s'_1$ and $t'_1 \ltimes s'_1$. Then we set s' to be $!s'_1$. Note that $!s_1 \to_{\mathsf{B}} !s'_1$ and $!t'_1 \ltimes !s'_1$, and we conclude.
   - Item 2.
     - 5.1 The reduction must be internal to t_1. Then $!t_1 \to_{\mathsf{gc!}} !t'_1 = t'$ follows from $t_1 \to_{\mathsf{gc!}} t'_1$. From the hypothesis $t_1 \ltimes s_1$ and the IH, there exists s'_1 such that $s_1 \to^*_{\mathsf{gc!}} s'_1$ and $t'_1 \ltimes s'_1$. Then we set s' to be $!s'_1$. Note that $!s_1 \to^*_{\mathsf{gc!}} !s'_1$ and $!t'_1 \ltimes !s'_1$, and we conclude.
6. $t \ltimes s$ is $\mathtt{der}(t_1) \ltimes z[z/s_1]$ and follows from $t_1 \ltimes s_1$.
   - Item 1. There are two cases.
     - 6.1 The reduction is internal to t_1. Then $\mathtt{der}(t_1) \to_{\neg\mathsf{gc!}} \mathtt{der}(t'_1) = t'$ follows from $t_1 \to_{\neg\mathsf{gc!}} t'_1$. From the hypothesis $t_1 \ltimes s_1$ and the IH, there exists s'_1 such that $s_1 \to_{\mathsf{B}} s'_1$ and $t'_1 \ltimes s'_1$. Then we set s' to be $z[z/s'_1]$. Note that $z[z/s_1] \to_{\mathsf{B}} z[z/s'_1]$ and $\mathtt{der}(t'_1) \ltimes z[z/s'_1]$, and we conclude.
     - 6.2 The reduction is at the root of t. Then the reduction step must be a $\mathsf{d!}$-step: $t = \mathtt{der}((!t_{11})\mathsf{L}) \mapsto_{\mathsf{d!}} t_{11}\mathsf{L} = s$. From $t_1 = (!t_{11})\mathsf{L} \ltimes s_1$. and Lem. G.2(4), $s_2 = (!s_{21})\mathsf{L}'$ and $t_{11} \ltimes s_{21}$ and $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{L}'$ and $\mathcal{X} \cap \mathsf{fv}(!s_{21}) = \emptyset$. Then $z[z/(!s_{21})\mathsf{L}'] \to_{\mathsf{ls!}} s_{21}[z/!s_{21}]\mathsf{L}'$. Moreover, $t_{11}\mathsf{L} \ltimes s_{21}[z/!s_{21}]\mathsf{L}'$ follows from $t_{11} \ltimes s_{21}$, then $t_{11} \ltimes s_{21}[z/!s_{21}]$, and finally from the latter and $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{L}'$ by Lem. G.2(8). We set s' to be $s_{21}[x/!s_{21}]\mathsf{L}'$ and conclude.
   - Item 2.
     - 6.1 The reduction is internal to t_1. Same as above.
7. $t \ltimes s$ is $t \ltimes s_1[z/!s_2]$ follows from $t \ltimes s_1$ and $z \notin \mathsf{fv}(s_1)$,
   - Item 1.
     - 7.1 Suppose $t \to_{\neg\mathsf{gc!}} t'$. From the hypothesis $t \ltimes s_1$ and the IH, there exists s'_1 such that $s_1 \to_{\mathsf{B}} s'_1$ and $t' \ltimes s'_1$. Then we set s' to be $s'_1[x/!s_2]$. Note that $s_1[x/!s_2] \to_{\mathsf{B}} s'_1[x/!s_2]$ and $t' \ltimes s'_1[x/!s_2]$, and we conclude.
   - Item 2.

7.1 Suppose $t \to_{\mathsf{gc!}} t'$. From the hypothesis $t \ltimes s_1$ and the IH, there exists s'_1 such that $s_1 \to^*_{\mathsf{gc!}} s'_1$ and $t' \ltimes s'_1$. Then we set s' to be $s'_1[x/!s_2]$. Note that $s_1[z/!s_2] \to^*_{\mathsf{gc!}} s'_1[z/!s_2]$ and $t' \ltimes s'_1[z/!s_2]$, and we conclude.

Lemma G.4. *1. $t \ltimes (\lambda x.\, s)\mathtt{L}'$ implies $t = (\lambda x.\, t_1)\mathtt{L}$ with $t_1 \ltimes s$ and $\mathtt{L} \ltimes_{\mathcal{X}} \mathtt{L}'$ and $\mathcal{X} \cap \mathsf{fv}(\lambda x.\, t_1) = \emptyset$.*

2. *$t \ltimes \mathtt{C}'\langle\langle x\rangle\rangle$ implies $t = \mathtt{C}\langle\langle x\rangle\rangle$ and $\mathtt{C} \ltimes_{\mathcal{X}} \mathtt{C}'$ and $x \notin \mathcal{X}$.*
3. *$t \ltimes (!s)\mathtt{L}'$ implies $t = (!t_1)\mathtt{L}$ and $t_1 \ltimes s$ and $\mathtt{L} \ltimes_{\mathcal{X}} \mathtt{L}'$ and $\mathcal{X} \cap \mathsf{fv}(!s) = \emptyset$.*
4. *$t \ltimes \mathtt{D}\langle\langle x\rangle\rangle$ implies*
   4.1 either $x \in \mathsf{fv}(t)$ and $t = \mathtt{C}\langle\langle x\rangle\rangle$ and $\mathtt{C} \ltimes_{\mathcal{X}} \mathtt{D}$;
   4.2 or $x \notin \mathsf{fv}(t)$ and $\mathtt{D} = \mathtt{D}_1\langle s_1[y/!\mathtt{D}_2\langle\langle x\rangle\rangle]\rangle$ and $y \notin \mathsf{fv}(s_1)$ and $t = \mathtt{C}\langle t_1\rangle$ and $\mathtt{C}\langle t_1\rangle \ltimes \mathtt{D}_1\langle s_1\rangle$.

Proof. Item 1. By induction on the derivation of $t \ltimes (\lambda x.\, s)\mathtt{L}'$.
Item 2. By induction on the derivation of $t \ltimes \mathtt{C}'\langle\langle x\rangle\rangle$.
Item 3. By induction on the derivation of $t \ltimes (!s)\mathtt{L}'$.
Item 4. By induction on the derivation of $t \ltimes \mathtt{D}\langle\langle x\rangle\rangle$.

Lemma G.5 (Backwards simulation of dereliction).

$$
\begin{tikzcd}
t \arrow[d, dashed, "\mathsf{B}^{\mathsf{der}}"', "="] \arrow[r, phantom, "\ltimes"] & s \arrow[d, "\mathfrak{r}"] \\
t' \arrow[r, phantom, "\ltimes"] & s'
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[d, dashed, "\mathsf{gc!}"', "="] \arrow[r, phantom, "\ltimes"] & s \arrow[d, "\mathsf{gc!}"] \\
t' \arrow[r, phantom, "\ltimes"] & s'
\end{tikzcd}
\tag{3}
$$

where $\mathfrak{r} \in \{\mathsf{db}, \mathsf{ls!}\}$.

Proof. We prove both items by induction on the derivation of $t \ltimes s$.

1. $t \ltimes s$ is $x \ltimes x$. Both items are immediate since there are no $\to_{\mathsf{B}}$ steps from x.
2. $t \ltimes s$ is $\lambda x.\, t_1 \ltimes \lambda x.\, s_1$ and follows from $t_1 \ltimes s_1$.
   – Item 1. The reduction must be internal: $\lambda x.\, s_1 \to_{\neg\mathsf{gc!}} \lambda x.\, s'_1 = s'$ follows from $s_1 \to_{\neg\mathsf{gc!}} s'_1$. From the IH there exists t'_1 such that $t_1 \to^{=}_{\mathsf{B}^{\mathsf{der}}} t'_1$ and $t'_1 \ltimes s'_1$. Then $\lambda x.\, t_1 \to^{=}_{\mathsf{B}^{\mathsf{der}}} \lambda x.\, t'_1$ and moreover $\lambda x.\, t'_1 \ltimes \lambda x.\, s'_1$. Thus we conclude by setting t' to be $\lambda x.\, t'_1$.
   – Item 2. Similar to Item 1.
3. $t \ltimes s$ is $t_1\, t_2 \ltimes s_1\, s_2$ and follows from $t_1 \ltimes s_1$ and $t_2 \ltimes s_2$
   – Item 1. There are three cases.
   3.1 The reduction is internal to s_1: $s = s_1\, s_2 \to_{\neg\mathsf{gc!}} s'_1 s_2 = s'$ follows from $s_1 \to_{\neg\mathsf{gc!}} s'_1$. From the IH there exists t'_1 such that $t_1 \to^{=}_{\mathsf{B}^{\mathsf{der}}} t'_1$ and $t'_1 \ltimes s'_1$. Then $t_1\, t_2 \to^{=}_{\mathsf{B}^{\mathsf{der}}} t'_1\, t_2$ and moreover $t'_1\, t_2 \ltimes s'_1\, s_2$. Thus we conclude by setting t' to be $t'_1\, t_2$.
   3.2 The reduction is internal to s_2: $s = s_1\, s_2 \to_{\neg\mathsf{gc!}} s_1 s'_2 = s'$ follows from $s_2 \to_{\neg\mathsf{gc!}} s'_2$. Similar to the previous case.
   3.3 The reduction is at the root of s. Then the step is a db step: $s = (\lambda x.\, s_{11})\mathtt{L}'\, s_2 \to_{\mathsf{db}} s_{11}[x/s_2]\mathtt{L}' = s'$. By Lem. G.4(1) there exists $t_{11}, \mathtt{L}$ such that $t_1 = (\lambda x.\, t_{11})\mathtt{L}$ with $t_{11} \ltimes s_{11}$ and $\mathtt{L} \ltimes_{\mathcal{X}} \mathtt{L}'$ and $\mathcal{X} \cap \mathsf{fv}(\lambda x.\, t_{11}) = \emptyset$.
   We set t' to be $t_{11}[x/t_2]\mathtt{L}$. Note that $t = (\lambda x.\, t_{11})\mathtt{L}\, t_2 \to_{\mathsf{db}} t_{11}[x/t_2]\mathtt{L}$. From $t_{11} \ltimes s_{11}$ and $t_2 \ltimes s_2$, $t_{11}[x/t_2] \ltimes s_{11}[x/s_2]$ holds. Finally, since $\mathtt{L} \ltimes_{\mathcal{X}} \mathtt{L}'$ and $\mathcal{X} \cap \mathsf{fv}((\lambda x.\, s_{11})\, s_2) = \emptyset$, we obtain $t_{11}[x/t_2]\mathtt{L} \ltimes s_{11}[x/s_2]\mathtt{L}'$ from Lem. G.2(8).

- Item 2. There are two cases.
  3.1 The reduction is internal to s_1: $s = s_1\, s_2 \to_{\mathsf{gc!}} s'_1 s_2 = t'$ follows from $s_1 \to_{\mathsf{gc!}} s'_1$. From the IH there exists t'_1 such that $t_1 \to^{=}_{\mathsf{gc!}} t'_1$ and $t'_1 \ltimes s'_1$. Then $t_1\, t_2 \to^{=}_{\mathsf{gc!}} t'_1\, t_2$ and moreover $t'_1\, t_2 \ltimes s'_1\, s_2$. Thus we conclude by setting t' to be $t'_1\, t_2$.
  3.2 The reduction is internal to s_2: $s = s_1\, s_2 \to_{\mathsf{gc!}} s_1 s'_2 = s'$ follows from $s_2 \to_{\mathsf{gc!}} s'_2$. Similar to the previous case.

4. $t \ltimes s$ is $t_1[x/t_2] \ltimes s_1[x/s_2]$ and follows from $t_1 \ltimes s_1$ and $t_2 \ltimes s_2$.
- Item 1. There are three cases.
  4.1 The reduction is internal to s_1. Then $s_1[x/s_2] \to_{\neg\mathsf{gc!}} s'_1[x/s_2] = s'$ follows from $s_1 \to_{\neg\mathsf{gc!}} s'_1$. From the hypothesis $t_1 \ltimes s_1$ and the IH, there exists t'_1 such that $t_1 \to^{=}_{\mathsf{B^{der}}} t'_1$ and $t'_1 \ltimes s'_1$. Then we set t' to be $t'_1[x/t_2]$. Note that $t'_1[x/t_2] \to^{=}_{\mathsf{B^{der}}} t'_1[x/t_2]$ and $t'_1[x/t_2] \ltimes s'_1[x/s_2]$, and we conclude.
  4.2 The reduction is internal to s_2. Similar to the previous case.
  4.3 The reduction is at the root of s. Then the reduction step must be a $\mathsf{ls!}$-step: $s = \mathsf{C}'\langle\langle x\rangle\rangle[x/(!s_{21})\mathsf{L}'] \mapsto_{\mathsf{ls!}} \mathsf{C}'\langle\langle s_{21}\rangle\rangle[x/!s_{21}]\mathsf{L}' = s'$ and $x \notin \mathsf{fv}(s_{21})$ and $\mathsf{fv}(\mathsf{C}') \cap \mathsf{dom}(\mathsf{L}') = \varnothing$. From $t_1 \ltimes \mathsf{C}'\langle\langle x\rangle\rangle$ and Lem. G.4(4), there are two possible cases:
    4.3.1 $x \in \mathsf{fv}(t_1)$ and $t_1 = \mathsf{C}\langle\langle x\rangle\rangle$ and $\mathsf{C} \ltimes_{\mathcal{X}} \mathsf{C}'$. Similarly, from $t_2 \ltimes (!s_{21})\mathsf{L}$ and Lem. G.4(3), $t_2 = (!t_{21})\mathsf{L}$ and $t_{21} \ltimes s_{21}$ and $\mathsf{L} \ltimes_{\mathcal{Y}} \mathsf{L}'$ and $\mathcal{Y} \cap \mathsf{fv}(!s_{21}) = \varnothing$. Then $t = \mathsf{C}\langle\langle x\rangle\rangle[x/(!t_{21})\mathsf{L}] \to^{=}_{\mathsf{B^{der}}} \mathsf{C}\langle\langle t_{21}\rangle\rangle[x/!t_{21}]\mathsf{L}$. From $t_{21} \ltimes s_{21}$ and $\mathsf{C} \ltimes_{\mathcal{X}} \mathsf{C}'$, we deduce from Lem. G.2(7), that $\mathsf{C}\langle\langle t_{21}\rangle\rangle \ltimes \mathsf{C}'\langle\langle s_{21}\rangle\rangle$. We may assume that $\mathsf{dom}(\mathsf{L}') \cap \mathsf{fv}(\mathsf{C}\langle\langle t_{21}\rangle\rangle) = \varnothing$. Finally, we set t' to be $\mathsf{C}\langle\langle t_{21}\rangle\rangle[x/!t_{21}]\mathsf{L}$ and conclude with $\mathsf{C}\langle\langle t_{21}\rangle\rangle[x/!t_{21}]\mathsf{L} \ltimes \mathsf{C}'\langle\langle s_{21}\rangle\rangle[x/!s_{21}]\mathsf{L}'$ by Lem. G.2(8).
    4.3.2 $x \notin \mathsf{fv}(t_1)$ and $\mathsf{C}' = \mathsf{C}'_1\langle s_1[y/!\mathsf{C}'_2\langle\langle x\rangle\rangle]\rangle$ and $y \notin \mathsf{fv}(s_1)$ and $t_1 = \mathsf{C}\langle t_{11}\rangle$ and $\mathsf{C}\langle t_{11}\rangle \ltimes \mathsf{C}'_1\langle s_1\rangle$. The step $s \mapsto_{\mathsf{ls!}} s'$ is thus of the form:

$$s = \mathsf{C}'_1\langle s_1[y/!\mathsf{C}'_2\langle\langle x\rangle\rangle]\rangle[x/(!s_{21})\mathsf{L}'] \mapsto_{\mathsf{ls!}} \mathsf{C}'_1\langle s_1[y/!\mathsf{C}'_2\langle\langle s_{21}\rangle\rangle]\rangle[x/!s_{21}]\mathsf{L}' = s'$$

Just like in the previous case, from $t_2 \ltimes (!s_{21})\mathsf{L}$ and Lem. G.4(3), $t_2 = (!t_{21})\mathsf{L}$ and $t_{21} \ltimes s_{21}$ and $\mathsf{L} \ltimes_{\mathcal{Y}} \mathsf{L}'$ and $\mathcal{Y} \cap \mathsf{fv}(!s_{21}) = \varnothing$. Then

$$\begin{tikzcd}
\mathsf{C}\langle t_{11}\rangle[x/(!t_{21})\mathsf{L}] \arrow[r, phantom, "\ltimes"] \arrow[d, equal] & \mathsf{C}'_1\langle s_1[y/!\mathsf{C}'_2\langle\langle x\rangle\rangle]\rangle[x/(!s_{21})\mathsf{L}'] \arrow[d, "\mathsf{ls!}"] \\
\mathsf{C}\langle t_{11}\rangle[x/(!t_{21})\mathsf{L}] \arrow[r, phantom, "\ltimes"] & \mathsf{C}'_1\langle s_1[y/!\mathsf{C}'_2\langle\langle s_{21}\rangle\rangle]\rangle[x/!s_{21}]\mathsf{L}'
\end{tikzcd}$$

- Item 2. There are three cases.
  4.1 The reduction is internal to s_1: $s = s_1[x/s_2] \to_{\mathsf{gc!}} s'_1[x/s_2] = t'$ follows from $s_1 \to_{\mathsf{gc!}} s'_1$. From the IH there exists t'_1 such that $t_1 \to^{=}_{\mathsf{gc!}} t'_1$ and $t'_1 \ltimes s'_1$. Then $t_1[x/t_2] \to^{=}_{\mathsf{gc!}} t'_1[x/t_2]$ and moreover $t'_1[x/t_2] \ltimes s'_1[x/s_2]$. Thus we conclude by setting t' to be $t'_1[x/t_2]$.
  4.2 The reduction is internal to s_2. Same as above.
  4.3 The reduction is at the root of s. Then the reduction step must be a $\mathsf{gc!}$-step: $s = s_1[x/(!s_{21})\mathsf{L}] \mapsto_{\mathsf{gc!}} s_1\mathsf{L} = t'$ and $x \notin \mathsf{fv}(s_1)$. From $t_2 \ltimes (!s_{21})\mathsf{L}$ and Lem. G.4(3), $t_2 = (!t_{21})\mathsf{L}'$ and $t_{21} \ltimes s_{21}$ and $\mathsf{L} \ltimes_{\mathcal{X}} \mathsf{L}'$ and $\mathcal{X} \cap \mathsf{fv}(!t_{21}) =$

$\emptyset$. Moreover, we may assume that not only $\mathcal{X} \cap \mathsf{fv}(!s_{21}) = \emptyset$, but also $\mathcal{X} \cap \mathsf{fv}(s_1) = \emptyset$. Consider $s_1[x/(!s_{21})\mathtt{L}']$. From $x \notin \mathsf{fv}(s_1)$ and Rem. G.2(1), also $x \notin \mathsf{fv}(t_1)$. Then $t = t_1[x/(!t_{21})\mathtt{L}'] \mapsto_{\mathsf{gc!}} t_1\mathtt{L}' = t'$. Note that from $\mathtt{L} \ltimes_{\mathcal{X}} \mathtt{L}'$, $\mathsf{dom}(\mathtt{L}') \cap \mathsf{fv}(s_1) = \emptyset$ and Lem. G.2(8), we have $t_1\mathtt{L} \ltimes s_1\mathtt{L}'$. We thus set t' to be $t_1\mathtt{L}'$ and conclude.

5. $t \ltimes s$ is $!t_1 \ltimes !s_1$ and follows from $t_1 \ltimes s_1$.
   - Item 1.
     5.1 The reduction must be internal to s_1. Then $!s_1 \to_{\neg\mathsf{gc!}} !s_1' = s'$ follows from $s_1 \to_{\neg\mathsf{gc!}} s_1'$. From the hypothesis $t_1 \ltimes s_1$ and the IH, there exists t_1' such that $t_1 \to^{=}_{\mathsf{B^{der}}} t_1'$ and $t_1' \ltimes s_1'$. Then we set t' to be $!t_1'$. Note that $!t_1 \to^{=}_{\mathsf{B^{der}}} !t_1'$ and $!t_1' \ltimes !s_1'$, and we conclude.
   - Item 2.
     5.1 The reduction must be internal to s_1. Then $!s_1 \to_{\mathsf{gc!}} !s_1' = s'$ follows from $s_1 \to_{\mathsf{gc!}} s_1'$. From the hypothesis $t_1 \ltimes s_1$ and the IH, there exists t_1' such that $t_1 \to^{=}_{\mathsf{gc!}} t_1'$ and $t_1' \ltimes s_1'$. Then we set t' to be $!s_1'$. Note that $!t_1 \to^{=}_{\mathsf{gc!}} !t_1'$ and $!t_1' \ltimes !s_1'$, and we conclude.
6. $t \ltimes s$ is $\mathtt{der}(t_1) \ltimes z[z/s_1]$ and follows from $t_1 \ltimes s_1$.
   - Item 1. There are two cases.
     6.1 The reduction is internal to s_1. Then $z[z/s_1] \to_{\neg\mathsf{gc!}} z[z/s_1'] = s'$ follows from $s_1 \to_{\neg\mathsf{gc!}} s_1'$. From the hypothesis $t_1 \ltimes s_1$ and the IH, there exists t_1' such that $t_1 \to^{=}_{\mathsf{B^{der}}} t_1'$ and $t_1' \ltimes s_1'$. Then we set t' to be $\mathtt{der}(t_1')$. Note that $\mathtt{der}(t_1) \to^{=}_{\mathsf{B^{der}}} \mathtt{der}(t_1')$ and $\mathtt{der}(t_1') \ltimes z[z/s_1']$, and we conclude.
     6.2 The reduction is at the root of s. Then the reduction step must be a $\mathsf{ls!}$-step: $s = z[z/(!s_{11})\mathtt{L}'] \to_{\mathsf{ls!}} s_{11}[z/!s_{11}]\mathtt{L}'$. From $t_1 \ltimes (!s_{11})\mathtt{L}'$ and Lem. G.4(3), $t_1 = (!t_{12})\mathtt{L}$ and $t_{12} \ltimes s_{11}$ and $\mathtt{L} \ltimes_{\mathcal{X}} \mathtt{L}'$ and $\mathcal{X} \cap \mathsf{fv}(!t_{12}) = \emptyset$. We set t' to be $\mathtt{der}((!t_{12})\mathtt{L})$. Note that $\mathtt{der}((!t_{12})\mathtt{L}) \mapsto_{\mathsf{d!}} t_{12}\mathtt{L}$. Moreover, from $t_{12} \ltimes s_{11}$ we deduce $t_{12} \ltimes s_{11}[z/!s_{11}]$ using gc; and from the latter we obtain $t_{12}\mathtt{L} \ltimes s_{11}[z/!s_{11}]\mathtt{L}'$ using Lem. G.2(8).
   - Item 2. There is one case.
     6.1 The reduction is internal to s_1. Then $z[z/s_1] \to_{\mathsf{gc!}} z[z/s_1'] = s'$ follows from $s_1 \to_{\mathsf{gc!}} s_1'$. From the hypothesis $t_1 \ltimes s_1$ and the IH, there exists t_1' such that $t_1 \to^{=}_{\mathsf{gc!}} t_1'$ and $t_1' \ltimes s_1'$. Then we set t' to be $\mathtt{der}(t_1')$. Note that $\mathtt{der}(t_1) \to^{=}_{\mathsf{gc!}} \mathtt{der}(t_1')$ and $\mathtt{der}(t_1') \ltimes z[z/s_1']$, and we conclude.
7. $t \ltimes s$ is $t \ltimes s_1[z/!s_2]$ follows from $t \ltimes s_1$ and $z \notin \mathsf{fv}(s_1)$,
   - Item 1. There are two cases.
     7.1 The reduction is internal to s_1. Suppose $s_1[z/!s_2] \to_{\neg\mathsf{gc!}} s_1'[z/!s_2]$ follows from $s_1 \to_{\neg\mathsf{gc!}} s_1'$. From the hypothesis $t \ltimes s_1$ and the IH, there exists t' such that $t \to^{=}_{\mathsf{B^{der}}} t'$ and $t' \ltimes s_1'$. We conclude since $t' \ltimes s_1'[z/!s_2]$ follows from gc.
     7.2 The reduction is internal to s_2. Suppose $s_1[z/!s_2] \to_{\neg\mathsf{gc!}} s_1[z/!s_2']$ follows from $s_2 \to_{\neg\mathsf{gc!}} s_2'$. We set t' to be t since $t' \ltimes s_1[z/!s_2']$ follows from gc.
   - Item 2. There are three cases.
     7.1 The reduction is internal to s_1. Suppose $s_1[z/!s_2] \to_{\mathsf{gc!}} s_1'[z/!s_2]$ follows from $s_1 \to_{\mathsf{gc!}} s_1'$. From the hypothesis $t \ltimes s_1$ and the IH, there exists t' such that $t \to^{=}_{\mathsf{gc!}} t'$ and $t' \ltimes s_1'$. We conclude since $t' \ltimes s_1'[z/!s_2]$ follows from gc.
     7.2 The reduction is internal to s_2. Suppose $s_1[z/!s_2] \to_{\mathsf{gc!}} s_1[z/!s_2']$ follows from $s_2 \to_{\mathsf{gc!}} s_2'$. We set t' to be t since $t' \ltimes s_1[z/!s_2']$ follows from gc.

7.3 The reduction is at the root. Then the reduction step must be a $\mathsf{gc!}$-step: $s = s_1[z/!s_{21}] \mapsto_{\mathsf{gc!}} s_1 = t'$. We set t' to be t and conclude from $t \ltimes s_1$.

Proposition G.2 (Bang typing [proof of Prop. 7.2]). *If $\Gamma \vdash t : A$ then $\Gamma^{\mathsf{B}}; \cdot \vdash t^{\mathsf{B}} : A^{\mathsf{B}}$.*

Proof. By induction on the derivation of $\Gamma \vdash t : A$:

1. $\mathtt{b\text{-}var}$: Let $\Gamma, x : !A \vdash x : A$ be derived from the $\mathtt{b\text{-}var}$ rule. Then:

$$\dfrac{\dfrac{}{\Gamma^{\mathsf{B}}, x : A^{\mathsf{B}}; \cdot \vdash x : \bullet A^{\mathsf{B}}}\,\mathtt{uvar}}{\Gamma^{\mathsf{B}}, x : A^{\mathsf{B}}; \cdot \vdash \mathtt{req}(x) : A^{\mathsf{B}}}\,\mathtt{request}$$

2. $\mathtt{b\text{-}abs}$: Let $\Gamma \vdash \lambda x.t : !A \to B$ be derived from $\Gamma, x : !A \vdash t : B$. Let a be a fresh linear variable, such that $a \notin \mathsf{fv}(t^{\mathsf{B}})$. Then:

$$\dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{B}}, x : A^{\mathsf{B}}; \cdot \vdash t^{\mathsf{B}} : B^{\mathsf{B}}} \qquad \dfrac{}{\Gamma^{\mathsf{B}}; a : !\bullet A^{\mathsf{B}} \vdash a : !\bullet A^{\mathsf{B}}}\,\mathtt{lvar}}{\Gamma^{\mathsf{B}}; a : !\bullet A^{\mathsf{B}} \vdash t^{\mathsf{B}}[x/a] : B^{\mathsf{B}}}\,\mathtt{sub}}{\Gamma^{\mathsf{B}}; \cdot \vdash \lambda a.t^{\mathsf{B}}[x/a] : !\bullet A^{\mathsf{B}} \multimap B^{\mathsf{B}}}\,\mathtt{abs}$$

3. $\mathtt{b\text{-}app}$: Let $\Gamma \vdash t\,s : B$ be derived from $\Gamma \vdash t : !A \to B$ and $\Gamma \vdash s : !A$. Then:

$$\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{B}}; \cdot \vdash t^{\mathsf{B}} : !\bullet A^{\mathsf{B}} \multimap B^{\mathsf{B}}} \qquad \dfrac{\text{IH}}{\Gamma^{\mathsf{B}}; \cdot \vdash s^{\mathsf{B}} : !\bullet A^{\mathsf{B}}}}{\Gamma^{\mathsf{B}}; \cdot \vdash t^{\mathsf{B}}\,s^{\mathsf{B}} : B^{\mathsf{B}}}\,\mathtt{app}$$

4. $\mathtt{b\text{-}prom}$: Let $\Gamma \vdash !t : !A$ be derived from $\Gamma \vdash t : A$. Then:

$$\dfrac{\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{B}}; \cdot \vdash t^{\mathsf{B}} : A^{\mathsf{B}}}}{\Gamma^{\mathsf{B}}; \cdot \vdash \bullet t^{\mathsf{B}} : \bullet A^{\mathsf{B}}}\,\mathtt{grant}}{\Gamma^{\mathsf{B}}; \cdot \vdash !\bullet t^{\mathsf{B}} : !\bullet A^{\mathsf{B}}}\,\mathtt{prom}$$

5. $\mathtt{b\text{-}es}$: Let $\Gamma \vdash t[x/s] : !B$ be derived from $\Gamma, x : !A \vdash t : B$ and $\Gamma \vdash s : !A$. Then:

$$\dfrac{\dfrac{\text{IH}}{\Gamma^{\mathsf{B}}, x : A^{\mathsf{B}}; \cdot \vdash t^{\mathsf{B}} : !\bullet B^{\mathsf{B}}} \qquad \dfrac{\text{IH}}{\Gamma^{\mathsf{B}}; \cdot \vdash s^{\mathsf{B}} : !\bullet A^{\mathsf{B}}}}{\Gamma^{\mathsf{B}}; \cdot \vdash t^{\mathsf{B}}[x/s^{\mathsf{B}}] : B^{\mathsf{B}}}$$

Definition G.1. *We define a subset $\mathsf{Ctxs}^{\mathsf{B}}_{\bullet} \subseteq \mathsf{Ctxs}_{\bullet}$ of the set of contexts, called* Bang contexts*:*

$$\underline{\mathsf{C}} ::= \mathtt{req}(\Box) \mid \mathtt{req}(\bullet\underline{\mathsf{C}}) \mid \lambda a.\,\underline{\mathsf{C}}[u/a] \mid \underline{\mathsf{C}}\,\underline{t} \mid \underline{t}\,\underline{\mathsf{C}} \mid !\bullet\underline{\mathsf{C}} \mid \underline{\mathsf{C}}[u/\underline{t}] \mid \underline{t}[u/\underline{\mathsf{C}}]$$

where, in the production $\underline{\mathsf{C}} ::= \lambda a.\,\underline{\mathsf{C}}[u/a]$ we assume that a is fresh, that is $a \notin \mathsf{fv}(\underline{\mathsf{C}})$. Furthermore, we define a subset $\mathsf{SCtxs}^{\mathsf{B}}_{\bullet} \subseteq \mathsf{SCtxs}_{\bullet}$ of the set of substitution contexts, called Bang substitution contexts*:*

$$\underline{\mathsf{L}} ::= \Box \mid \underline{\mathsf{L}}[u/\underline{t}]$$

The inverse translation can be extended to Bang contexts and substitution contexts, setting $\mathtt{req}(\Box)^{-\mathsf{B}} := \Box$ for Bang contexts, and $\Box^{-\mathsf{B}} := \Box$ for Bang substitution contexts.

Lemma G.6 (Context decomposition for the inverse Bang translation).

1. *$t\mathsf{L} \in \mathcal{T}_{\mathsf{B}}$ if and only if $t \in \mathcal{T}_{\mathsf{B}}$ and $\mathsf{L} \in \mathsf{SCtxs}^{\mathsf{B}}_{\bullet}$.*
2. *$\mathsf{C}\langle\langle x\rangle\rangle \in \mathcal{T}_{\mathsf{B}}$ if and only if $\mathsf{C} \in \mathsf{Ctxs}^{\mathsf{B}}_{\bullet}$.*
3. *If $\underline{\mathsf{C}} \in \mathsf{Ctxs}^{\mathsf{B}}_{\bullet}$ and $\underline{t} \in \mathcal{T}_{\mathsf{B}}$ then $\underline{\mathsf{C}}\langle\langle \bullet\underline{t}\rangle\rangle \in \mathcal{T}_{\mathsf{B}}$.*
4. *If $\underline{t} \in \mathcal{T}_{\mathsf{B}}$ and $\underline{\mathsf{L}} \in \mathsf{SCtxs}^{\mathsf{B}}_{\bullet}$, then $(\underline{t}\,\underline{\mathsf{L}})^{-\mathsf{B}} = \underline{t}^{-\mathsf{B}}\underline{\mathsf{L}}^{-\mathsf{N}}$.*
5. *If $\underline{\mathsf{C}} \in \mathsf{Ctxs}^{\mathsf{B}}_{\bullet}$ then $\underline{\mathsf{C}}\langle\langle x\rangle\rangle^{-\mathsf{B}} = \underline{\mathsf{C}}^{-\mathsf{B}}\langle\langle x\rangle\rangle$.*
6. *If $\underline{\mathsf{C}} \in \mathsf{Ctxs}^{\mathsf{B}}_{\bullet}$ and $\underline{t} \in \mathcal{T}_{\mathsf{B}}$, then $\underline{\mathsf{C}}\langle\langle \bullet\underline{t}\rangle\rangle^{-\mathsf{B}} = \underline{\mathsf{C}}^{-\mathsf{B}}\langle\langle \underline{t}^{-\mathsf{B}}\rangle\rangle$.*

Proof. By induction on the first judgement in the statement of each item, except the first and fourth items which are by induction on L and $\underline{\mathsf{L}} \in \mathsf{SCtxs}^{\mathsf{B}}_{\bullet}$, resp.

Remark G.3. $\mathsf{fv}(\underline{t}^{-\mathsf{B}}) = \mathsf{fv}(\underline{t})$.

Lemma G.7 (Inverse Bang simulation [PROOF OF LEMMA 7.2]**).** *Let $\underline{t} \in \mathcal{T}_{\mathsf{B}}$ and $s \in \mathcal{T}_{\bullet}$. If $\underline{t} \to_{\bullet} s$ then $s \in \mathcal{T}_{\mathsf{B}}$ and $\underline{t}^{-\mathsf{B}} \to^{=}_{\mathsf{B}} s^{-\mathsf{B}}$.*

Proof. By induction on the (unique) derivation of $\underline{t} \in \mathcal{T}_{\mathsf{B}}$:

1. $\underline{t} = \mathtt{req}(x)$: Impossible, as there are no steps $\underline{t} \to_{\bullet} s$.
2. $\underline{t} = \mathtt{req}(\bullet\underline{t}_1)$: We consider two subcases, depending on whether the step is at the root of the term or internal to $\bullet\underline{t}_1$:
   2.1 If the step is at the root, we have that $\underline{t} = \mathtt{req}(\bullet\underline{t}_1) \mapsto_{\bullet\mathsf{req}} \underline{t}_1 = s$, so $s = \underline{t}_1 \in \mathcal{T}_{\mathsf{B}}$ and $\underline{t}^{-\mathsf{B}} = \mathtt{req}(\bullet\underline{t}_1)^{-\mathsf{B}} = \underline{t}_1^{-\mathsf{B}} = s^{-\mathsf{B}}$.
   2.2 If the step is internal to $\bullet\underline{t}_1$, note that it cannot be at the root of $\bullet\underline{t}_1$, since this term does not match the left-hand side of any rewriting rule. Then the step must be internal to $\underline{t}_1$, that is, $\underline{t} = \mathtt{req}(\bullet\underline{t}_1) \to_{\bullet} \mathtt{req}(\bullet s') = s$ with $\underline{t}_1 \to_{\bullet} s'$. By IH, $s' \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, so $s = \mathtt{req}(\bullet s') \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, and $\underline{t}^{-\mathsf{B}} = \mathtt{req}(\bullet\underline{t}_1)^{-\mathsf{B}} = \underline{t}_1^{-\mathsf{B}} \to^{=}_{\mathsf{B}} s'^{-\mathsf{B}} = \mathtt{req}(\bullet s')^{-\mathsf{B}} = s^{-\mathsf{B}}$.
3. $\underline{t} = \lambda a.\,\underline{t}'[x/a]$: Note that the step cannot be at the root, and that that there cannot be a $\to_{\bullet\mathsf{ls}}$ nor a $\to_{\bullet\mathsf{gc}}$ step involving the substitution $[x/a]$, since these rules would require that a be of the form $(!r)\mathsf{L}$, but a is a linear variable. This means that the step must be internal to $\underline{t}'$, that is, $\underline{t} = \lambda a.\,\underline{t}'[x/a] \to_{\bullet} \lambda a.\,s'[x/a] = s$ with $\underline{t}' \to_{\bullet} s'$. By IH, $s' \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, so $s = \lambda a.\,s'[x/a] \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, and $\underline{t}^{-\mathsf{B}} = (\lambda a.\,\underline{t}'[x/a])^{-\mathsf{B}} = \lambda x.\,\underline{t}'^{-\mathsf{B}} \to^{=}_{\mathsf{N}} \lambda x.\,s'^{-\mathsf{B}} = (\lambda a.\,s'[x/a])^{-\mathsf{B}} = s^{-\mathsf{B}}$.
4. $\underline{t} = \underline{t}_1\,\underline{t}_2$: We consider three subcases, depending on whether the step is at the root of the term, internal to $\underline{t}_1$, or internal to $\underline{t}_2$:
   4.1 If the step is at the root of the term, it must be a $\bullet\mathsf{db}$ step, that is, $\underline{t}_1$ must be of the form $(\lambda a.\,t)\mathsf{L}$. By Lem. G.6, this means that $\lambda a.\,t \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ and $\mathsf{L} \in \mathsf{SCtxs}^{\mathsf{N}}_{\bullet}$. In particular, t must be of the form $t = \underline{t}'_1[x/a]$. Then we have that $\underline{t} = (\lambda a.\,\underline{t}'_1[x/a])\mathsf{L}\,!\bullet\underline{t}_2 \mapsto_{\bullet\mathsf{db}} \underline{t}'_1[x/!\bullet\underline{t}_2]\mathsf{L} = s$. Note that $s = \underline{t}'_1[x/!\bullet\underline{t}_2]\mathsf{L} \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ by Lem. G.6. By IH, we have that

$$\begin{array}{rl} \underline{t}^{-\mathsf{B}} = & ((\lambda a.\,\underline{t}'_1[x/a])\mathsf{L}\,!\bullet\underline{t}_2)^{-\mathsf{B}} \\ = & (\lambda x.\,\underline{t}'^{-\mathsf{B}}_1)\mathsf{L}^{-\mathsf{B}}\,\underline{t}_2^{-\mathsf{B}} \\ \mapsto_{\mathsf{db}} & \underline{t}'^{-\mathsf{B}}_1[x/\underline{t}_2^{-\mathsf{B}}]\mathsf{L}^{-\mathsf{B}} \\ = & (\underline{t}'_1[x/!\bullet\underline{t}_2]\mathsf{L})^{-\mathsf{B}} \\ = & s^{-\mathsf{B}} \end{array}$$

   4.2 If the step is internal to $\underline{t}_1$, we have that $\underline{t} = \underline{t}_1\,\underline{t}_2 \to_{\bullet} s_1\,\underline{t}_2 = s$ with $\underline{t}_1 \to_{\bullet} s_1$. By IH, $s_1 \in \mathcal{T}^{\mathsf{N}}_{\bullet}$, so $s = s_1\,\underline{t}_2 \in \mathcal{T}^{\mathsf{N}}_{\bullet}$ and $\underline{t}^{-\mathsf{B}} = \underline{t}_1^{-\mathsf{B}}\,\underline{t}_2^{-\mathsf{B}} \to^{=}_{\mathsf{N}} s_1^{-\mathsf{B}}\,\underline{t}_2^{-\mathsf{B}} = s^{-\mathsf{B}}$.

4.3 If the step is internal to $\underline{t}_2$, we have that $\underline{t} = \underline{t}_1\,\underline{t}_2 \to_\bullet \underline{t}_1\,s_2 = s$ with $\underline{t}_2 \to_\bullet s_2$. Similar to the previous case.

5. $\underline{t} = !\bullet\underline{t}_1$. If the step is internal to $!\bullet\underline{t}_1$, note that it cannot be at the root of $!\bullet\underline{t}_1$ nor at the root of $\bullet\underline{t}_1$, since these terms do not match the left-hand side of any rewriting rule. So the step must be internal to $\underline{t}_1$, that is, we have that $\underline{t} = !\bullet\underline{t}_1 \to_\bullet !\bullet s_1 = s$ with $\underline{t}_1 \to_\bullet s_1$. By IH, $s_1 \in \mathcal{T}_{\mathsf{B}}$, so $s = !\bullet s_1 \in \mathcal{T}_{\mathsf{B}}$ and $\underline{t}^{-\mathsf{B}} = \underline{t}_1^{-\mathsf{B}} \to_{\mathsf{B}}^{=} s_1^{-\mathsf{B}} = s^{-\mathsf{B}}$.

6. $\underline{t} = \underline{t}_1[x/\underline{t}_2]$: We consider three subcases, depending on whether the step is at the root of the term, internal to $\underline{t}_1$, or internal to $\underline{t}_2$:

6.1 If the step is at the root of the term, it must be either a $\bullet\mathsf{ls}$ or a $\bullet\mathsf{gc}$ step. We consider two further subcases:

6.1.1 If the step is a $\bullet\mathsf{ls}$ step, then $\underline{t}_2 = (!\bullet\underline{t}_3)\underline{\mathsf{L}}$ and we have that $\underline{t}_1 = \mathsf{C}\langle\!\langle x\rangle\!\rangle \in \mathcal{T}_{\mathsf{B}}$ and $\underline{t} = \mathsf{C}\langle\!\langle x\rangle\!\rangle[x/(!\bullet\underline{t}_3)\underline{\mathsf{L}}] \mapsto_{\bullet\mathsf{ls}} \mathsf{C}\langle\!\langle \bullet\underline{t}_2\rangle\!\rangle[x/(!\bullet\underline{t}_3)]\underline{\mathsf{L}} = s$. Note that, by Lem. G.6, $\mathsf{C} \in \mathsf{Ctxs}_\bullet^{\mathsf{B}}$, so, again by Lem. G.6, $s = \mathsf{C}\langle\!\langle \bullet\underline{t}_2\rangle\!\rangle[x/!\bullet\underline{t}_2]\underline{\mathsf{L}} \in \mathcal{T}_\bullet^{\mathsf{N}}$. Moreover, also using Lem. G.6, we have that $\underline{t}^{-\mathsf{B}} = (\mathsf{C}\langle\!\langle x\rangle\!\rangle[x/(!\bullet\underline{t}_2)\underline{\mathsf{L}}])^{-\mathsf{B}} = \mathsf{C}^{-\mathsf{B}}\langle\!\langle x\rangle\!\rangle[x/\underline{t}_2^{-\mathsf{B}}] \mapsto_{\mathsf{ls}} \mathsf{C}^{-\mathsf{B}}\langle\!\langle \underline{t}_2^{-\mathsf{B}}\rangle\!\rangle[x/\underline{t}_2^{-\mathsf{B}}] = (\mathsf{C}\langle\!\langle \bullet\underline{t}_2\rangle\!\rangle[x/!\bullet\underline{t}_2])^{-\mathsf{B}} = s^{-\mathsf{B}}$.

6.1.2 If the step is a $\bullet\mathsf{gc}$ step, we have that $x \notin \mathsf{fv}(\underline{t}_1)$ and $\underline{t} = \underline{t}_1[x/!\bullet\underline{t}_2] \mapsto_{\bullet\mathsf{gc}} \underline{t}_1 = s$. Note that $s = \underline{t}_1 \in \mathcal{T}_\bullet^{\mathsf{N}}$. Moreover, $\underline{t}^{-\mathsf{B}} = \underline{t}_1[x/!\bullet\underline{t}_2]^{-\mathsf{B}} = \underline{t}_1^{-\mathsf{B}}[x/\underline{t}_2^{-\mathsf{B}}] \mapsto_{\mathsf{gc}} \underline{t}_1^{-\mathsf{B}} = s^{-\mathsf{B}}$. Note that $x \notin \underline{t}_1^{-\mathsf{B}}$ because $\mathsf{fv}(\underline{t}_1^{-\mathsf{B}}) = \mathsf{fv}(\underline{t}_1)$, as noted in Rem. G.3.

6.2 If the step is internal to $\underline{t}_1$, we have that $\underline{t} = \underline{t}_1[x/\underline{t}_2] \to_\bullet s_1[x/\underline{t}_2] = s$ with $\underline{t}_1 \to_\bullet s_1$. By IH, $s_1 \in \mathcal{T}_{\mathsf{B}}$, so $s = s_1[x/\underline{t}_2] \in \mathcal{T}_{\mathsf{B}}$ and $\underline{t}^{-\mathsf{B}} = \underline{t}_1^{-\mathsf{B}}[x/\underline{t}_2^{-\mathsf{B}}] \to_{\mathsf{N}}^{=} s_1^{-\mathsf{B}}[x/\underline{t}_2^{-\mathsf{B}}] = s^{-\mathsf{B}}$.

6.3 If the step is internal to $\underline{t}_2$, we have that $\underline{t} = \underline{t}_1[x/\underline{t}_2] \to_\bullet \underline{t}_1[x/s_2] = s$ with $\underline{t}_2 \to_\bullet s_2$. By IH, $s_2 \in \mathcal{T}_{\mathsf{B}}$, so $s = \underline{t}_1[x/s_2] \in \mathcal{T}_{\mathsf{B}}$ and $\underline{t}^{-\mathsf{B}} = \underline{t}_1^{-\mathsf{B}}[x/\underline{t}_2^{-\mathsf{B}}] \to_{\mathsf{B}}^{=} \underline{t}_1^{-\mathsf{B}}[x/s_2^{-\mathsf{B}}] = s^{-\mathsf{B}}$.