Two Decreasing Measures for Simply Typed λ-Terms (Extended Version)

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Abstract

This paper defines two decreasing measures for terms of the simply typed λ-calculus, called the $\mathcal{W}$-measure and the $\mathcal{T}^{\mathbf{m}}$-measure. A decreasing measure is a function that maps each typable λ-term to an element of a well-founded ordering, in such a way that contracting any β-redex decreases the value of the function, entailing strong normalization. Both measures are defined constructively, relying on an auxiliary calculus, a non-erasing variant of the λ-calculus. In this system, dubbed the $\lambda^{\mathbf{m}}$-calculus, each β-step creates a "wrapper" containing a copy of the argument that cannot be erased and cannot interact with the context in any other way. Both measures rely crucially on the observation, known to Turing and Prawitz, that contracting a redex cannot create redexes of higher degree, where the degree of a redex is defined as the height of the type of its λ-abstraction. The $\mathcal{W}$-measure maps each λ-term to a natural number, and it is obtained by evaluating the term in the $\lambda^{\mathbf{m}}$-calculus and counting the number of remaining wrappers. The $\mathcal{T}^{\mathbf{m}}$-measure maps each λ-term to a structure of nested multisets, where the nesting depth is proportional to the maximum redex degree.

2012 ACM Subject Classification Theory of computation → Equational logic and rewriting; Theory of computation → Lambda calculus

Keywords and phrases Lambda Calculus, Rewriting, Termination, Strong Normalization, Simple Types

Funding This work was partially supported by project grants PICT-2021-I-A-00090, PICT-2021-I-INVI-00602, PIP 11220200100368CO, PICT 2019-1272, PUNQ 2218/22, and PUNQ 2219/22.

Acknowledgements To the anonymous reviewers, Giulio Manzonetto, and Eduardo Bonelli for feedback on earlier versions of this paper. The second author would like to thank his advisors Alejandro Díaz-Caro and Pablo E. Martínez-López.

1 Introduction

In this paper we revisit a fundamental question, that of **strong normalization** of the simply typed λ-calculus (STLC). We begin by recalling that a reduction relation is *weakly normalizing* (WN) if every term can be reduced to normal form in a finite number of steps, whereas it is *strongly normalizing* (SN) if there are no infinite reduction sequences ($a_1 \to a_2 \to a_3 \to \dots$). Let us review three proof techniques for proving strong normalization of the STLC.

One of the better known ways to prove that the STLC is SN is through arguments based on **reducibility models**. The idea is to interpret each type A as a set $[\![A]\!]$ of strongly normalizing terms, and to prove that each term M of type A is an element of $[\![A]\!]$. Many variants of these ideas can be found in the literature, including Girard's reducibility candidates [17] and Tait's saturated sets [30]. These techniques provide relatively succint proofs and they generalize well to extensions of the STLC, *e.g.* to dependent type theory [6] or classical calculi [13]. On the other hand, the abstract nature of reducibility arguments does not provide a "tangible" insight on why a β-reduction step brings a term closer to normal form. More specifically, reducibility arguments do not construct explicit **decreasing measures**. By decreasing measure we mean a function "#" mapping each λ-term to a well-founded ordering $(X, >)$ such that $M \to_\beta N$ implies $\#(M) > \#(N)$.

Another way to prove strong normalization is based on **redex degrees**. A *redex* in the STLC is an applied abstraction, *i.e.* a term of the form $(\lambda x. M)\, N$. The *degree* of a redex is defined as the *height* of the type of its abstraction. A crucial observation, that can be attributed to an unpublished note of Turing (as reported by Gandy [15]; see also [4]), is that *contracting a redex cannot create a redex of higher or equal degree.* Recall that a redex S is *created* by the contraction of a redex R if S has no *ancestor* before R. Indeed, as shown by Lévy [22], in the λ-calculus, redexes can be created in exactly one of the three ways below:

1 $\quad (\underline{\lambda} x. x)\, (\lambda y. M)\, N \quad \to_\beta \quad (\underline{\lambda} y. M)\, N$

2 $\quad (\underline{\lambda} x. \lambda y. M)\, N\, P \quad \to_\beta \quad (\underline{\lambda} y. M[x := N])\, P$

3 $\quad (\underline{\lambda} x. \ldots x\, M \ldots)\, (\lambda y. N) \quad \to_\beta \quad \ldots (\underline{\lambda} y. N)\, M[x := \lambda y. N] \ldots$

where we underline the λ of the contracted redex on the left, and the λ of the created redex on the right. In each of these cases, it can be seen that the degree of the created redex is strictly lower than the degree of the contracted redex. For instance, in creation case **1**, the type of the contracted redex is of the form $(A \to B) \to (A \to B)$, while the type of the created redex is $A \to B$, so the height strictly decreases.

With this fact in mind, for each term M one can define what we call **Turing's measure**, *i.e.* the multiset $\mathcal{T}(M)$ of the degrees of all the redexes of M. One may hope that any reduction step $M \to_\beta N$ decreases the measure, *i.e.* $\mathcal{T}(M) \succ \mathcal{T}(N)$, where "$\succ$" is the usual well-founded multiset ordering induced by the ordering $(\mathbb{N}, >)$ of its elements [12]. Unfortunately, this is not the case: even though contracting a redex can only create redexes of strictly lower degree, it can still make an arbitrary number of *copies* of redexes of arbitrarily large degrees.

In his notes, Turing observed that one can follow a reduction strategy that always selects the *rightmost* redex of highest degree. This strategy ensures that the contracted redex does not copy redexes of higher or equal degree, which makes the $\mathcal{T}(-)$ measure strictly decrease, thus proving that the λ-calculus is WN. An even simpler measure that also decreases, if one follows this strategy, is $\mathcal{T}'(M) = (D, n)$, where D is the maximum degree of the redexes in M and n is the number of redexes of degree D in M. Similar ideas were exploited by Prawitz [28] and Gentzen (as reported by von Plato [27]) to normalize proofs in natural deduction. After WN has been established, an indirect proof of SN can be obtained by translating each typable λ-term M to a typable term M' of the *λI-calculus*; see for instance [29, Section 3.5].

In summary, redex degrees can be used to define concrete measures such as $\mathcal{T}(M)$ and $\mathcal{T}'(M)$, that are computable in linear time and decrease when following a particular reduction strategy. As already mentioned, these measures do not necessarily decrease when contracting arbitrary β-redexes.

A third way to prove SN relies on an interpretation that maps terms to **increasing functionals**. This approach was pioneered by Gandy [16] and refined by de Vrijer [10]. Each type A is mapped to a partially ordered set $[\![A]\!]$. Specifically, base types are mapped to $(\mathbb{N}, \leq)$, and $[\![A \to B]\!]$ is defined as the set of strictly increasing functions $[\![A]\!] \to [\![B]\!]$, partially ordered by the point-wise order. Each term M of type A is interpreted as an element $[M] \in [\![A]\!]$. Moreover, an element $f \in [\![A]\!]$ can be projected to a natural number $f\star \in \mathbb{N}$ in such a way that $M \to_\beta N$ implies $[M]\star > [N]\star$. This indeed provides a decreasing measure. One of the downsides of this measure is that computing $[M]\star$ is essentially as difficult as evaluating M, because $[M]$ is defined as a higher-order functional with a similar structure as the λ-term M itself.

In this work we propose **two decreasing measures for the STLC**, that we dub the $\mathcal{W}$-measure and the $\mathcal{T}^{\mathbf{m}}$-measure, and we prove that they are decreasing. An ideal decreasing measure should fulfill multiple (partly subjective) requirements: **1.** the measure should be easy to calculate, in terms of computational complexity; **2.** its codomain (a well-founded ordering) should be simple, in terms of its ordinal type; **3.** it should give us insight on why β-reduction terminates; **4.** it should be easy to prove that the measure is decreasing. A measure that excels simultaneously at all these requirements is elusive, and perhaps unattainable. The proposed measures have different strengths and weaknesses.

Contributions and structure of this document The $\mathcal{W}$-measure and the $\mathcal{T}^{\mathbf{m}}$-measure are defined by means of on an auxiliary calculus that we dub the $\lambda^{\mathbf{m}}$-calculus. The remainder of the paper is structured as follows.

In **Section 2** we **define the $\lambda^{\mathbf{m}}$-calculus**. It is an extension of the STLC with terms[1] of the form $t\{s\}$, called *wrappers*. A wrapper $t\{s\}$ should be understood as essentially the term t in which s is a *memorized term*, that is, leftover garbage that can be reduced but cannot interact with the context in any way. The type of $t\{s\}$ is the same as the type of t, disregarding the type of s.

The β-reduction rule is modified so that contracting a redex $(\lambda x.\, t)\, s$, besides substituting the free occurrences of x by s in t, produces a wrapper that contains a copy of the argument s. The reduction rule is $(\lambda x.\, t)\{u_1\}\ldots\{u_n\}\, s \;\to_{\mathbf{m}}\; t[x := s]\{s\}\{u_1\}\ldots\{u_n\}$. Note that we allow the presence of an arbitrary number of memorized terms mediating between the abstraction and the application. This is to avoid memorized terms *blocking* redexes. For example, if $I = \lambda x.\, x$:

$$(\lambda x.\, x(xy))I \to_{\mathbf{m}} (I(Iy))\{I\} \to_{\mathbf{m}} (Iy)\{Iy\}\{I\} \to_{\mathbf{m}} (Iy)\{y\{y\}\}\{I\} \to_{\mathbf{m}} y\{y\}\{y\{y\}\}\{I\}$$

Then we study some syntactic properties of $\lambda^{\mathbf{m}}$. In particular, we define a relation $t \rhd s$ of *forgetful reduction*, meaning that s is obtained from t by erasing one memorized subterm. For example, $x\,\{x\{y\}\}\{y\{z\}\} \rhd x\,\{y\{z\}\}$. Forgetful reduction is used as a technical tool to prove that the measures are decreasing in the following sections.

In **Section 3**, we **propose the $\mathcal{W}$-measure** (Def. 12), and we prove that it is decreasing. To define the $\mathcal{W}$-measure, we resort to an operation $\mathtt{S}_d(t)$ that simultaneously contracts all the redexes of degree d in a term of the $\lambda^{\mathbf{m}}$-calculus, that is, the result of the *complete development* of all the redexes of degree d. The degree of a redex $(\lambda x.\, t)\{u_1\}\ldots\{u_n\}\, s$ is defined similarly as for the STLC, as the height of the type of the abstraction. To calculate the $\mathcal{W}$-measure of a λ-term M, let D be the maximum degree of the redexes in M, and define $\mathcal{W}(M)$ as the number of wrappers in $\mathtt{S}_1(\mathtt{S}_2(\ldots\mathtt{S}_D(M)))$. For example, if $M = (\lambda x.\, x\,(x\,y))\,(\lambda z.\, w)$, it turns out that $\mathtt{S}_1(\mathtt{S}_2(M)) = w\{w\{y\}\}\{\lambda z.\, w\}$ which has three wrappers, so $\mathcal{W}(M) = 3$. The $\mathcal{W}$-measure maps each typable λ-term to a natural number. The main result of Section 3 is Thm. 15, stating that $\mathcal{W}$ **is decreasing**, *i.e.* that $M \to_\beta N$ implies $\mathcal{W}(M) > \mathcal{W}(N)$.

In **Section 4** we study **reduction by degrees**, a restricted notion of reduction in the $\lambda^{\mathbf{m}}$-calculus, written $t \xrightarrow{d}_{\mathbf{m}} s$, meaning that t reduces to s by contracting a redex of degree d. This section contains technical commutation, termination, and postponement results.

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[1] Note that terms of the $\lambda^{\mathbf{m}}$-calculus are ranged over by $t, s, \ldots$ (rather than $M, N, \ldots$).

In **Section 5**, we **propose the $\mathcal{T}^{\mathbf{m}}$-measure**, and we prove that it is decreasing. To define the $\mathcal{T}^{\mathbf{m}}$-measure, we define two auxiliary measures $\mathcal{T}^{\mathbf{m}}_{\leq D}(t)$ and $\mathcal{R}^{\mathbf{m}}_{D}(t)$, indexed by a natural number $D \in \mathbb{N}_0$, mutually recursively:

- $\mathcal{T}^{\mathbf{m}}_{\leq D}(t)$ is the multiset of pairs $(d, \mathcal{R}^{\mathbf{m}}_{d}(t))$, for each redex occurrence of degree $d \leq D$ in t;
- $\mathcal{R}^{\mathbf{m}}_{D}(t)$ is the multiset of elements $\mathcal{T}^{\mathbf{m}}_{\leq D-1}(t')$, for each reduction sequence $t \xrightarrow{D}^{*}_{\mathbf{m}} t'$.

The measure $\mathcal{T}^{\mathbf{m}}_{\leq D}(t)$ is defined for every $D \geq 0$, while $\mathcal{R}^{\mathbf{m}}_{D}(t)$ is defined only for $D \geq 1$. Multisets are ordered according to the usual multiset ordering, and pairs according to the lexicographic ordering. To calculate the $\mathcal{T}^{\mathbf{m}}$-measure of a λ-term M, let D be the maximum degree of the redexes in M, and define $\mathcal{T}^{\mathbf{m}}_{\leq}(M) \stackrel{\text{def}}{=} \mathcal{T}^{\mathbf{m}}_{\leq D}(M)$. The measure $\mathcal{T}^{\mathbf{m}}_{\leq}(M)$ yields a structure of nested multisets of nesting depth at most $2D$. The main theorem of Section 3 is Thm. 32, stating that $\mathcal{T}^{\mathbf{m}}$ **is decreasing**, *i.e.* that $M \to_{\beta} N$ implies $\mathcal{T}^{\mathbf{m}}_{\leq}(M) > \mathcal{T}^{\mathbf{m}}_{\leq}(N)$.

Finally, in **Section 6**, we conclude.

2 The $\lambda^{\mathbf{m}}$-calculus

As mentioned in the introduction, the $\lambda^{\mathbf{m}}$-calculus is an extension of the STLC in which the β-reduction rule keeps an extra memorized copy of the argument in a "wrapper" $t\{s\}$, in such a way that contracting a redex like $(\lambda x.\, t)\, s$ does not erase s, even if x does not occur free in t. In this section we define the $\lambda^{\mathbf{m}}$-calculus and we prove some of the properties that are needed in the following sections to prove that the $\mathcal{W}$-measure and the $\mathcal{T}^{\mathbf{m}}$-measure are decreasing. In particular, we discuss *subject reduction* (Prop. 3) and *confluence* (Prop. 4); we define an operation of **simplification** (Def. 5) which turns out to calculate the normal form of a term (Prop. 7); and we define the relation called **forgetful reduction** (Def. 8), which is shown to commute with reduction (Prop. 10).

First we fix the notation and nomenclature. *Types* of the STLC are either base types $(\alpha, \beta, \ldots)$ or arrow types $(A \to B)$. *Terms* are either variables $(x^A, y^A, \ldots)$, abstractions $(\lambda x^A.\, M)$, or applications $(M\, N)$, with the usual typing rules. Terms are defined up to α-renaming of bound variables. We adopt an *à la* Church presentation of the STLC, but we omit most type decorations on variables as long as there is little danger of confusion. The β-reduction rule is $(\lambda x.\, M)\, N \to_{\beta} M[x := N]$ where $M[x := N]$ is the capture-avoiding substitution of the free occurrences of x in M by N.

The $\lambda^{\mathbf{m}}$-calculus: syntax and reduction The set of $\lambda^{\mathbf{m}}$-*terms* —or just *terms*— is given by $t, s, \ldots ::= x^A \mid \lambda x^A.\, t \mid t\, s \mid t\{s\}$. The four kinds of terms are respectively called *variables*, *abstractions*, *applications*, and *wrappers*. In a wrapper $t\{s\}$, the subterm t is called the *body* and s is called the *memorized term*. As in the STLC, we usually omit type annotations and terms are regarded up to α-renaming. A *context* is a term $\mathtt{C}$ with a single free occurrence of a distinguished variable $\Box$, and $\mathtt{C}[t]$ is the variable-capturing substitution of the occurrence of $\Box$ in $\mathtt{C}$ by t.

Typing judgments are of the form $\Gamma \vdash t : A$ where Γ is a partial function mapping variables to types. Derivable typing judgments are defined by the following rules:

$$\frac{}{\Gamma, x : A \vdash x^A : A} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.\, t : A \to B} \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash s : A}{\Gamma \vdash t\, s : B} \qquad \frac{\Gamma \vdash t : A \quad \Gamma \vdash s : B}{\Gamma \vdash t\{s\} : A}$$

A term t is *typable* if $\Gamma \vdash t : A$ holds for some Γ and some A. Unless otherwise specified, when we speak of "terms" we mean "typable terms". It is straightforward to show that a typable term has a unique type. We write $\mathsf{type}(t)$ for the type of t.

A *memory*, written $\mathtt{L}$, is a list of memorized terms, given by the grammar $\mathtt{L} ::= \Box \mid \mathtt{L}\{t\}$. If t is a term and $\mathtt{L}$ is a memory, we write $t\mathtt{L}$ for the term that results from appending all the memorized terms in $\mathtt{L}$ to t, that is, $(t)(\Box\{s_1\}\ldots\{s_n\}) = t\{s_1\}\ldots\{s_n\}$. We write $t[x := s]$ for the operation of capture-avoiding substitution of the free occurrences of x in t by s. The $\lambda^{\mathbf{m}}$-calculus is the rewriting system whose objects are typable $\lambda^{\mathbf{m}}$-terms, endowed with the following notion of reduction, closed by compatibility under arbitrary contexts:

▶ **Definition 1** (Reduction in the $\lambda^{\mathbf{m}}$-calculus). $(\lambda x.\, t)\mathtt{L}\, s \;\to_{\mathbf{m}}\; t[x := s]\{s\}\mathtt{L}$

Abstractions followed by lists of memorized terms, *i.e.* terms of the form $(\lambda x.\, t)\mathtt{L}$, are called $\mathbf{m}$-*abstractions*. Note that all abstractions are also $\mathbf{m}$-abstractions, as $\mathtt{L}$ may be empty. A *redex* is an expression matching the left-hand side of the $\to_{\mathbf{m}}$-reduction rule, which must be an *applied* $\mathbf{m}$-*abstraction*, *i.e.* a term of the form $(\lambda x.\, t)\mathtt{L}\, s$. The *height* of a type is given by $\mathsf{h}(\alpha) \stackrel{\text{def}}{=} 0$ and $\mathsf{h}(A \to B) \stackrel{\text{def}}{=} 1 + \max(\mathsf{h}(A), \mathsf{h}(B))$. The *degree* of a $\mathbf{m}$-abstraction $(\lambda x.\, t)\mathtt{L}$ is defined as the height of its type; note that this number is always strictly positive, since the type must be of the form $A \to B$. Moreover, this type is unique, so the operation is well-defined. The *degree* of a redex $(\lambda x.\, t)\mathtt{L}\, s$ is defined as the degree of the $\mathbf{m}$-abstraction $(\lambda x.\, t)\mathtt{L}$. The *max-degree* of a term t is written $\mathsf{maxdeg}(t)$ and it is defined as the maximum degree of the redexes in t, or 0 if t has no redexes. The *weight* $\mathsf{w}(t)$ of a $\lambda^{\mathbf{m}}$-term t is the number of wrappers in t.

▶ **Example 2.** Let 0 be a base type and let $t := (\lambda x^{0\to 0}.\, \lambda y^0.\, y^0\{x^{0\to 0}\,(x^{0\to 0}\, z^0)\})\, I\, w^0$, where $I := \lambda x^0.\, x^0$. One possible way to reduce t is:

$$\begin{aligned}(\lambda x.\, \lambda y.\, y\{x\,(x\, z)\})\, I\, w \;&\to_{\mathbf{m}}\; (\lambda y.\, y\{I\,(I\, z)\})\{I\}\, w \;\to_{\mathbf{m}}\; w\{I\,(I\, z)\}\{w\}\{I\} \\ &\to_{\mathbf{m}}\; w\{I\,(z\{z\})\}\,\{w\}\,\{I\} \;\to_{\mathbf{m}}\; w\{z\{z\}\{z\{z\}\}\}\{w\}\{I\} = s\end{aligned}$$

The degrees of the redexes contracted in each step are 2, 1, 1, and 1, in that order. Note that $\mathsf{maxdeg}(t) = 2$ and that the weight of the resulting term is $\mathsf{w}(s) = 6$.

Two basic properties of the $\lambda^{\mathbf{m}}$-calculus are subject reduction and confluence. These are immediate consequences of the fact that the $\lambda^{\mathbf{m}}$-calculus can be understood as an *orthogonal HRS* in the sense of Nipkow [26], *i.e.* a left-linear higher-order rewriting system without critical pairs.

▶ **Proposition 3** (Subject reduction). *Let $\Gamma \vdash t : A$ and $t \to_{\mathbf{m}} s$. Then $\Gamma \vdash s : A$.*

▶ **Proposition 4** (Confluence). *If $t_1 \to^*_{\mathbf{m}} t_2$ and $t_1 \to^*_{\mathbf{m}} t_3$, there exists a term t_4 such that $t_2 \to^*_{\mathbf{m}} t_4$ and $t_3 \to^*_{\mathbf{m}} t_4$.*

Full simplification Next, we define an operation written $\mathtt{S}_*(t)$ and called *full simplification*.

Let $d \geq 1$ be a natural number. The *simplification of degree d*, written $\mathtt{S}_d(t)$, is the result of simultaneously contracting all the redexes of degree d in t, that is, the result of the *complete development* of all redexes of degree d. Formally, for each $\lambda^{\mathbf{m}}$-term t we define $\mathtt{S}_d(t)$, and, for each memory $\mathtt{L}$, we define $\mathtt{S}_d(\mathtt{L})$ as follows:

▶ **Definition 5** (Simplification).

$$\begin{aligned}\mathtt{S}_d(x) &\stackrel{\text{def}}{=} x \\ \mathtt{S}_d(\lambda x.\, t) &\stackrel{\text{def}}{=} \lambda x.\, \mathtt{S}_d(t) \\ \mathtt{S}_d(t\, s) &\stackrel{\text{def}}{=} \begin{cases} \mathtt{S}_d(t')[x := \mathtt{S}_d(s)]\{\mathtt{S}_d(s)\}\mathtt{S}_d(\mathtt{L}) & \text{if } t = (\lambda x.\, t')\mathtt{L} \text{ and it is of degree } d \\ \mathtt{S}_d(t)\, \mathtt{S}_d(s) & \text{otherwise} \end{cases} \\ \mathtt{S}_d(t\{s\}) &\stackrel{\text{def}}{=} \mathtt{S}_d(t)\{\mathtt{S}_d(s)\}\end{aligned}$$

where if $\mathtt{L}$ is a memory, $\mathtt{S}_d(\mathtt{L})$ is defined by $\mathtt{S}_d(\Box) \stackrel{\text{def}}{=} \Box$ and $\mathtt{S}_d(\mathtt{L}\{t\}) \stackrel{\text{def}}{=} \mathtt{S}_d(\mathtt{L})\{\mathtt{S}_d(t)\}$. Furthermore, if t is a $\lambda^{\mathbf{m}}$-term of max-degree D, we define the *full simplification* of t as the term that results from iteratively taking the simplification of degree i from D down to 1. More precisely, $\mathtt{S}_*(t) \stackrel{\text{def}}{=} \mathtt{S}_1(\ldots\mathtt{S}_{D-1}(\mathtt{S}_D(t)))$.

▶ **Example 6.** Consider the λ-term $M = (\lambda x^{0\to 0}.\, x^{0\to 0}(x^{0\to 0}\, y^0))(\lambda z^0.\, w^0)$. It can be regarded also as a $\lambda^{\mathbf{m}}$-term, and we have:

$$\begin{array}{rcl} \mathtt{S}_2(M) & = & ((\lambda z^0.\, w^0)\,((\lambda z^0.\, w^0)\, y^0))\{\lambda z^0.\, w^0\} \\ \mathtt{S}_*(M) = \mathtt{S}_1(\mathtt{S}_2(M)) & = & w^0\{w^0\{y^0\}\}\{\lambda z^0.\, w^0\} \end{array}$$

Note that M has only one redex, whose abstraction is of type $(0 \to 0) \to 0$ and hence of degree 2, and that $\mathtt{S}_2(M)$ has two redexes, whose abstractions are of type $0 \to 0$ and hence of degree 1. Moreover, consider the λ-term $N = (\lambda z^0.\, w^0)\,((\lambda z^0.\, w^0)\, y^0)$. Then $\mathtt{S}_*(N) = \mathtt{S}_1(N) = w\{w\{y\}\}$. Note that N has two redexes whose abstraction is of type $0 \to 0$ and hence of degree 1. As an additional note, in the λ-calculus there is a reduction step $M \to_\beta N$, and we have that $\mathsf{w}(\mathtt{S}_*(M)) = 3 > 2 = \mathsf{w}(\mathtt{S}_*(N))$. So this example illustrates that the $\mathcal{W}$-measure (as defined in Def. 12) is decreasing (as we will show in Thm. 15).

As it turns out, **full simplification corresponds to reduction to normal form**. More precisely, we have the following result, which entails in particular that the $\lambda^{\mathbf{m}}$-calculus is weakly normalizing:

▶ **Proposition 7.** *$t \to^*_{\mathbf{m}} \mathtt{S}_*(t)$, and moreover $\mathtt{S}_*(t)$ is a $\to_{\mathbf{m}}$-normal form.*

Proof. To show that $t \to^*_{\mathbf{m}} \mathtt{S}_*(t)$, it suffices to prove a lemma stating that $t \to^*_{\mathbf{m}} \mathtt{S}_d(t)$ for all $d \geq 1$. This implies that $t \to^*_{\mathbf{m}} \mathtt{S}_D(t) \to^*_{\mathbf{m}} \mathtt{S}_{D-1}(\mathtt{S}_D(t)) \ldots \to^*_{\mathbf{m}} \mathtt{S}_1(\ldots\mathtt{S}_{D-1}(\mathtt{S}_D(t))) = \mathtt{S}_*(t)$, where D is the max-degree of t. The lemma itself is straightforward by induction on t.

To show that $\mathtt{S}_*(t)$ is a $\to_{\mathbf{m}}$-normal form, the key property is that, after performing a simplification of order d, no redexes of order d remain. The reason is that contracting a redex of order d can only create redexes of lower degree. More precisely, we prove a key lemma stating that if $d \geq 1$ and $\mathsf{maxdeg}(t) \leq d$, then $\mathsf{maxdeg}(\mathtt{S}_d(t)) < d$. If we let $\mathsf{maxdeg}(t) \leq D$, we can iterate this lemma, to obtain that $\mathsf{maxdeg}(\mathtt{S}_D(t)) < D$, and $\mathsf{maxdeg}(\mathtt{S}_{D-1}(\mathtt{S}_D(t))) < D-1$, …, and finally $\mathsf{maxdeg}(\mathtt{S}_1(\ldots\mathtt{S}_{D-1}(\mathtt{S}_D(t)))) < 1$. This means that $\mathtt{S}_*(t) = \mathtt{S}_1(\ldots\mathtt{S}_{D-1}(\mathtt{S}_D(t)))$ does not contain redexes, since there are no redexes of degree 0, so $\mathtt{S}_*(t)$ must be a $\to_{\mathbf{m}}$-normal form. See Prop. 43 in the appendix for detailed proofs. ◀

Forgetful reduction To conclude this section, we introduce the relation of *forgetful reduction* $t \rhd^+ s$, and we prove that it commutes with reduction.

▶ **Definition 8.** *A $\lambda^{\mathbf{m}}$-term t* reduces via a forgetful step *to s, written $t \rhd s$, according to the following axiom, closed by compatibility under arbitrary contexts:*

$$t\{s\} \rhd t$$

We say that t reduces via forgetful reduction *to s if and only if $t \rhd^+ s$, where $\rhd^+$ denotes the transitive closure of $\rhd$.*

▶ **Example 9.** $(\lambda x.\, x\{y\{y\}\})\{z\{z\}\} \rhd (\lambda x.\, x\{y\{y\}\})\{z\} \rhd (\lambda x.\, x)\{z\} \rhd \lambda x.\, x$.

▶ **Proposition 10** (Forgetful reduction commutes with reduction)**.** *If $t \rhd^+ s$ and $t \to^*_{\mathbf{m}} t'$, there exists a term s' such that $t' \rhd^+ s'$ and $s \to^*_{\mathbf{m}} s'$. Furthermore, if $t \rhd^+ s$ and t is a $\to_{\mathbf{m}}$-normal form, then s is also a normal form.*

Proof. The result can be reduced to a local commutation result, stating that if $t \rhd s$ and $t \to_{\mathbf{m}} t'$, there exists a term s' such that $t' \rhd^+ s'$ and $s \to_{\mathbf{m}}^{=} s'$, where $\to_{\mathbf{m}}^{=}$ is the reflexive closure of $\to_{\mathbf{m}}$. Local commutation can be proved by case analysis. The interesting cases are when a shrinking step $s \rhd s'$ lies inside the argument of a redex, and when a reduction step $r \to_{\mathbf{m}} r'$ is inside erased garbage:

$$\begin{array}{ccc} (\lambda x.\, t)\mathtt{L}\, s & \rhd & (\lambda x.\, t)\mathtt{L}\, s' \\ \downarrow & & \downarrow \\ t[x := s]\{s\}\mathtt{L} & \rhd^+ & t[x := s']\{s'\}\mathtt{L} \end{array} \qquad\qquad \begin{array}{ccc} u\{r\} & \rhd & u \\ \downarrow & & \| \\ u\{r'\} & \rhd^+ & u \end{array}$$

For the last part of the statement, it suffices to show that if $t \rhd s$ in one step and t is a $\to_{\mathbf{m}}$-normal form, then s is also a normal form, which is straightforward by induction on t. See Prop. 46 in the appendix for detailed proofs. ◀

Each step in the STLC has a *corresponding* step in the $\lambda^{\mathbf{m}}$-calculus, that contracts the redex in the same position. For instance the step $(\lambda x.\, x\, y)\, I \to_\beta I\, y$ in the STLC has a corresponding step $(\lambda x.\, x\, y)\, I \to_{\mathbf{m}} (I\, y)\{I\}$ in the $\lambda^{\mathbf{m}}$-calculus. In this example, $(I\, y)\{I\} \rhd I\, y$. The following easy lemma confirms that this is a general fact:

▶ **Lemma 11** (Reduce/forget lemma)**.** *Let $M \to_\beta N$ be a β-step, and let $M \to_{\mathbf{m}} s$ be the corresponding step in $\lambda^{\mathbf{m}}$. Then $s \rhd N$.*

3 The $\mathcal{W}$-measure

In this section, we **define the $\mathcal{W}$-measure** (Def. 12) and we **prove that it is decreasing** (Thm. 15). Let us try to convey some ideas that led to the definition of the $\mathcal{W}$-measure. Recall that an abstract rewriting system $(A, \to)$ is *weakly Church–Rosser* (WCR) if $\leftarrow\to \subseteq \to^*\leftarrow^*$, *Church–Rosser* (CR) if $\leftarrow^*\to^* \subseteq \to^*\leftarrow^*$, and *increasing* (Inc) if there exists a function $|\cdot| : A \to \mathbb{N}$ such that $a \to b$ implies $|a| < |b|$. Let us also recall *Klop–Nederpelt's lemma* [31, Theorem 1.2.3 (iii)], which states that $\text{Inc} \wedge \text{WCR} \wedge \text{WN} \implies \text{SN} \wedge \text{CR}$.

Let $(A, \to)$ be increasing and WCR. Given a reduction $a \to^* b$, where b is a normal form, we can find a *decreasing* measure for the set of objects reachable from a, that is, the set $\{c \in A \mid a \to^* c\}$. In fact, by Klop–Nederpelt's lemma, we know that for every $c \in A$ such that $a \to^* c$ we have that $c \to^* b$, which implies that $|c| \leq |b|$, and hence we can define $\#(c) := |b| - |c|$. It is easy to see that $\#(-)$ is a decreasing measure, since $c \to c'$ implies that $|c| < |c'|$ so $\#(c) := |b| - |c| > |b| - |c'| = \#(c')$. Furthermore, the value of $\#(c)$ does not depend on the choice of a, by uniqueness of normal forms.

The idea behind the $\mathcal{W}$-measure is that the construction of a *decreasing* measure can be based on an *increasing* measure, according to the previous observation. It is not possible to build an increasing measure directly for the STLC; *e.g.* the following infinite sequence of expansions $t \leftarrow I\, t \leftarrow I\, (I\, t) \leftarrow \ldots$ would induce an infinite decreasing chain of natural numbers $|t| > |I\, t| > |I\, (I\, t)| > \ldots$.

One could try to define an increasing measure in a variant of the STLC such as Endrullis *et al.*'s clocked λ-calculus [14], in which the β-rule becomes $(\lambda x.\, t)\, s \to \tau(t[x := s])$, that is, contracting a β-redex produces a counter "τ" that keeps track of the number of contracted redexes. One could then count the number of τ's: for example, in the reduction sequence $(\lambda x.\, x\, (x\, y))\, I \to \tau(I\, (I\, y)) \to \tau\tau(I\, y) \to \tau\tau\tau y$ the number of counters strictly increases with each step. Unfortunately, this does not define an increasing measure, due to *erasure*. For example, $(\lambda x.\, y)\, t \to \tau y$ erases all the counters in t.

This is the motivation behind the definition of the $\lambda^{\mathbf{m}}$-calculus, which avoids erasure by always keeping an extra copy of the argument in a wrapper. The $\lambda^{\mathbf{m}}$-calculus is indeed increasing: in a step $t \to_{\mathbf{m}} s$ one has that $\mathsf{w}(t) < \mathsf{w}(s)$, where we recall that $\mathsf{w}(t)$ denotes the *weight*, *i.e.* the number of wrappers in t. For example, the step $(\lambda x.\, y)\,(z\{z\}) \to_{\mathbf{m}} y\{z\{z\}\}$ increases the number of wrappers. The decreasing measure $\mathcal{W}(M)$ is defined essentially by reducing M to normal form in the $\lambda^{\mathbf{m}}$-calculus and counting the number of wrappers in the result:

▶ **Definition 12** (The $\mathcal{W}$-measure)**.** *For each typable λ-term M, define $\mathcal{W}(M) \stackrel{\text{def}}{=} \mathsf{w}(\mathtt{S}_*(M))$.*

As we show below, $\mathtt{S}_*(M)$ turns out to be exactly the normal form of M in the $\lambda^{\mathbf{m}}$-calculus. We insist in writing $\mathtt{S}_*(M)$ to emphasize that the *definition* of the $\mathcal{W}$-measure does not require to prove that the $\lambda^{\mathbf{m}}$-calculus is weakly normalizing. Indeed, the simplification $\mathtt{S}_d(t)$ can be defined by structural induction on t, and the full simplification $\mathtt{S}_*(t) = \mathtt{S}_1(\mathtt{S}_2(\ldots \mathtt{S}_D(t)))$ can be calculated in exactly D iterations. On the other hand, the *proof* that the $\mathcal{W}$-measure is decreasing does rely on the fact that $\mathtt{S}_*(M)$ is the normal form of M.

In the remainder of this section, we prove that the $\mathcal{W}$-measure is indeed decreasing. The following lemma states that forgetful reduction decreases weight, and it is straightforward to prove:

▶ **Lemma 13.** *If $t \rhd^+ s$ then $\mathsf{w}(t) > \mathsf{w}(s)$.*

The proof that the $\mathcal{W}$-measure decreases relies on the two following properties that relate full simplification $\mathtt{S}_*(-)$ respectively with reduction $(\to_{\mathbf{m}})$ and forgetful reduction $(\rhd^+)$:

▶ **Lemma 14.** **1.** *If $t \to_{\mathbf{m}} s$ then $\mathtt{S}_*(t) = \mathtt{S}_*(s)$.* **2.** *If $t \rhd^+ s$ then $\mathtt{S}_*(t) \rhd^+ \mathtt{S}_*(s)$.*

Proof. For the first item, note that by Prop. 7, we know that $t \to_{\mathbf{m}}^* \mathtt{S}_*(t)$ and that $t \to_{\mathbf{m}} s \to_{\mathbf{m}}^* \mathtt{S}_*(s)$, where moreover $\mathtt{S}_*(t)$ and $\mathtt{S}_*(s)$ are $\to_{\mathbf{m}}$-normal forms. By confluence (Prop. 4), this means that $\mathtt{S}_*(t) = \mathtt{S}_*(s)$.

For the second item, note that by Prop. 7, we know that $t \to_{\mathbf{m}}^* \mathtt{S}_*(t)$. Since we also know $t \rhd^+ s$ by hypothesis, and since forgetful reduction commutes with reduction (Prop. 10), there exists a term u such that $s \to_{\mathbf{m}}^* u$ and $\mathtt{S}_*(t) \rhd^+ u$. By Prop. 7 we know that $\mathtt{S}_*(t)$ is in normal form, so by Prop. 10 u must also be a normal form. On the other hand, by Prop. 7 we know that $s \to_{\mathbf{m}}^* \mathtt{S}_*(s)$, where $\mathtt{S}_*(s)$ must also be a normal form. In summary, we have that $s \to_{\mathbf{m}}^* u$ and $s \to_{\mathbf{m}}^* \mathtt{S}_*(s)$, where both u and $\mathtt{S}_*(s)$ are normal forms. By confluence (Prop. 4) $u = \mathtt{S}_*(s)$, and from this we obtain that $\mathtt{S}_*(t) \rhd^+ u = \mathtt{S}_*(s)$, as required. ◀

▶ **Theorem 15.** *Let M, N be typable λ-terms such that $M \to_\beta N$. Then $\mathcal{W}(M) > \mathcal{W}(N)$.*

Proof. Given the step $M \to_\beta N$, consider the corresponding step $M \to_{\mathbf{m}} s$, and note that $s \rhd^+ N$ by the reduce/forget lemma (Lem. 11). Since $M \to_{\mathbf{m}} s \rhd^+ N$, by Lem. 14, we have that $\mathtt{S}_*(M) = \mathtt{S}_*(s) \rhd^+ \mathtt{S}_*(N)$. Finally, by Lem. 13, $\mathcal{W}(M) = \mathsf{w}(\mathtt{S}_*(M)) > \mathsf{w}(\mathtt{S}_*(N)) = \mathcal{W}(N)$. ◀

The following is one example that the $\mathcal{W}$-measure decreases (see Ex. 6 for another example):

▶ **Example 16.** Let $M = (\lambda x^0.\, y^{0\to0\to0}\, x^0\, x^0)\,((\lambda x^{0\to0}.\, x^{0\to0}\, z^0)\, f^{0\to0})$, consider the step $M = (\lambda x.\, y\, x\, x)((\lambda x.\, x\, z)\, f) \to_\beta (\lambda x.\, y\, x\, x)\,(f\, z) = N$, and note that $\mathcal{W}(M) = \mathsf{w}(\mathtt{S}_*(M)) = 4 > 1 = \mathcal{W}(N)$, since:

$$\mathtt{S}_*(M) = (y\,(f\,z)\{f\}\,(f\,z)\{f\})\{(f\,z)\{f\}\} \qquad \mathtt{S}_*(N) = (y\,(f\,z)\,(f\,z))\{f\,z\}$$

4 Reduction by degrees

This section is of purely technical nature. The aim is to develop tools that we use in the following section to reason about the $\mathcal{T}^{\mathbf{m}}$-measure. To do so, we need to introduce witnesses of steps and reduction sequences, treating the $\lambda^{\mathbf{m}}$-calculus as an *abstract rewriting system* in the sense of [31, Def. 8.2.2] or as a *transition system* in the sense of [24, Def. 1]. *Objects* are $\lambda^{\mathbf{m}}$-terms, *steps* are 5-uples $R = (\mathtt{C}, x, t, \mathtt{L}, s)$ witnessing the reduction step $\mathtt{C}[(\lambda x.\,t)\mathtt{L}\,s] \to_{\mathbf{m}} \mathtt{C}[t[x := s]\{s\}\mathtt{L}]$ under a context $\mathtt{C}$, and *reductions* $(\rho, \sigma, \ldots)$ are sequences of composable steps. Similarly, *forgetful steps* are triples $R = (\mathtt{C}, t, s)$ witnessing the forgetful reduction $\mathtt{C}[t\{s\}] \rhd \mathtt{C}[t]$, and *forgetful reductions* (also written $\rho, \sigma, \ldots$) are sequences of composable forgetful steps. We write ρ^{src} and ρ^{tgt} respectively for the source and target terms of ρ.

For each $d \in \mathbb{N}_0$, we define **reduction of degree** d as follows:

▶ **Definition 17.** *$t \xrightarrow{d}_{\mathbf{m}} s$ if and only if $t \to_{\mathbf{m}} s$ by contracting a redex of degree d.*

We write $R : t \xrightarrow{d}_{\mathbf{m}} s$ if R is a step witnessing a reduction step of degree d, and $\rho : t \xrightarrow{d}{}^{*}_{\mathbf{m}} s$ if ρ is a reduction witnessing a sequence of reduction steps of degree d.

The following results require to explicitly manipulate steps and reductions. We only give sketches of the proofs for lack of space. See Section A.2 in the appendix for detailed proofs.

▶ **Proposition 18** (Commutation of reduction by degrees)**.** *For any two reductions $\rho : t_1 \xrightarrow{d}{}^{*}_{\mathbf{m}} t_2$ and $\sigma : t_1 \xrightarrow{D}{}^{*}_{\mathbf{m}} t_3$, there exists a term t_4 and one can construct reductions $\sigma/\rho : t_2 \xrightarrow{D}{}^{*}_{\mathbf{m}} t_4$ and $\rho/\sigma : t_3 \xrightarrow{d}{}^{*}_{\mathbf{m}} t_4$ such that, furthermore, if $d \neq D$, then* **1.** *ρ/σ contains at least as many steps as ρ; and* **2.** *ρ/σ determines ρ, that is, $\rho_1/\sigma = \rho_2/\sigma$ implies $\rho_1 = \rho_2$.*

Proof. This is reduced to the fact that the $\lambda^{\mathbf{m}}$-calculus can be understood as an orthogonal higher-order rewriting system in the sense of Nipkow [26]. Indeed, ρ/σ and σ/ρ can be taken to be the standard notion of projection based on residuals for orthogonal HRSs. Note that item **1.** holds because the $\lambda^{\mathbf{m}}$-calculus is non-erasing while item **2.** is a consequence of the *unique ancestor* property, *i.e.* each redex *descends* from at most one redex. ◀

▶ **Corollary 19** (Termination of reduction by degrees)**.** *The relation $\xrightarrow{d}_{\mathbf{m}}$ is strongly normalizing.*

Proof. This is a consequence of the fact that HRSs enjoy the Finite Developments property [31, Theorem 11.5.11], observing that reduction of degree d does not create redexes of degree d. Alternatively, it can be easily shown that $t \xrightarrow{d}{}^{*}_{\mathbf{m}} \mathtt{S}_d(t)$ and $\mathtt{S}_d(t)$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form, so $\xrightarrow{d}_{\mathbf{m}}$ is WN. Moreover, one can observe that $\xrightarrow{d}_{\mathbf{m}}$ is *uniformly normalizing* [19], given that there is no erasure, which entails that $\xrightarrow{d}_{\mathbf{m}}$ is SN. ◀

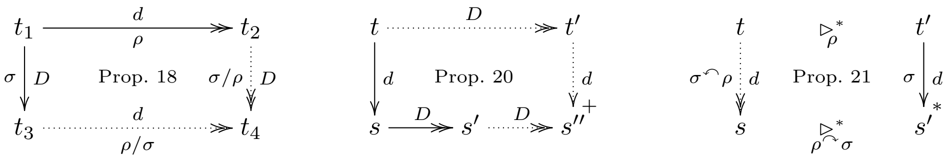
▶ **Proposition 20** (Lifting property for lower steps)**.** *Let $d < D$ and $t \xrightarrow{d}_{\mathbf{m}} s \xrightarrow{D}{}^{*}_{\mathbf{m}} s'$. Then there exist terms t', s'' such that $t \xrightarrow{D}{}^{*}_{\mathbf{m}} t'$ and $s' \xrightarrow{D}{}^{*}_{\mathbf{m}} s''$ and $t' \xrightarrow{d}{}^{+}_{\mathbf{m}} s''$.*

Proof. Note that $t \xrightarrow{D}{}^{*}_{\mathbf{m}} \mathtt{S}_D(t)$. By Prop. 18, there exists a term u such that $s \xrightarrow{D}{}^{*}_{\mathbf{m}} u$ and $\mathtt{S}_D(t) \xrightarrow{d}{}^{+}_{\mathbf{m}} u$. Again by Prop. 18, there exists s'' such that $u \xrightarrow{D}{}^{*}_{\mathbf{m}} s''$ and $s' \xrightarrow{D}{}^{*}_{\mathbf{m}} s''$. Moreover, $\mathtt{S}_D(t)$ is in $\xrightarrow{D}_{\mathbf{m}}$-normal form. Since $\mathtt{S}_D(t) \xrightarrow{d}{}^{*}_{\mathbf{m}} u$ with $d < D$ and reduction does not create redexes of higher degree, u is also in $\xrightarrow{D}_{\mathbf{m}}$-normal form, so $u = s''$, and we are done. ◀

▶ **Proposition 21** (Postponement of forgetful reduction). *For any two reductions $\rho : t \rhd^* t'$ and $\sigma : t' \xrightarrow{d}{}^*_{\mathbf{m}} s'$, there exists a term s and reductions $\rho^\frown\sigma : s \rhd^* s'$ and $\sigma^\frown\rho : t \xrightarrow{d}{}^*_{\mathbf{m}} s$. Furthermore, $\sigma^\frown\rho$ determines σ, that is, $\sigma_1^\frown\rho = \sigma_2^\frown\rho$ implies $\sigma_1 = \sigma_2$.*

Proof. This can be reduced to an analysis of the critical pairs between the rewriting rules defining $\rhd^{-1}$ and $\to_{\mathbf{m}}$. Critical pairs are of the form $(\lambda x.\, t)\mathtt{L}_1\{\!\!\{s\}\!\!\}\mathtt{L}_2\, u \rhd (\lambda x.\, t)\mathtt{L}_1\mathtt{L}_2\, u \to_{\mathbf{m}} t[x := u]\{\!\!\{u\}\!\!\}\mathtt{L}_1\mathtt{L}_2$ and can be closed by $(\lambda x.\, t)\mathtt{L}_1\{\!\!\{s\}\!\!\}\mathtt{L}_2\, u \to_{\mathbf{m}} t[x := u]\{\!\!\{u\}\!\!\}\mathtt{L}_1\{\!\!\{s\}\!\!\}\mathtt{L}_2 \rhd t[x := u]\{\!\!\{u\}\!\!\}\mathtt{L}_1\mathtt{L}_2$. ◀

The following diagrams depict the statements of the three preceding propositions:



5 The $\mathcal{T}^{\mathbf{m}}$-measure

In this section, we **define the $\mathcal{T}^{\mathbf{m}}$-measure** (Def. 25) and we **prove that it is decreasing** (Thm. 32). We start with some preliminary notions.

A partially ordered set $(X, >)$ is *well-founded* if there are no infinite decreasing chains. $\mathbb{M}(X)$ denotes the set of *finite multisets* over a set X, which are functions $\mathfrak{m} : X \to \mathbb{N}_0$ such that $\mathfrak{m}(x) > 0$ for finitely many values of $x \in X$. We write $\mathfrak{m} + \mathfrak{n}$ for the sum of multisets, and $x \in \mathfrak{m}$ if $\mathfrak{m}(x) > 0$. We write $[x_1, \dots, x_n]$ for the multiset of elements $x_1, \dots, x_n$, taking multiplicities into account. If X is a finite set and $f : X \to Y$ is a function, we use the "multiset builder" notation $[f(x) \mid\mid x \in X]$ to denote the multiset $\sum_{x \in X}[f(x)]$. If $(X, >)$ is a partially ordered set, we define a binary relation $\succ^1$ on multisets by declaring that $\mathfrak{m} + [x] \succ^1 \mathfrak{m} + \mathfrak{n}$ if $x > y$ for every $y \in \mathfrak{n}$. The *multiset order* induced by $(X, >)$ is the strict order relation on multisets defined by declaring that $\mathfrak{m} \succ \mathfrak{n}$ if and only if $\mathfrak{m}\ (\succ^1)^+\ \mathfrak{n}$. We recall the following widely known theorem by Dershowitz and Manna [12]:

▶ **Theorem 22.** *If $(X, >)$ is well-founded, then $(\mathbb{M}(X), \succ)$ is well-founded.*

As usual, $\mathfrak{m} \succeq \mathfrak{n}$ stands for $(\mathfrak{m} = \mathfrak{n} \vee \mathfrak{m} \succ \mathfrak{n})$, and $\mathfrak{m} \preceq \mathfrak{n}$ stands for $\mathfrak{n} \succeq \mathfrak{m}$. We define an operation $k \otimes \mathfrak{m}$ by the recursive equations $0 \otimes \mathfrak{m} \stackrel{\text{def}}{=} [\,]$ and $(1 + k) \otimes \mathfrak{m} \stackrel{\text{def}}{=} \mathfrak{m} + k \otimes \mathfrak{m}$. The relation $\mathfrak{m} \mathrel{:\succ:} \mathfrak{n}$, called the *pointwise multiset order*, is defined to hold if $\mathfrak{m}$ and $\mathfrak{n}$ can be written as of the forms $\mathfrak{m} = [x_1, \dots, x_n]$ and $\mathfrak{n} = [y_1, \dots, y_n]$ in such a way that $x_i > y_i$ for all $i \in 1..n$. Observe that if $\mathfrak{m} \mathrel{:\succ:} \mathfrak{n}$ then for all $k \in \mathbb{N}_0$ we have that $\mathfrak{m} \succeq k \otimes \mathfrak{n}$. Another easy-to-check property is that if $\mathfrak{m} \mathrel{:\succ:} \mathfrak{n}$ and $\mathfrak{m}$ is non-empty then $\mathfrak{m} \succ \mathfrak{n}$.

A first frustrated attempt As mentioned in the introduction, Turing's measure, given by $\mathcal{T}(M) \stackrel{\text{def}}{=} [d \mid\mid R \text{ is a redex occurrence of degree } d \text{ in } M]$, decreases when contracting the rightmost redex of highest degree. Our goal is to mend the $\mathcal{T}$-measure in such a way that contracting *any* redex decreases the measure. The difficulty is that a redex of degree d may copy redexes of a higher or equal degree $d' \geq d$. So one can wonder: whenever a redex R of degree d makes n copies of a redex S of degree $d' \geq d$, in what sense can the copies of S be considered "smaller" than S? To address this, we generalize the $\mathcal{T}$-measure to a family of measures $\mathcal{T}_D(M) \stackrel{\text{def}}{=} [(d, \mathcal{T}_{d-1}(M)) \mid\mid R \text{ is a redex occurrence of degree } d \leq D \text{ in } M]$ indexed by a degree $D \in \mathbb{N}_0$. Note that $\mathcal{T}_0(M)$ is the empty multiset because there are no redexes of degree 0.

Let us try to argue that if $d \leq D$ and $M \xrightarrow{d}_\beta N$ then $\mathcal{T}_D(M) \succ \mathcal{T}_D(N)$. Here $M \xrightarrow{d}_\beta N$ means that $M \to_\beta N$ by contracting a redex of degree d. Suppose that the contraction of the redex $R : M \xrightarrow{d}_\beta N$ copies a redex S of degree d', where we assume that $d < d' \leq D$, producing n copies $S_1, \ldots, S_n$. Note that the contribution of S to the multiset is $(d', \mathcal{T}_{d'-1}(M))$, and the contribution of each S_i is $(d', \mathcal{T}_{d'-1}(N))$. By induction on D, we could inductively argue that $\mathcal{T}_{d'-1}(M) \succ \mathcal{T}_{d'-1}(N)$, since $d' - 1 < d' \leq D$. So far the property would seem to hold.

The problem with this proposal is that a redex R of degree d may still make copies of redexes of degree *exactly* d, whose contribution does not necessarily decrease[2].

A second frustrated attempt The difficulty is to deal with the situation in which a redex R of degree d makes n copies of a redex S of the same degree d. A key observation is that a reduction sequence $M \xrightarrow{d}{}^{*}_\beta N$ must be a *development*[3] of the set of redexes of degree d. This is because contracting a redex of degree d can only create redexes of degree strictly less than d, so any redex of degree d that remains after one $\xrightarrow{d}_\beta$-step must be a *residual* of a preexisting redex. This motivates our second attempt to define a measure, consisting of two families of measures $\mathcal{T}^\beta_{\leq D}(-)$ and $\mathcal{R}^\beta_D(-)$, indexed by $D \in \mathbb{N}_0$ and defined mutually recursively:

$$\mathcal{T}^\beta_{\leq D}(M) \overset{\text{def}}{=} [(d, \mathcal{R}^\beta_d(M)) \mid\mid R \text{ is a } \beta\text{-redex occurrence of degree } d \leq D \text{ in } M]$$

$$\mathcal{R}^\beta_D(M) \overset{\text{def}}{=} [\mathcal{T}^\beta_{\leq D-1}(M') \mid\mid \rho : M \xrightarrow{D}{}^{*}_\beta M']$$

Note that there are no redexes of degree 0, so $\mathcal{T}^\beta_{\leq D}(M)$ may not depend on $\mathcal{R}^\beta_0(M)$. In fact, $\mathcal{R}^\beta_D(M)$ is defined only for $D \geq 1$. The recursive definition is well-founded because $\mathcal{T}^\beta_{\leq D}(M)$ may depend on $\mathcal{R}^\beta_1(M), \ldots, \mathcal{R}^\beta_D(M)$ which in turn may only depend on $\mathcal{T}^\beta_{\leq d}(M')$ for $d < D$. The multiplicity of $\mathcal{T}^\beta_{\leq D-1}(M')$ in the multiset $\mathcal{R}^\beta_D(M)$ is given by the number of reduction sequences that contract only redexes of degree D, that is, the number of different paths $M \xrightarrow{D}{}^{*}_{\mathbf{m}} M'$. One important point is that, for the measure $\mathcal{R}^\beta_D(t)$ to be well defined, one needs to argue that the number of paths $M \xrightarrow{D}{}^{*}_{\mathbf{m}} M'$ is finite. Since $M \xrightarrow{D}{}^{*}_{\mathbf{m}} M'$ is a development, this is a consequence of the *finite developments* (FD) property for orthogonal HRSs [31, Theorem 11.5.11].[4]

Let us try to argue that if $d \leq D$ and $M \xrightarrow{d}_\beta N$ then $\mathcal{T}^\beta_{\leq D}(M) \succ \mathcal{T}^\beta_{\leq D}(N)$. On the first hand, if a redex $R : M \xrightarrow{d}_\beta N$ of degree d copies a redex S of *exactly* the same degree d making n copies $S_1, \ldots, S_n$, the contribution of S to the multiset is $(d, \mathcal{R}^\beta_d(M))$, whereas each S_i contributes $(d, \mathcal{R}^\beta_d(N))$, and we can argue that $\mathcal{R}^\beta_d(M) \succ \mathcal{R}^\beta_d(N)$, because we can injectively map each reduction sequence $\rho : N \xrightarrow{d}{}^{*}_\beta N'$ to the reduction sequence $R\rho : M \xrightarrow{d}_\beta N \xrightarrow{d}{}^{*}_\beta N'$,

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[2] For example, in $M = (\lambda x^0. y^{0\to0\to0}\, x^0\, x^0)\,((\lambda z^0. z^0)\, w^0) \xrightarrow{1}_\beta y^{0\to0\to0}\,((\lambda z^0. z^0)\, w^0)\,((\lambda z^0. z^0)\, w^0) = N$ the measure does not decrease, as $\mathcal{T}_1(M) = [(1, []), (1, [])] = \mathcal{T}_1(N)$.

[3] Recall that a development of a set of redexes X is a reduction sequence $M \to^*_\beta N$ in which each step contracts a *residual* of a redex in X. The residuals of a redex $S : t \to_\beta s$ after the contraction of a redex $R : t \to_\beta t'$ are, informally speaking, the "copies" left of S in t'. For formal definitions see [3, Section 11.2].

[4] Note that FD only ensures that developments are finite. To see that the set $\{\rho \mid M \xrightarrow{D}{}^{*}_{\mathbf{m}} M'\}$ is finite, one should resort to König's lemma, together with the fact that the STLC is finitely branching. For a constructive proof, one can use a computable decreasing measure, such as in de Vrijer's proof of FD [9].

where $R\rho$ denotes the composition of R and ρ. Furthermore, there is an empty reduction sequence $M \xrightarrow{d}_{\beta}^{*} M$ contributing an element $\mathcal{T}^{\beta}_{\leq d-1}(M)$ to $\mathcal{R}^{\beta}_{d}(M)$ but not to $\mathcal{R}^{\beta}_{d}(N)$.

On the other hand, if the contraction of a redex $R : M \xrightarrow{d}_{\beta} N$ of degree d copies a redex S of *strictly* greater degree $d' > d$ making n copies $S_1, \ldots, S_n$, the weight of S is $(d', \mathcal{R}^{\beta}_{d'}(M))$ and the weight of each S_i is $(d', \mathcal{R}^{\beta}_{d'}(N))$, and we would need to show that $\mathcal{R}^{\beta}_{d'}(M) \succ \mathcal{R}^{\beta}_{d'}(N)$. One way to do so would be to map each reduction sequence $\rho : N \xrightarrow{d}_{\beta}^{*} N'$ to a reduction sequence $\sigma : M \xrightarrow{d}_{\beta}^{*} M'$ such that $\mathcal{T}^{\beta}_{\leq d'-1}(M') \succ \mathcal{T}^{\beta}_{\leq d'-1}(N')$. However, there does not seem to be a way to rule out the possibility that σ might erase R and that $M' = N'$, which would yield $\mathcal{T}^{\beta}_{\leq d'-1}(M') = \mathcal{T}^{\beta}_{\leq d'-1}(N')$, rather than a strict inequality. The root of the problem seems again to be *erasure*.

Definition of the $\mathcal{T}^{\mathbf{m}}$-measure The $\mathcal{T}^{\mathbf{m}}$-measure is based on the ideas described above, but considering reduction in the $\lambda^{\mathbf{m}}$-calculus rather than in the STLC, to ensure that there is no erasure. Informally, the $\mathcal{T}^{\mathbf{m}}$-measure is defined by means of the two following equations. These equations are exactly as the ones defining $\mathcal{T}^{\beta}_{\leq D}(-)$ and $\mathcal{R}^{\beta}_{D}(-)$ above, with the only difference that they deal with $\lambda^{\mathbf{m}}$-terms and $\to_{\mathbf{m}}$-reduction rather than with pure λ-terms and $\to_{\beta}$-reduction:

$$\mathcal{T}^{\mathbf{m}}_{\leq D}(t) \stackrel{\text{def}}{=} [(d, \mathcal{R}^{\mathbf{m}}_{d}(t)) \mid\mid R \text{ is a } \mathbf{m}\text{-redex occurrence of degree } d \leq D \text{ in } t]$$

$$\mathcal{R}^{\mathbf{m}}_{D}(t) \stackrel{\text{def}}{=} [\mathcal{T}^{\mathbf{m}}_{\leq D-1}(t') \mid\mid \rho : t \xrightarrow{D}_{\mathbf{m}}^{*} t']$$

To be able to reason about these measures inductively, it will be convenient to define an auxiliary measure $\mathcal{T}^{\mathbf{m}}_{d}(t_0, t)$ as the multiset of elements of the form $(d, \mathcal{R}^{\mathbf{m}}_{d}(t_0))$ for each $\mathbf{m}$-redex occurrence of degree *exactly* d in t. This auxiliary measure takes two arguments t_0 and t, and it is defined by structural recursion on the second argument (t), while the first argument (t_0) is used to keep track of the original term. Note that, with this auxiliary definition, we can write $\mathcal{T}^{\mathbf{m}}_{\leq D}(t)$ as the sum $\mathcal{T}^{\mathbf{m}}_{\leq D}(t) = \mathcal{T}^{\mathbf{m}}_{1}(t,t) + \ldots + \mathcal{T}^{\mathbf{m}}_{D}(t,t)$.

To define the measure formally, we start by precisely defining the codomain of the measure.

▶ **Definition 23** (Codomain of the $\mathcal{T}^{\mathbf{m}}$-measure)**.** *For each $d \geq 0$, we define a set $\mathbb{T}_d$, and for $d \geq 1$ we define a set $\mathbb{R}_d$, mutually recursively:*

$$\mathbb{T}_d \stackrel{\text{def}}{=} \mathbb{M}(\{(i,b) \mid 1 \leq i \leq d,\ b \in \mathbb{R}_i\}) \qquad \mathbb{R}_d \stackrel{\text{def}}{=} \mathbb{M}(\mathbb{T}_{d-1})$$

The sets $\mathbb{T}_d$ and $\mathbb{R}_d$ are partially ordered by the induced multiset ordering on their elements. Tuples (i,b) are ordered with the lexicographic order, that is, $(i,b) > (i',b')$ if and only if $i > i' \vee (i = i' \wedge b \succ b')$. Note that $\mathbb{T}_0 = \{[\,]\}$ and that if $d \leq d'$ then $\mathbb{T}_d \subseteq \mathbb{T}_{d'}$ and $\mathbb{R}_d \subseteq \mathbb{R}_{d'}$. Moreover, $(\mathbb{T}_d, \succ)$ and $(\mathbb{R}_d, \succ)$ are well-founded partial orders by Thm. 22.

Given typable $\lambda^{\mathbf{m}}$-terms t_0, t, and $d \in \mathbb{N}_0$, we define $\mathcal{T}^{\mathbf{m}}_{d}(t_0,t) \in \mathbb{T}_d$ and $\mathcal{T}^{\mathbf{m}}_{\leq d}(t) \in \mathbb{T}_d$, and if $d > 0$ we define $\mathcal{R}^{\mathbf{m}}_{d}(t) \in \mathbb{R}_d$, by induction on d as follows. Note that $\mathcal{T}^{\mathbf{m}}_{d}(t_0,t)$ is defined by a nested induction on t, and it is also defined on memories ($\mathcal{T}^{\mathbf{m}}_{d}(t_0, \mathtt{L})$):

▶ **Definition 24** (The measures $\mathcal{T}^{\mathbf{m}}_{d}(-,-)$, $\mathcal{T}^{\mathbf{m}}_{\leq d}(-)$, and $\mathcal{R}^{\mathbf{m}}_{d}(-)$)**.**

$$
\begin{aligned}
\mathcal{T}^{\mathbf{m}}_{d}(t_0, x) &\stackrel{\text{def}}{=} [\,] \\
\mathcal{T}^{\mathbf{m}}_{d}(t_0, \lambda x.\, s) &\stackrel{\text{def}}{=} \mathcal{T}^{\mathbf{m}}_{d}(t_0, s) \\
\mathcal{T}^{\mathbf{m}}_{d}(t_0, s\, u) &\stackrel{\text{def}}{=} \begin{cases} \mathcal{T}^{\mathbf{m}}_{d}(t_0, s') + \mathcal{T}^{\mathbf{m}}_{d}(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_{d}(t_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_{d}(t_0))] \\ \qquad\qquad \textit{if } s = (\lambda x.\, s')\mathtt{L} \textit{ and it is of degree } d \\ \mathcal{T}^{\mathbf{m}}_{d}(t_0, s) + \mathcal{T}^{\mathbf{m}}_{d}(t_0, u) \qquad \textit{otherwise} \end{cases} \\
\mathcal{T}^{\mathbf{m}}_{d}(t_0, s\{u\}) &\stackrel{\text{def}}{=} \mathcal{T}^{\mathbf{m}}_{d}(t_0, s) + \mathcal{T}^{\mathbf{m}}_{d}(t_0, u)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}^{\mathbf{m}}_{d}(t_0, \Box) &\stackrel{\text{def}}{=} [\,] \\
\mathcal{T}^{\mathbf{m}}_{d}(t_0, \mathtt{L}\{t\}) &\stackrel{\text{def}}{=} \mathcal{T}^{\mathbf{m}}_{d}(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_{d}(t_0, t)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}^{\mathbf{m}}_{\leq d}(t) &\stackrel{\text{def}}{=} \textstyle\sum_{i=1}^{d} \mathcal{T}^{\mathbf{m}}_{i}(t, t) \\
\mathcal{R}^{\mathbf{m}}_{d}(t) &\stackrel{\text{def}}{=} [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(t') \mid\mid \rho : t \xrightarrow{d}{}^{*}_{\mathbf{m}} t']
\end{aligned}
$$

Moreover, the $\mathcal{T}^{\mathbf{m}}$**-measure** itself is defined for λ-terms as follows:

▶ **Definition 25.** *If M is a typable λ-term, $\mathcal{T}^{\mathbf{m}}(M) \stackrel{\text{def}}{=} \mathcal{T}^{\mathbf{m}}_{\leq D}(M)$ where $D := \mathsf{maxdeg}(M)$.*

When we write $\mathcal{T}^{\mathbf{m}}_{\leq D}(M)$, we implicitly regard M as a $\lambda^{\mathbf{m}}$-term without any memorized terms.

From a higher-level perspective, the $\mathcal{T}^{\mathbf{m}}_{d}(t_0, t)$ measure defined above is the multiset of pairs of the form $(d, \mathcal{R}^{\mathbf{m}}_{d}(t_0))$ for each redex of degree d in t. Similarly, $\mathcal{T}^{\mathbf{m}}_{\leq D}(t)$ is the multiset of pairs of the form $(d, \mathcal{R}^{\mathbf{m}}_{d}(t))$ for each redex of degree $d \leq D$ in t. In particular, $\mathcal{T}^{\mathbf{m}}_{0}(t_0, t)$ and $\mathcal{T}^{\mathbf{m}}_{\leq 0}(t)$ are empty multisets, because there are no redexes of degree 0. Two easy remarks are that $D \leq D'$ implies $\mathcal{T}^{\mathbf{m}}_{\leq D}(t) \preceq \mathcal{T}^{\mathbf{m}}_{\leq D'}(t)$, and that $\mathcal{T}^{\mathbf{m}}_{d}(t_0, t\mathtt{L}) = \mathcal{T}^{\mathbf{m}}_{d}(t_0, t) + \mathcal{T}^{\mathbf{m}}_{d}(t_0, \mathtt{L})$.

▶ Remark 26. As mentioned in the preceding discussion, one important point is that for $\mathcal{R}^{\mathbf{m}}_{d}(-)$ to be well-defined we need to argue that the set $\{\rho \mid \exists t'.\ \rho : t \xrightarrow{d}{}^{*}_{\mathbf{m}} t'\}$ is finite. This is a consequence of Coro. 19.

▶ **Example 27.** Let $\Delta := \lambda x^{0\to 0}.\, x^{0\to 0}(x^{0\to 0} z^0)$ and $W := \lambda y^0.\, w^0$ and consider the diagram:

$$
t_0 = \Delta\, W \xrightarrow{2} t_1 = (W(Wz))\{W\}
$$
$$
t_1 \xrightarrow{1} t_2 = w\{Wz\}\{W\} \xrightarrow{1} t_4 = w\{w\{z\}\}\{W\}
$$
$$
t_1 \xrightarrow{1} t_3 = (W(w\{z\}))\{W\} \xrightarrow{1} t_4
$$

Then $\mathcal{T}^{\mathbf{m}}_{\leq 0}(t_1) = \mathcal{T}^{\mathbf{m}}_{\leq 0}(t_2) = \mathcal{T}^{\mathbf{m}}_{\leq 0}(t_3) = \mathcal{T}^{\mathbf{m}}_{\leq 0}(t_4) = \mathcal{T}^{\mathbf{m}}_{\leq 1}(t_4) = \mathcal{T}^{\mathbf{m}}_{\leq 2}(t_4) = [\,]$, and:

$$
\begin{array}{ll}
\mathcal{T}^{\mathbf{m}}_{\leq 2}(t_0) = [(2, \mathcal{R}^{\mathbf{m}}_{2}(t_0))] & \mathcal{R}^{\mathbf{m}}_{2}(t_0) = [\mathcal{T}^{\mathbf{m}}_{\leq 1}(t_0), \mathcal{T}^{\mathbf{m}}_{\leq 1}(t_1)] \\
\mathcal{T}^{\mathbf{m}}_{\leq 2}(t_1) = \mathcal{T}^{\mathbf{m}}_{\leq 1}(t_1) = [(1, \mathcal{R}^{\mathbf{m}}_{1}(t_1)), (1, \mathcal{R}^{\mathbf{m}}_{1}(t_1))] & \mathcal{R}^{\mathbf{m}}_{1}(t_1) = [\mathcal{T}^{\mathbf{m}}_{\leq 0}(t_1), \mathcal{T}^{\mathbf{m}}_{\leq 0}(t_2), \mathcal{T}^{\mathbf{m}}_{\leq 0}(t_3), \mathcal{T}^{\mathbf{m}}_{\leq 0}(t_4)] \\
\mathcal{T}^{\mathbf{m}}_{\leq 2}(t_2) = \mathcal{T}^{\mathbf{m}}_{\leq 1}(t_2) = [(1, \mathcal{R}^{\mathbf{m}}_{1}(t_2))] & \mathcal{R}^{\mathbf{m}}_{1}(t_2) = [\mathcal{T}^{\mathbf{m}}_{\leq 0}(t_2), \mathcal{T}^{\mathbf{m}}_{\leq 0}(t_4)] \\
\mathcal{T}^{\mathbf{m}}_{\leq 2}(t_3) = \mathcal{T}^{\mathbf{m}}_{\leq 1}(t_3) = [(1, \mathcal{R}^{\mathbf{m}}_{1}(t_3))] & \mathcal{R}^{\mathbf{m}}_{1}(t_3) = [\mathcal{T}^{\mathbf{m}}_{\leq 0}(t_3), \mathcal{T}^{\mathbf{m}}_{\leq 0}(t_4)]
\end{array}
$$

In particular, $\mathcal{T}^{\mathbf{m}}_{\leq 2}(t_0) \succ \mathcal{T}^{\mathbf{m}}_{\leq 2}(t_1) \succ \mathcal{T}^{\mathbf{m}}_{\leq 2}(t_2) \succ \mathcal{T}^{\mathbf{m}}_{\leq 2}(t_4)$ and $\mathcal{T}^{\mathbf{m}}_{\leq 2}(t_1) \succ \mathcal{T}^{\mathbf{m}}_{\leq 2}(t_3) \succ \mathcal{T}^{\mathbf{m}}_{\leq 2}(t_4)$.

The $\mathcal{T}^{\mathbf{m}}$-measure is decreasing Lastly, we show the main theorem of this section, stating that if $M \to_\beta N$ then $\mathcal{T}^{\mathbf{m}}(M) \succ \mathcal{T}^{\mathbf{m}}(N)$. This theorem is based on three technical results, that we call *high/increase*, *low/decrease*, and *forget/decrease*:

1. **High/increase** (Prop. 29) establishes —perhaps confusingly— that $\mathcal{T}^{\mathbf{m}}_{\leq d}(-)$ (non-strictly) **increases** if one contracts a redex of higher degree $D > d$. More precisely, if $0 \leq d < D$ and $t \xrightarrow{D}_{\mathbf{m}} t'$ then $\mathcal{T}^{\mathbf{m}}_{\leq d}(t) \preceq \mathcal{T}^{\mathbf{m}}_{\leq d}(t')$. Note that $\mathcal{T}^{\mathbf{m}}_{\leq d}(t)$ only looks at redexes of degree $i \leq d$, and contracting a redex of degree $D > d$ *cannot erase* a redex of any degree $i \leq d$, because the $\lambda^{\mathbf{m}}$-calculus is non-erasing. Contracting a redex of degree D can, at most, replicate redexes of degree i. This property is needed for a technical reason to prove the low/decrease property, and it relies crucially on the commutation result of the previous section (Prop. 18).
2. **Low/decrease** (Prop. 30) establishes that $\mathcal{T}^{\mathbf{m}}_{\leq D}(-)$ **strictly** decreases if one contracts a redex of lower degree $d < D$. More precisely, if $1 \leq d \leq D$ and $t \xrightarrow{d}_{\mathbf{m}} t'$ then $\mathcal{T}^{\mathbf{m}}_{\leq D}(t) \succ \mathcal{T}^{\mathbf{m}}_{\leq D}(t')$. This is the core of the argument, and the most technically difficult part to prove. It relies crucially on the lifting property of the previous section (Prop. 20).
3. **Forget/decrease** (Prop. 31) establishes that forgetful reduction (non-strictly) decreases the measure. More precisely, if $t \rhd t'$ then $\mathcal{T}^{\mathbf{m}}_{\leq d}(t) \succeq \mathcal{T}^{\mathbf{m}}_{\leq d}(t')$. This property is used as a final step in the main theorem, and it relies crucially on postponement of forgetful reduction, as studied in the previous section (Prop. 21).

Below we sketch the proofs of these three properties. See Prop. 65, Prop. 68, and Prop. 69 in the appendix for detailed proofs. Let us first mention a straightforward lemma.

▶ **Lemma 28** (Measure of a substitution). **1.** *$\mathcal{T}^{\mathbf{m}}_{d}(t_0, t) \preceq \mathcal{T}^{\mathbf{m}}_{d}(t_0, t[x := s])$.* **2.** *If s is not a $\mathbf{m}$-abstraction of degree d, then $\mathcal{T}^{\mathbf{m}}_{d}(t_0, t[x := s]) = \mathcal{T}^{\mathbf{m}}_{d}(t_0, t) + k \otimes \mathcal{T}^{\mathbf{m}}_{d}(t_0, s)$ for some $k \in \mathbb{N}_0$.*

Proof. By induction on t. See Lem. 64 and Lem. 67 in the appendix for details. ◀

▶ **Proposition 29** (High/increase). *Let $D \in \mathbb{N}_0$. Then the following hold:*
1. *If $1 \leq d < D$ and $t \xrightarrow{D}_{\mathbf{m}} t'$ then $\mathcal{R}^{\mathbf{m}}_{d}(t) \preceq \mathcal{R}^{\mathbf{m}}_{d}(t')$.*
2. *If $0 \leq d < D$ and $t_0 \xrightarrow{D}_{\mathbf{m}} t'_0$ then $\mathcal{T}^{\mathbf{m}}_{d}(t_0, t) \preceq \mathcal{T}^{\mathbf{m}}_{d}(t'_0, t)$.*
3. *If $0 \leq d < D$ and $t_0 \xrightarrow{D}_{\mathbf{m}} t'_0$ and $t \xrightarrow{D}_{\mathbf{m}} t'$ then $\mathcal{T}^{\mathbf{m}}_{d}(t_0, t) \preceq \mathcal{T}^{\mathbf{m}}_{d}(t'_0, t')$.*
4. *If $0 \leq d < D$ and $t \xrightarrow{D}_{\mathbf{m}} t'$ then $\mathcal{T}^{\mathbf{m}}_{\leq d}(t) \preceq \mathcal{T}^{\mathbf{m}}_{\leq d}(t')$.*

Proof. The four items are proved simultaneously by induction on d, where item **1** resorts to the IH, and the following items may resort to the previous items without decreasing d. Items **2** and **3** proceed by a nested induction on t. Most cases are straightforward.

One interesting situation occurs in item **3** when $t = (\lambda x.\, s)\mathtt{L}\, u$ is the redex of degree D contracted by the step $t \xrightarrow{D}_{\mathbf{m}} t'$. Then we resort to the first part of Lem. 28.

Another interesting part of the proof is item **1**. Let $1 \leq d < D$ and $t \xrightarrow{D}_{\mathbf{m}} t'$ and let us show that $\mathcal{R}^{\mathbf{m}}_{d}(t) \preceq \mathcal{R}^{\mathbf{m}}_{d}(t')$. Indeed, let $X := \{\rho \mid (\exists s)\ \rho : t \xrightarrow{d}{}^{*}_{\mathbf{m}} s\}$ and $Y := \{\sigma \mid (\exists s')\ \sigma : t' \xrightarrow{d}{}^{*}_{\mathbf{m}} s'\}$, and let $R : t \xrightarrow{D}_{\mathbf{m}} t'$. Using Prop. 18, we can define an injective function $\varphi : X \to Y$ by $\varphi(\rho) := \rho/R$. Note that $\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \preceq \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}})$ holds for every $\rho \in X$ using item **4** of the IH (noting that $1 \leq d - 1 < D$ holds because $1 \leq d < D$), resorting to the IH as many times as the length of the reduction $s \xrightarrow{D}{}^{*}_{\mathbf{m}} s'_{\rho}$. To conclude the proof, let $Z = Y \setminus \varphi(X)$. Then:

$$\mathcal{R}^{\mathbf{m}}_{d}(t) = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in X] \preceq^{(\star)} [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}}) \mid\mid \rho \in X] =^{(\star\star)} [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in \varphi(X)]$$
$$\preceq [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in \varphi(X)] + [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Z] = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y] = \mathcal{R}^{\mathbf{m}}_{d}(t')$$

To justify the step marked with $(\star)$, note that $[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in X] = \sum_{\rho\in X}[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}})] \preceq \sum_{\rho\in X}[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}})] = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}}) \mid\mid \rho \in X]$ because $\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \preceq \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}})$, as we have already claimed. To justify the step marked with $(\star\star)$, note that φ is injective. ◀

▶ **Proposition 30** (Low/decrease). *Let $D \in \mathbb{N}_0$. Then the following hold:*

1. *If $1 \leq d \leq j \leq D$ and $t \xrightarrow{d}_{\mathbf{m}} t'$ then $\mathcal{R}^{\mathbf{m}}_j(t) \succ \mathcal{R}^{\mathbf{m}}_j(t')$.*
2. *If $1 \leq d \leq j \leq D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t'_0$ then $\mathcal{T}^{\mathbf{m}}_j(t_0, t) \mathrel{:\succ:} \mathcal{T}^{\mathbf{m}}_j(t'_0, t)$.*
3. *If $1 \leq d \leq D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t'_0$ and $t \xrightarrow{d}_{\mathbf{m}} t'$, then for all $\mathfrak{m} \in \mathbb{T}_{d-1}$ we have $\mathcal{T}^{\mathbf{m}}_d(t_0, t) \succ \mathcal{T}^{\mathbf{m}}_d(t'_0, t') + \mathfrak{m}$.*
4. *If $1 \leq d < j \leq D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t'_0$ and $t \xrightarrow{d}_{\mathbf{m}} t'$ then $\mathcal{T}^{\mathbf{m}}_j(t_0, t) \succeq \mathcal{T}^{\mathbf{m}}_j(t'_0, t')$.*
5. *If $1 \leq d \leq D$ and $t \xrightarrow{d}_{\mathbf{m}} t'$ then $\mathcal{T}^{\mathbf{m}}_{\leq D}(t) \succ \mathcal{T}^{\mathbf{m}}_{\leq D}(t')$.*

Proof. The five items are proved simultaneously by induction on D, where item **1** resorts to the IH, and the following items may resort to the previous items without decreasing d. Items **2**–**4** proceed by a nested induction on t. We mention some of the interesting parts of the proof.

For item **1**, let $1 \leq d \leq j \leq D$ and $t \xrightarrow{d}_{\mathbf{m}} t'$ and let us show that $\mathcal{R}^{\mathbf{m}}_j(t) \succ \mathcal{R}^{\mathbf{m}}_j(t')$. Let $X := \{\rho \mid (\exists s)\ \rho : t \xrightarrow{j}{}^{*}_{\mathbf{m}} s\}$ and $Y := \{\sigma \mid (\exists s')\ \sigma : t' \xrightarrow{j}{}^{*}_{\mathbf{m}} s'\}$, and consider two subcases:

- If $d = j$, let $R : t \xrightarrow{d}_{\mathbf{m}} t'$, define an injective function $\varphi : Y \to X$ by $\varphi(\sigma) = R\,\sigma$, let $Z = X \setminus \varphi(Y)$, and note that:
$$\begin{aligned}\mathcal{R}^{\mathbf{m}}_j(t) &= [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in \varphi(Y)] + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z]\\ &= [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(R\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y] + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z] \quad \text{since } \varphi \text{ is injective}\\ &= [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y] + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z] = \mathcal{R}^{\mathbf{m}}_j(t') + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z]\end{aligned}$$
To conclude that $\mathcal{R}^{\mathbf{m}}_j(t) \succ \mathcal{R}^{\mathbf{m}}_j(t')$, note that Z is non-empty because it contains the empty reduction $\epsilon : t \xrightarrow{d}{}^{*}_{\mathbf{m}} t$.
- If $d < j$, we construct a function $\varphi : Y \to X$ as follows. By Prop. 20, for each reduction $\sigma : t' \xrightarrow{j}{}^{*}_{\mathbf{m}} s'$ there exist s_σ, u_σ, and reductions $\varphi(\sigma) : t \xrightarrow{j}{}^{*}_{\mathbf{m}} s_\sigma$ and $s' \xrightarrow{j}{}^{*}_{\mathbf{m}} u_\sigma$ and $s_\sigma \xrightarrow{d}{}^{+}_{\mathbf{m}} u_\sigma$. Note that for every $\sigma \in Y$ we have $\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\varphi(\sigma)^{\mathsf{tgt}}) = \mathcal{T}^{\mathbf{m}}_{\leq j-1}(s_\sigma) \succ^{\dagger} \mathcal{T}^{\mathbf{m}}_{\leq j-1}(u_\sigma) \succeq^{\ddagger} \mathcal{T}^{\mathbf{m}}_{\leq j-1}(s') = \mathcal{T}^{\mathbf{m}}_{\leq j-1}(\sigma^{\mathsf{tgt}})$ where † holds by item 5 of the IH observing that $1 \leq d \leq j-1 < D$ because $d < j \leq D$, and ‡ holds by high/increase (Prop. 29) observing that $0 \leq j-1 < j$. To conclude the proof, let $Z = X \setminus \varphi(Y)$, and note that:
$$\begin{aligned}\mathcal{R}^{\mathbf{m}}_j(t) &= [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in \varphi(Y)] + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z]\\ &= [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\varphi(\sigma)^{\mathsf{tgt}}) \mid\mid \sigma \in Y] + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z]\\ &\succeq [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\varphi(\sigma)^{\mathsf{tgt}}) \mid\mid \sigma \in Y] \succ^{(\star)} [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y] = \mathcal{R}^{\mathbf{m}}_j(t')\end{aligned}$$
For the step marked with $(\star)$, note that $[\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\varphi(\sigma)^{\mathsf{tgt}}) \mid\mid \sigma \in Y] \mathrel{:\succ:} [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y]$ because $\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\varphi(\sigma)^{\mathsf{tgt}}) \succ \mathcal{T}^{\mathbf{m}}_{\leq j-1}(\sigma^{\mathsf{tgt}})$ holds by the claim above where, moreover, Y is non-empty because it contains the empty reduction $\epsilon : t' \xrightarrow{j}{}^{*}_{\mathbf{m}} t'$.

Another interesting situation occurs in item **3**, when $t = (\lambda x.\,s)\mathtt{L}\,u$ is the redex of degree d contracted by the step $t \xrightarrow{d}_{\mathbf{m}} t'$. The step is of the form $t = (\lambda x.\,s)\mathtt{L}\,u \xrightarrow{d}_{\mathbf{m}} s[x := u]\{u\}\mathtt{L} = t'$. Note that u is not an abstraction of degree d, because it is the argument of an abstraction of degree d. So by Lem. 28 there exists $k \in \mathbb{N}_0$ such that $\mathcal{T}^{\mathbf{m}}_d(t'_0, s[x := u]) = \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + k \otimes \mathcal{T}^{\mathbf{m}}_d(t'_0, u)$. The crucial observation is that $\mathcal{T}^{\mathbf{m}}_d(t_0, u) \succeq (1 + k) \otimes \mathcal{T}^{\mathbf{m}}_d(t'_0, u)$, which is because by item 2 we have that $\mathcal{T}^{\mathbf{m}}_d(t_0, u) \mathrel{:\succ:} \mathcal{T}^{\mathbf{m}}_d(t'_0, u)$.

Finally, for item **5**, let $1 \leq d \leq D$ and $t \xrightarrow{d}_{\mathbf{m}} t'$ and let us show that $\mathcal{T}^{\mathbf{m}}_{\leq D}(t) \succ \mathcal{T}^{\mathbf{m}}_{\leq D}(t')$. Indeed:

$$\begin{aligned}\mathcal{T}^{\mathbf{m}}_{\leq D}(t) &= \textstyle\sum_{i=1}^{D} \mathcal{T}^{\mathbf{m}}_{i}(t,t) \succeq \mathcal{T}^{\mathbf{m}}_{d}(t,t) + \sum_{j=d+1}^{D} \mathcal{T}^{\mathbf{m}}_{j}(t,t)\\ &\succ \mathcal{T}^{\mathbf{m}}_{\leq d-1}(t') + \mathcal{T}^{\mathbf{m}}_{d}(t',t') + \textstyle\sum_{j=d+1}^{D} \mathcal{T}^{\mathbf{m}}_{j}(t,t) \text{ by item } \mathbf{3}, \text{ taking } \mathfrak{m} := \mathcal{T}^{\mathbf{m}}_{\leq d-1}(t')\\ &\succeq \mathcal{T}^{\mathbf{m}}_{\leq d-1}(t') + \mathcal{T}^{\mathbf{m}}_{d}(t',t') + \textstyle\sum_{j=d+1}^{D} \mathcal{T}^{\mathbf{m}}_{j}(t',t') = \mathcal{T}^{\mathbf{m}}_{\leq D}(t') \text{ by item } \mathbf{4}.\end{aligned}$$ ◀

▶ **Proposition 31** (Forget/decrease). *Let $d \in \mathbb{N}_0$. Then the following hold:*

1. *If $t \rhd t'$ then $\mathcal{R}^{\mathbf{m}}_{d}(t) \succeq \mathcal{R}^{\mathbf{m}}_{d}(t')$.*
2. *If $t_0 \rhd t'_0$ then $\mathcal{T}^{\mathbf{m}}_{d}(t_0,t) \succeq \mathcal{T}^{\mathbf{m}}_{d}(t'_0,t)$.*
3. *If $t_0 \rhd t'_0$ and $t \rhd t'$ then $\mathcal{T}^{\mathbf{m}}_{d}(t_0,t) \succeq \mathcal{T}^{\mathbf{m}}_{d}(t'_0,t')$.*
4. *If $t \rhd t'$ then $\mathcal{T}^{\mathbf{m}}_{\leq d}(t) \succeq \mathcal{T}^{\mathbf{m}}_{\leq d}(t')$.*

Proof. The four items are proved simultaneously by induction on D, where item **1** resorts to the IH, and the following items may resort to the previous items without decreasing d. Items **2** and **3** proceed by a nested induction on t.

The interesting part is item **1**, so let $t \rhd t'$ and let us show that $\mathcal{R}^{\mathbf{m}}_{d}(t) \succeq \mathcal{R}^{\mathbf{m}}_{d}(t')$. Let $X := \{\rho \mid (\exists s)\ \rho : t \xrightarrow{d}{}^{*}_{\mathbf{m}} s\}$ and $Y := \{\sigma \mid (\exists s')\ \sigma : t' \xrightarrow{d}{}^{*}_{\mathbf{m}} s'\}$. Define an injective function $\varphi : Y \to X$ by $\varphi(\sigma) := \sigma \frown R$, resorting to Prop. 21, where $\sigma \frown R : t \xrightarrow{d}{}^{*}_{\mathbf{m}} s_\sigma$. and $s_\sigma \rhd^{*} s'$. Note that for every $\sigma \in Y$ we have $\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}}) = \mathcal{T}^{\mathbf{m}}_{\leq d-1}(s_\sigma) \succeq^{\dagger} \mathcal{T}^{\mathbf{m}}_{\leq d-1}(s') = \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}})$, where † holds by item **4** of the IH, observing that $d-1 < d$. To conclude the proof, let $Z = X \setminus \varphi(Y)$, and note that:

$$\begin{aligned}\mathcal{R}^{\mathbf{m}}_{d}(t) &= [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in \varphi(Y)] + [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z] \succeq [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in \varphi(Y)]\\ &=^{(\star)} [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}}) \mid\mid \sigma \in Y] \succeq^{(\star\star)} [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y] = \mathcal{R}^{\mathbf{m}}_{d}(t')\end{aligned}$$

For the step marked with $(\star)$, note that φ is injective. For the step marked with $(\star\star)$, note that $[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}}) \mid\mid \sigma \in Y] = \sum_{\sigma\in Y}[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}})] \succeq \sum_{\sigma\in Y}[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}})] = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y]$ because $\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}}) \succeq \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}})$, as we have already justified. ◀

Finally, we prove the main theorem in this section:

▶ **Theorem 32.** *Let M, N be typable λ-terms such that $M \to_\beta N$. Then $\mathcal{T}^{\mathbf{m}}(M) > \mathcal{T}^{\mathbf{m}}(N)$.*

Proof. Let $D = \mathsf{maxdeg}(M)$ and $D' = \mathsf{maxdeg}(N)$. Let $M \to_{\mathbf{m}} s$ be the step corresponding to $M \to_\beta N$. By Lem. 11 note that $s \rhd N$. Then:

$$\mathcal{T}^{\mathbf{m}}(M) = \mathcal{T}^{\mathbf{m}}_{\leq D}(M) \succ^{\text{Prop. 30}} \mathcal{T}^{\mathbf{m}}_{\leq D}(s) \succeq^{\text{Prop. 31}} \mathcal{T}^{\mathbf{m}}_{\leq D}(N) \succeq \mathcal{T}^{\mathbf{m}}_{\leq D'}(N) = \mathcal{T}^{\mathbf{m}}(N)$$

The last inequality holds because $D \geq D'$ since, as is well-known, contraction of a β-redex in the simply typed λ-calculus cannot create a redex of higher degree. ◀

6 Conclusion

We have defined two decreasing measures for the STLC, the $\mathcal{W}$-measure (Def. 12) and the $\mathcal{T}^{\mathbf{m}}$-measure (Def. 25). These measures are decreasing (Thm. 15 and Thm. 32 respectively) and, to the best of our knowledge, they provide two new proofs of strong normalization for the STLC. Both measures are defined constructively and by purely syntactic methods, using the $\lambda^{\mathbf{m}}$-calculus as an auxiliary tool.

The problem of finding a "straightforward" decreasing measure for β-reduction in the simply typed λ-calculus is posed as Problem #26 in the TLCA list of open problems [5], and as Problem #19 in the RTA list of open problems [11].

One strength of the $\mathcal{W}$-measure is that its codomain is simple: each term is mapped to a natural number. One weakness is that the definition of the $\mathcal{W}$-measure relies on reduction in the $\lambda^{\mathbf{m}}$-calculus, and computing the $\mathcal{W}$-measure is at least as costly as evaluating the

λ-term itself. Measures based on Gandy's [16, 10] have similar characteristics. One question is whether the values of the $\mathcal{W}$-measure and measures based on Gandy's can be related. It is not immediate to establish a precise correspondence.

On the other hand, one strength of the $\mathcal{T}^{\mathbf{m}}$-measure is that it shows how to extend Turing's measure $\mathcal{T}(-)$ so that it decreases when contracting *any* redex. The proof is based on a delicate analysis of how contracting a redex of degree d may create and copy redexes of degree d', depending on whether $d < d'$, or $d = d'$, or $d > d'$. We hope that this may provide novel insights on why the STLC is SN. The codomain of the $\mathcal{T}^{\mathbf{m}}$-measure is not so simple, as the $\mathcal{T}^{\mathbf{m}}$-measure maps each term to a structure of nested multisets. Yet, it is "reasonably simple": the fact that the partial orders $\mathbb{T}_d$ and $\mathbb{R}_d$ are well-founded only relies on the ordinary multiset and lexicographic orderings. The $\mathcal{T}^{\mathbf{m}}$-measure is costly to compute; in particular $\mathcal{R}^{\mathbf{m}}_d(t)$ is defined as a sum over all reductions $\rho : t \xrightarrow{d}^{*}_{\mathbf{m}} t'$, which may produce a combinatorial explosion. Another weakness is that our proofs make use of relatively heavy rewriting machinery, as we have to keep explicit track of witnesses (*e.g.* in Section 4).

Besides the techniques mentioned in the introduction, other proofs of SN of the STLC can be found in the literature. For example, David [7] gives a purely syntactic proof of SN relying on the standardization theorem; Loader [23], as well as Joachimski and Matthes [18], give combinatorial proofs of SN based on inductive predicates characterizing strongly normalizing terms. As far as we know, the only proofs that explicitly construct decreasing measures are those based on Gandy's.

The idea of keeping "leftover garbage" can be traced back to at least the works of Nederpelt [21] and Klop [20], who studied non-erasing variants of (possibly) erasing rewriting systems, in order to relate weak and strong normalization. Many variations of these ideas have been explored in the past, such as in de Groote's notion of β_S reduction [8] or Neergaard and Sørensen calculus with memory [25]. Instead of using the $\lambda^{\mathbf{m}}$-calculus, it is possible that other non-erasing systems may be used. For instance, Gandy [16] translates λ-terms to the terms of λI-calculus to avoid erasing arguments.

The definition of reduction in the $\lambda^{\mathbf{m}}$-calculus, which allows arbitrary memory in between the abstraction and the application, is inspired by Accattoli and Kesner's work on calculi with explicit substitutions "at a distance" [1]. This mechanism can be traced back, again, to at least the work of Nederpelt [21].

The definition of the $\lambda^{\mathbf{m}}$-calculus as a means to obtain an *increasing* measure was inspired by the fact that, in explicit substitution calculi without erasure, labeled reduction (in the sense of Lévy labels [22]) increases the sum of the sizes of all the labels in the term [2].

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A Technical appendix

A.1 Proofs of Section 2 — The $\lambda^{\mathbf{m}}$-calculus

In this section we give detailed proofs of the results about the $\lambda^{\mathbf{m}}$-calculus stated in Section 2.

▶ Remark 33. $t\{s\}$ is a $\mathbf{m}$-abstraction if and only if t is a $\mathbf{m}$-abstraction.

▶ **Lemma 34** (Substitution lemma). *Let* $\Gamma, x : A \vdash t : B$ *and* $\Gamma \vdash s : A$. *Then* $\Gamma \vdash t[x := s] : B$.

Proof. Straightforward by induction on t. ◀

▶ **Proposition 35** (Subject reduction). *Let* $\Gamma \vdash t : A$ *and* $t \to_{\mathbf{m}} s$. *Then* $\Gamma \vdash s : A$.

Proof. Straightforward by induction on the derivation of the step $t \to_{\mathbf{m}} s$, resorting to Lem. 34 for the base case, when there is a $\mathbf{m}$-reduction step at the root. ◀

A.1.1 Confluence of the $\lambda^{\mathbf{m}}$-calculus

▶ **Proposition 36** (Confluence). *The* $\lambda^{\mathbf{m}}$*-calculus is confluent. That is, if* $t_1 \to^*_{\mathbf{m}} t_2$ *and* $t_1 \to^*_{\mathbf{m}} t_3$*, there exists a term* t_4 *such that* $t_2 \to^*_{\mathbf{m}} t_4$ *and* $t_3 \to^*_{\mathbf{m}} t_4$.

Proof. The proof can be done following standard techniques. For example, following Tait and Martin-Löf's technique, we may define a notion of simultaneous reduction $\Rightarrow_{\mathbf{m}}$ that allows to contract many redexes simultaneously, *i.e.* allowing the complete development of any set of redexes on the starting term. Then it suffices to show that $\to_{\mathbf{m}} \subseteq \Rightarrow_{\mathbf{m}} \subseteq \to^*_{\mathbf{m}}$ and that $\Rightarrow_{\mathbf{m}}$ enjoys the diamond property, *i.e.* that if $t_1 \Rightarrow_{\mathbf{m}} t_2$ and $t_1 \Rightarrow_{\mathbf{m}} t_3$ there exists a term t_4 such that $t_2 \Rightarrow_{\mathbf{m}} t_4$ and $t_3 \Rightarrow_{\mathbf{m}} t_4$. The key lemma is:

$$t \Rightarrow_{\mathbf{m}} t' \text{ and } s \Rightarrow_{\mathbf{m}} s' \text{ implies } t[x := s] \Rightarrow_{\mathbf{m}} t'[x := s']$$

The key diagrams in the proof that $\Rightarrow_{\mathbf{m}}$ enjoys the diamond property are:

$$\begin{array}{ccc} (\lambda x.\, t_1)\mathtt{L}_1\, s_2 & \Longrightarrow & (\lambda x.\, t_2)\mathtt{L}_2\, s_2 \\ \Downarrow & & \Downarrow \\ t_3[x := s_3]\{s_3\}\mathtt{L}_3 & \Longrightarrow & t_4[x := s_4]\{s_4\}\mathtt{L}_4 \end{array} \qquad \begin{array}{ccc} (\lambda x.\, t_1)\mathtt{L}_1\, s_2 & \Longrightarrow & t_2[x := s_2]\{s_2\}\mathtt{L}_2 \\ \Downarrow & & \Downarrow \\ t_3[x := s_3]\{s_3\}\mathtt{L}_3 & \Longrightarrow & t_4[x := s_4]\{s_4\}\mathtt{L}_4 \end{array}$$

◀

A.1.2 Simplification of a $\lambda^{\mathbf{m}}$-term

▶ **Definition 37** (Generalization of notions to memories). *We generalize some of the notions to memories as follows:*

1. *The reduction relation* $\to_{\mathbf{m}}$ *is extended to operate on memories with the two following inductively defined rules:*
   1.1 *If* $t \to_{\mathbf{m}} t'$ *then* $\mathtt{L}\{t\} \to_{\mathbf{m}} \mathtt{L}\{t'\}$.
   1.2 *If* $\mathtt{L} \to_{\mathbf{m}} \mathtt{L}'$ *then* $\mathtt{L}\{t\} \to_{\mathbf{m}} \mathtt{L}'\{t\}$.
2. *The max-degree is extended to memories as follows:* $\mathsf{maxdeg}(\Box) = 0$ *and* $\mathsf{maxdeg}(\mathtt{L}\{t\}) = \max(\mathsf{maxdeg}(\mathtt{L}), \mathsf{maxdeg}(t))$.

▶ **Lemma 38** (Terms reduce to its simplification). *For every term* t *and for all* $k \geq 1$ *we have that* $t \to^*_{\mathbf{m}} \mathtt{S}_k(t)$.

Proof. To prove it by induction, we generalize the statement to memories, *i.e.* $\mathtt{L} \to_{\mathbf{m}}^* \mathtt{S}_k(\mathtt{L})$. We proceed by simultaneous induction on t and $\mathtt{L}$:

1. $t = x$: Immediate, as $x \to_{\mathbf{m}}^* x = \mathtt{S}_k(x)$ in zero steps.
2. $t = \lambda x.\, s$: Then $\lambda s. \to_{\mathbf{m}}^* \lambda x.\, \mathtt{S}_k(s) = \mathtt{S}_k(\lambda x.\, s)$ by IH.
3. $t = (\lambda x.\, s)\mathtt{L}\, u$ where $(\lambda x.\, s)\mathtt{L}$ is a $\mathbf{m}$-abstraction of degree k: By IH $(\lambda x.\, s)\mathtt{L}\, u \to_{\mathbf{m}}^* (\lambda x.\, \mathtt{S}_k(s))\mathtt{S}_k(\mathtt{L})\, \mathtt{S}_k(u) \to_{\mathbf{m}} \mathtt{S}_k(s)[x := \mathtt{S}_k(u)]\{\mathtt{S}_k(u)\}\mathtt{S}_k(\mathtt{L}) = \mathtt{S}_k((\lambda x.\, s)\mathtt{L}\, u) = \mathtt{S}_k(t)$.
4. $t = s\, u$ where s is not a $\mathbf{m}$-abstraction of degree k: By IH $s\, u \to_{\mathbf{m}}^* \mathtt{S}_k(s)\, \mathtt{S}_k(u) = \mathtt{S}_k(s\, u) = \mathtt{S}_k(t)$.
5. $t = s\{u\}$: By IH $s\{u\} \to_{\mathbf{m}}^* \mathtt{S}_k(s)\{\mathtt{S}_k(u)\} = \mathtt{S}_k(s\{u\}) = \mathtt{S}_k(t)$.
6. $\mathtt{L} = \Box$: Immediate, as $\Box \to_{\mathbf{m}}^* \Box = \mathtt{S}_k(\Box)$ in zero steps.
7. $\mathtt{L} = \mathtt{L}'\{t\}$: By IH $\mathtt{L}'\{t\} \to_{\mathbf{m}}^* \mathtt{S}_k(\mathtt{L}')\{\mathtt{S}_k(t)\} = \mathtt{S}_k(\mathtt{L}'\{t\})$.

◀

▶ **Lemma 39** (Substitution of terms of lower type does not create abstractions)**.** *If* $\mathsf{h}(\mathsf{type}(t)) > \mathsf{h}(\mathsf{type}(s))$ *and t not a $\mathbf{m}$-abstraction, then $t[x := s]$ is not a $\mathbf{m}$-abstraction.*

Proof. By induction on t:

1. $t = y$: We claim that $y \neq x$. Indeed, note that the type of y is $\mathsf{type}(t)$ but the type of x is $\mathsf{type}(s)$. By contradiction, suppose that $x = y$. Then $\mathsf{type}(t) = \mathsf{type}(s)$ and in particular $\mathsf{h}(\mathsf{type}(t)) > \mathsf{h}(\mathsf{type}(s)) = \mathsf{h}(\mathsf{type}(t))$, which is impossible. Then we have that $y \neq x$, so $t[x := s] = y[x := s] = y$, which is not a $\mathbf{m}$-abstraction.
2. $t = \lambda x.\, t'$: Impossible, since t is not a $\mathbf{m}$-abstraction by hypothesis.
3. $t = t_1\, t_2$: Then $t[x := s] = t_1[x := s]\, t_2[x := s]$ is trivially not an abstraction.
4. $t = t_1\{t_2\}$: Note that t_1 is not a $\mathbf{m}$-abstraction by Rem. 33. Furthermore, note that $\mathsf{type}(t_1) = \mathsf{type}(t)$ so in particular $\mathsf{h}(\mathsf{type}(t_1)) = \mathsf{h}(\mathsf{type}(t)) > \mathsf{h}(\mathsf{type}(s))$. We are under the conditions to apply the IH on t_1, hence $t_1[x := s]$ is not a $\mathbf{m}$-abstraction. To conclude, note that $t[x := s] = t_1[x := s]\{t_2[x := s]\}$ cannot be a $\mathbf{m}$-abstraction by Rem. 33.

◀

▶ **Lemma 40** (Simplification does not create abstractions)**.** *If* $\mathsf{h}(\mathsf{type}(t)) \geq k$ *and* $\mathsf{maxdeg}(t) \leq k$ *and t is not a $\mathbf{m}$-abstraction, then $\mathtt{S}_k(t)$ is not a $\mathbf{m}$-abstraction.*

Proof. By induction on t:

1. $t = x$: Then $\mathtt{S}_k(x) = x$ is not a $\mathbf{m}$-abstraction.
2. $t = \lambda x.\, s$: Impossible, since t is not a $\mathbf{m}$-abstraction by hypothesis.
3. $t = (\lambda x.\, s)\mathtt{L}\, u$ where $(\lambda x.\, s)\mathtt{L}$ is a $\mathbf{m}$-abstraction of degree k: We claim that this case is impossible. Writing the types explicitly, we have that $\mathsf{type}(t) = B$ with $\mathsf{h}(B) = \mathsf{h}(\mathsf{type}(t)) \geq k$ by hypothesis. Then the type of the function must be of the form $\mathsf{type}(\lambda x.\, s) = A \to B$. But note that $\mathsf{h}(\mathsf{type}(\lambda x.\, s)) = \mathsf{h}(A \to B) > \mathsf{h}(B) \geq k$. This means that $(\lambda x.\, s)\mathtt{L}$ cannot be of degree k, contradicting the hypothesis of this case.
4. $t = s\, u$ where s is not a $\mathbf{m}$-abstraction of degree k: Then $\mathtt{S}_k(t) = s\, u = \mathtt{S}_k(s)\, \mathtt{S}_k(u)$ is not a $\mathbf{m}$-abstraction.
5. $t = s\{u\}$: Note that $\mathsf{type}(s) = \mathsf{type}(t)$, so in particular $\mathsf{h}(\mathsf{type}(s)) = \mathsf{h}(\mathsf{type}(t)) \geq k$. Moreover, $\mathsf{maxdeg}(s) \leq \mathsf{maxdeg}(t) \leq k$ and s is not a $\mathbf{m}$-abstraction by Rem. 33. We are under the conditions to apply the IH on s, hence $\mathtt{S}_k(s)$ is not a $\mathbf{m}$-abstraction. To conclude, note that $\mathtt{S}_k(t) = \mathtt{S}_k(s)\{\mathtt{S}_k(u)\}$ cannot be a $\mathbf{m}$-abstraction by Rem. 33.

◀

▶ **Lemma 41** (Properties of the max-degree)**.**

1. $\mathsf{maxdeg}(t\mathtt{L}) = \max(\mathsf{maxdeg}(t), \mathsf{maxdeg}(\mathtt{L}))$

2. *If* $\mathsf{maxdeg}(t) < k$ *and* $\mathsf{maxdeg}(s) < k$ *and* $\mathsf{h}(\mathsf{type}(s)) < k$ *then* $\mathsf{maxdeg}(t[x := s]) < k$.

Proof. Item 1 is straightforward by induction on $\mathtt{L}$, since $\mathsf{maxdeg}(t\{s\}) = \max(\mathsf{maxdeg}(t), \mathsf{maxdeg}(s))$. For item 2, we proceed by induction on t:

1. $t = y$: We consider two subcases, depending on whether $y = x$ or not. If $y = x$, then $\mathsf{maxdeg}(x[x := s]) = \mathsf{maxdeg}(s) < k$. If $y \neq x$, then $\mathsf{maxdeg}(y[x := s]) = \mathsf{maxdeg}(y) = \mathsf{maxdeg}(t) < k$.
2. $t = \lambda y.\, t'$: By α-conversion we assume that $x \neq y$. Note that $\mathsf{maxdeg}(t') = \mathsf{maxdeg}(\lambda y.\, t') = \mathsf{maxdeg}(t) < k$ by hypothesis. Then $\mathsf{maxdeg}((\lambda y.\, t')[x := s]) = \mathsf{maxdeg}(\lambda y.\, t'[x := s]) = \mathsf{maxdeg}(t'[x := s]) < k$ by IH.
3. $t = t_1\, t_2$: Note that $\mathsf{maxdeg}(t_1) \leq \mathsf{maxdeg}(t_1\, t_2) = \mathsf{maxdeg}(t) < k$ and, similarly, $\mathsf{maxdeg}(t_2) < k$. This means that we can apply the IH to obtain that $\mathsf{maxdeg}(t_1[x := s]) < k$ and $\mathsf{maxdeg}(t_2[x := s]) < k$. We proceed by case analysis, depending on whether $\mathsf{h}(\mathsf{type}(t_1)) < k$ or $\mathsf{h}(\mathsf{type}(t_1)) \geq k$:
   - **3.1** If $\mathsf{h}(\mathsf{type}(t_1)) < k$ then, by the substitution lemma (Lem. 34), the terms $t_1[x := s]$ and t_1 have the same type. In particular, $t_1[x := s]\, t_2[x := s]$ cannot be a redex of degree k or greater, since $\mathsf{h}(\mathsf{type}(t_1[x := s])) < k$. As a consequence, if $t_1[x := s]\, t_2[x := s]$ is a redex, its degree is at most $k - 1$. Hence $\mathsf{maxdeg}((t_1\, t_2)[x := s]) = \mathsf{maxdeg}(t_1[x := s]\, t_2[x := s]) \leq \max(k - 1, \mathsf{maxdeg}(t_1[x := s]), \mathsf{maxdeg}(t_2[x := s])) < k$.
   - **3.2** If $\mathsf{h}(\mathsf{type}(t_1)) \geq k$, note that t_1 cannot be a $\mathbf{m}$-abstraction, because then $t = t_1\, t_2$ would be a redex of degree k or greater, but by hypothesis we know that $\mathsf{maxdeg}(t) < k$. Note that we are under the conditions of Lem. 39, so we know that $t_1[x := s]$ is not a $\mathbf{m}$-abstraction. In particular, $t_1[x := s]\, t_2[x := s]$ cannot be a redex. Hence $\mathsf{maxdeg}((t_1\, t_2)[x := s]) = \mathsf{maxdeg}(t_1[x := s]\, t_2[x := s]) = \max(\mathsf{maxdeg}(t_1[x := s]), \mathsf{maxdeg}(t_2[x := s])) < k$.
4. $t = t_1\{t_2\}$: Note that $\mathsf{maxdeg}(t_1) \leq \mathsf{maxdeg}(t_1\{t_2\}) = \mathsf{maxdeg}(t) < k$ and, similarly, $\mathsf{maxdeg}(t_2) < k$. This means that we can apply the IH to obtain that $\mathsf{maxdeg}(t_1[x := s]) < k$ and $\mathsf{maxdeg}(t_2[x := s]) < k$. Hence $\mathsf{maxdeg}((t_1\{t_2\})[x := s]) = \mathsf{maxdeg}(t_1[x := s]\{t_2[x := s]\}) = \max(\mathsf{maxdeg}(t_1[x := s]), \mathsf{maxdeg}(t_2[x := s])) < k$.

◀

▶ **Lemma 42** (Simplification decreases the max-degree)**.** *Suppose that* $k \geq 1$. *If* $\mathsf{maxdeg}(t) \leq k$ *then* $\mathsf{maxdeg}(\mathtt{S}_k(t)) < k$.

Proof. Let $k \geq 1$ be such that $\mathsf{maxdeg}(t) \leq k$. We argue that $\mathsf{maxdeg}(\mathtt{S}_k(t)) < k$, that is, all the redexes in t have degree less than k. To prove it by induction, we generalize the statement to memories, proving also that $\mathsf{maxdeg}(\mathtt{S}_k(\mathtt{L})) < k$. We prove the statement simultaneously by induction on t and $\mathtt{L}$:

1. $t = x$: Then $\mathtt{S}_k(x) = x$ has no redexes, so $\mathsf{maxdeg}(x) = 0 < 1 \leq k$.
2. $t = \lambda x.\, s$: Note that $\mathsf{maxdeg}(s) = \mathsf{maxdeg}(\lambda x.\, s) \leq k$ so by IH $\mathsf{maxdeg}(\mathtt{S}_k(s)) < k$. Moreover, $\mathsf{maxdeg}(\mathtt{S}_k(\lambda x.\, s)) = \mathsf{maxdeg}(\lambda x.\, \mathtt{S}_k(s)) = \mathsf{maxdeg}(\mathtt{S}_k(s)) < k$.
3. $t = (\lambda x.\, s)\mathtt{L}\, u$ where $(\lambda x.\, s)\mathtt{L}$ is a $\mathbf{m}$-abstraction of degree k: Note that $\mathsf{maxdeg}(s) \leq \mathsf{maxdeg}(t)$ because any redex in the subterm s is also a redex in the whole term t, so in particular $\mathsf{maxdeg}(s) \leq k$ and we may apply the IH on s to conclude that $\mathsf{maxdeg}(\mathtt{S}_k(s)) < k$. Similarly, by IH, we have that $\mathsf{maxdeg}(\mathtt{S}_k(\mathtt{L})) < k$ and $\mathsf{maxdeg}(\mathtt{S}_k(u)) < k$.
   Since t is typable, its type is of the form $\mathsf{type}(t) = B$ with $\mathsf{type}((\lambda x.\, s)\mathtt{L}) = A \to B$ and $\mathsf{type}(u) = A$. Note that $\mathsf{h}(\mathsf{type}(u)) = \mathsf{h}(A) < \mathsf{h}(A \to B) = k$ since $(\lambda x.\, s)\mathtt{L}$ is of degree k by hypothesis of this case.

To conclude this case, note that:

$$\begin{array}{ll} & \mathsf{maxdeg}(\mathtt{S}_k(t)) \\ = & \mathsf{maxdeg}(\mathtt{S}_k(s)[x := \mathtt{S}_k(u)]\{\mathtt{S}_k(u)\}\mathtt{S}_k(\mathtt{L})) \\ & \quad \text{by definition} \\ \leq & \max(\mathsf{maxdeg}(\mathtt{S}_k(s)[x := \mathtt{S}_k(u)]), \mathsf{maxdeg}(\mathtt{S}_k(u)), \mathsf{maxdeg}(\mathtt{S}_k(\mathtt{L}))) \\ & \quad \text{by Lem. 41 (1)} \\ < & \max(k, k, k) \\ & \quad \text{by Lem. 41 (2) and the IH} \\ = & k \end{array}$$

For the last inequality, we use the fact that $\mathsf{h}(\mathsf{type}(u)) < k$.

4. $t = s\,u$ where s is not a $\mathbf{m}$-abstraction of degree k: Note that $\mathsf{maxdeg}(s) \leq \mathsf{maxdeg}(t)$ because any redex in the subterm s is also a redex in the whole term t, so in particular $\mathsf{maxdeg}(s) \leq k$ and we may apply the IH on s to conclude that $\mathsf{maxdeg}(\mathtt{S}_k(s)) < k$. Similarly, by IH, we have that $\mathsf{maxdeg}(\mathtt{S}_k(u)) < k$.
We proceed by case analysis, depending on whether $\mathsf{h}(\mathsf{type}(s)) < k$ or $\mathsf{h}(\mathsf{type}(s)) \geq k$:
  4.1 If $\mathsf{h}(\mathsf{type}(s)) < k$, then by Lem. 38 we know that $s \to^*_{\mathbf{m}} \mathtt{S}_k(s)$ and by subject reduction (Prop. 3) we have that $\mathsf{type}(s) = \mathsf{type}(\mathtt{S}_k(s))$. In particular, $\mathtt{S}_k(s)\,\mathtt{S}_k(u)$ cannot be a redex of degree k or greater, because $\mathsf{h}(\mathsf{type}(\mathtt{S}_k(s))) = \mathsf{h}(\mathsf{type}(s)) < k$. That is, if $\mathtt{S}_k(s)\,\mathtt{S}_k(u)$ is a redex, its degree is at most $k-1$. Hence we have that $\mathsf{maxdeg}(\mathtt{S}_k(t)) = \mathsf{maxdeg}(\mathtt{S}_k(s)\,\mathtt{S}_k(u)) \leq \max(k-1, \mathsf{maxdeg}(\mathtt{S}_k(s)), \mathsf{maxdeg}(\mathtt{S}_k(u))) < k$.
  4.2 If $\mathsf{h}(\mathsf{type}(s)) \geq k$, note that s cannot be a $\mathbf{m}$-abstraction. Indeed, we know by hypothesis of this case that s is not an abstraction of degree k. Furthermore, s cannot be an abstraction of degree $k' > k$, because then $t = s\,u$ would be a redex of degree $k' > k$, but then we would have that $k < k' \leq \mathsf{maxdeg}(t) \leq k$, which is a contradiction. Since s is not a $\mathbf{m}$-abstraction, $\mathsf{maxdeg}(s) \leq k$, and $\mathsf{h}(\mathsf{type}(s)) \geq k$, we are under the conditions to apply Lem. 40 to conclude that $\mathtt{S}_k(s)$ is not a $\mathbf{m}$-abstraction. This means that $\mathtt{S}_k(s)\,\mathtt{S}_k(u)$ cannot be a redex. Hence we have that $\mathsf{maxdeg}(\mathtt{S}_k(t)) = \mathsf{maxdeg}(\mathtt{S}_k(s)\,\mathtt{S}_k(u)) = \max(\mathsf{maxdeg}(\mathtt{S}_k(s)), \mathsf{maxdeg}(\mathtt{S}_k(u))) < k$.

5. $t = s\{u\}$: Note that $\mathsf{maxdeg}(s) \leq \mathsf{maxdeg}(t)$, so in particular $\mathsf{maxdeg}(s) \leq k$ and we may apply the IH on s to conclude that $\mathsf{maxdeg}(\mathtt{S}_k(s)) < k$. Similarly, by IH, we have that $\mathsf{maxdeg}(\mathtt{S}_k(u)) < k$. Hence we have that $\mathsf{maxdeg}(\mathtt{S}_k(t)) = \mathsf{maxdeg}(\mathtt{S}_k(s)\{\mathtt{S}_k(u)\}) = \max(\mathsf{maxdeg}(\mathtt{S}_k(s)), \mathsf{maxdeg}(\mathtt{S}_k(u))) < k$.

6. $\mathtt{L} = \Box$: Immediate, as $\mathsf{maxdeg}(\Box) = 0 < 1 \leq k$.

7. $\mathtt{L} = \mathtt{L}'\{t\}$: Similar to case 5 of this lemma.

◀

▶ **Proposition 43** (Simplification is normalization). *$t \to^*_{\mathbf{m}} \mathtt{S}_*(t)$ and $\mathtt{S}_*(t)$ is a $\to_{\mathbf{m}}$-normal form.*

Proof. Let k be the max-degree of t. For each $0 \leq i \leq k$ we define $\mathtt{S}_{>i}(t)$ as follows, by induction on $k - i$:

$$\begin{array}{lcl} \mathtt{S}_{>k}(t) & \stackrel{\text{def}}{=} & t \\ \mathtt{S}_{>i}(t) & \stackrel{\text{def}}{=} & \mathtt{S}_{i+1}(\mathtt{S}_{>i+1}(t)) \quad \text{for each } 0 \leq i < k \end{array}$$

That is, $\mathtt{S}_{>i}(t) \stackrel{\text{def}}{=} \mathtt{S}_{i+1}(\ldots \mathtt{S}_{k-1}(\mathtt{S}_k(t)))$. Note that $\mathtt{S}_{>k}(t) = t$ and $\mathtt{S}_{>0}(t) = \mathtt{S}_*(t)$. Let us prove each of the two parts of the statement:

1. To show that $t \to_{\mathbf{m}}^* \mathtt{S}_*(t)$, note that for each $1 \leq i \leq k$ we have that $\mathtt{S}_{>i}(t) \to_{\mathbf{m}}^* \mathtt{S}_i(\mathtt{S}_{>i}(t)) = \mathtt{S}_{>i-1}(t)$ by Lem. 38. Hence:

$$t = \mathtt{S}_{>k}(t) \to_{\mathbf{m}}^* \mathtt{S}_{>k-1}(t) \ldots \to_{\mathbf{m}}^* \mathtt{S}_{>i}(t) \to_{\mathbf{m}}^* \mathtt{S}_{>i-1}(t) \ldots \to_{\mathbf{m}}^* \mathtt{S}_{>0}(t) = \mathtt{S}_*(t)$$

2. To show that $\mathtt{S}_*(t)$ is a $\to_{\mathbf{m}}$-normal form, we claim that for each $0 \leq i \leq k$ we have that $\mathsf{maxdeg}(\mathtt{S}_{>i}(t)) \leq i$. We proceed by induction on $k - i$. In the base case, we have that $i = k$, so $\mathsf{maxdeg}(\mathtt{S}_{>k}(t)) = \mathsf{maxdeg}(t) = k$ since k is the max-degree of t. For the induction step, let $k - i > 0$, so $0 \leq i < k$. By IH we have that $\mathsf{maxdeg}(\mathtt{S}_{>i+1}(t)) \leq i + 1$. Then $\mathsf{maxdeg}(\mathtt{S}_{>i}(t)) = \mathsf{maxdeg}(\mathtt{S}_{i+1}(\mathtt{S}_{>i+1}(t))) < i + 1$ by Lem. 42. This means that $\mathsf{maxdeg}(\mathtt{S}_{>i}(t)) \leq i$, as required.

◀

A.1.3 Forgetful reduction

The forgetful reduction relation is generalized to operate on substitution contexts so that, for example, $(\Box\{x\}\{y\}) \rhd (\Box\{y\})$.

▶ **Lemma 44** (Properties of forgetful reduction).
1. *If $t \rhd t'$ then $t\mathtt{L} \rhd t'\mathtt{L}$.*
2. *If $\mathtt{L} \rhd \mathtt{L}'$ then $t\mathtt{L} \rhd t\mathtt{L}'$.*
3. *If $t \rhd t'$ then $t[x := s] \rhd t'[x := s]$.*
4. *If $s \rhd s'$ then $t[x := s] \rhd^* t[x := s']$ (in zero or more steps).*

Proof.
- Items 1 and 2 are straightforward by induction on $\mathtt{L}$.
- Items 3 and 4 are straightforward by induction on t. For item 4, note that when $t = y$ with $y \neq x$, we have that $y[x := s] = y \rhd^* y = y[x := s']$ in exactly zero steps. Note also that more that one step of $\rhd$ may be required when t is an application or a wrapper.

◀

▶ **Lemma 45** (Local commutation of reduction and forgetful reduction). *If $t \rhd s$ and $t \to_{\mathbf{m}} t'$, there exists a term s' such that $t' \rhd^+ s'$ and $s \to_{\mathbf{m}}^= s'$, where $\rhd^+$ is the transitive closure of $\rhd$, and $\to_{\mathbf{m}}^=$ is the reflexive closure of $\to_{\mathbf{m}}$. Graphically:*

$$\begin{array}{ccc} t & \rhd & s \\ \downarrow & & \downarrow{=} \\ t' & \rhd^+ & s' \end{array}$$

Proof. By induction on t:
1. $t = x$: Note that this case is impossible, since there are no steps $x \to_{\mathbf{m}} t'$.
2. $t = \lambda x.\, t_1$: Since $\lambda x.\, t_1 \to_{\mathbf{m}} t'$, we know that t' must be of the form $t' = \lambda x.\, t_1'$ with $t_1 \to_{\mathbf{m}} t_1'$. Note that the $\rhd$ step is internal, that is, $\lambda x.\, t_1 \rhd \lambda x.\, s_1 = s$ with $t_1 \rhd s_1$. By IH there exists s_1' such that $t_1' \rhd^+ s_1'$ and $s_1 \to_{\mathbf{m}}^= s_1'$. Taking $s' := \lambda x.\, s_1'$ we have:

$$\begin{array}{ccc} \lambda x.\, t_1 & \rhd & \lambda x.\, s_1 \\ \downarrow & & \downarrow{=} \\ \lambda x.\, t_1' & \rhd^+ & \lambda x.\, s_1' \end{array}$$

3. $t = t_1\, t_2$: We consider three subcases, depending on whether the step $t_1\, t_2 \to_{\mathbf{m}} t'$ is a $\to_{\mathbf{m}}$ step at the root, internal to t_1, or internal to t_2:

3.1 If the $\to_{\mathbf{m}}$ step is at the root, then t_1 is a **m**-abstraction of the form $t_1 = (\lambda x.\, t_{11})\mathtt{L}$ and the step is of the form $t = (\lambda x.\, t_{11})\mathtt{L}\, t_2 \to_{\mathbf{m}} t_{11}[x := t_2]\{\!\{t_2\}\!\}\mathtt{L} = t'$. Moreover, since $t = (\lambda x.\, t_{11})\mathtt{L}\, t_2 \rhd t'$, we consider three further subcases, depending on whether the step $t \rhd s$ is internal to t_{11}, internal to $\mathtt{L}$, or internal to t_2:

3.1.1 If the $\rhd$ step is internal to t_{11}, then $s = (\lambda x.\, s_{11})\mathtt{L}\, t_2$ with $t_{11} \rhd s_{11}$. Taking $s' := s_{11}[x := t_2]\{\!\{t_2\}\!\}\mathtt{L}$ we have:

$$\begin{array}{ccc}
(\lambda x.\, t_{11})\mathtt{L}\, t_2 & \rhd & (\lambda x.\, s_{11})\mathtt{L}\, t_2 \\
\downarrow & & \downarrow \\
t_{11}[x := t_2]\{\!\{t_2\}\!\}\mathtt{L} & \rhd & s_{11}[x := t_2]\{\!\{t_2\}\!\}\mathtt{L}
\end{array}$$

For the $\rhd$ step at the bottom, by Lem. 44 (1) it suffices to show that $t_{11}[x := t_2] \rhd s_{11}[x := t_2]$. This is a consequence of Lem. 44 (3).

3.1.2 If the $\rhd$ step is internal to $\mathtt{L}$, then $s = (\lambda x.\, t_{11})\mathtt{L}'\, t_2$ with $\mathtt{L} \rhd \mathtt{L}'$. Taking $s' := t_{11}[x := t_2]\{\!\{t_2\}\!\}\mathtt{L}'$ we have:

$$\begin{array}{ccc}
(\lambda x.\, t_{11})\mathtt{L}\, t_2 & \rhd & (\lambda x.\, t_{11})\mathtt{L}'\, t_2 \\
\downarrow & & \downarrow \\
t_{11}[x := t_2]\{\!\{t_2\}\!\}\mathtt{L} & \rhd & t_{11}[x := t_2]\{\!\{t_2\}\!\}\mathtt{L}'
\end{array}$$

The $\rhd$ step at the bottom holds by Lem. 44 (2).

3.1.3 If the $\rhd$ step is internal to t_2, then $s = (\lambda x.\, t_{11})\mathtt{L}\, s_2'$ with $t_2 \rhd s_2$. Taking $s' := t_{11}[x := t_2]\{\!\{s_2'\}\!\}\mathtt{L}$ we have:

$$\begin{array}{ccc}
(\lambda x.\, t_{11})\mathtt{L}\, t_2 & \rhd & (\lambda x.\, t_{11})\mathtt{L}\, s_2 \\
\downarrow & & \downarrow \\
t_{11}[x := t_2]\{\!\{t_2\}\!\}\mathtt{L} & \rhd^+ & t_{11}[x := s_2]\{\!\{s_2\}\!\}\mathtt{L}
\end{array}$$

For the bottom of the diagram, note that: $t_{11}[x := t_2] \rhd^* t_{11}[x := s_2]$ by Lem. 44 (4). Hence $t_{11}[x := t_2]\{\!\{t_2\}\!\} \rhd^* t_{11}[x := s_2]\{\!\{t_2\}\!\} \rhd t_{11}[x := s_2]\{\!\{s_2\}\!\}$. Resorting to Lem. 44 (1) we conclude.

3.2 If the $\to_{\mathbf{m}}$ step is internal to t_1, the step is of the form $t_1\, t_2 \to_{\mathbf{m}} t_1'\, t_2$ with $t_1 \to_{\mathbf{m}} t_1'$. We consider two further subcases, depending on whether the $\rhd$ step is internal to t_1 or internal to t_2:

3.2.1 If the $\rhd$ step is internal to t_1, then $s = s_1\, t_2$ with $t_1 \rhd s_1$. By IH there exists s_1' such that $t_1' \rhd^+ s_1'$ and $s_1 \to_{\mathbf{m}}^{=} s_1'$. Taking $s' := s_1'\, t_2$ we have:

$$\begin{array}{ccc}
t_1\, t_2 & \rhd & s_1\, t_2 \\
\downarrow & & \downarrow^{=} \\
t_1'\, t_2 & \rhd^+ & s_1'\, t_2
\end{array}$$

3.2.2 If the $\rhd$ step is internal to t_2, then $s = t_1\, s_2$ with $t_2 \rhd s_2$. Taking $s' := t_1'\, s_2$ we have:

$$\begin{array}{ccc}
t_1\, t_2 & \rhd & t_1\, s_2 \\
\downarrow & & \downarrow \\
t_1'\, t_2 & \rhd & t_1'\, s_2
\end{array}$$

3.3 If the $\to_{\mathbf{m}}$ step is internal to t_2, the proof is similar to the previous case.

4. $t = t_1\{t_2\}$: We consider two subcases, depending on whether the step $t_1\{t_2\} \to_{\mathbf{m}} t'$ is internal to t_1 or internal to t_2:

4.1 If the $\to_{\mathbf{m}}$ step is internal to t_1, then $t_1\{t_2\} \to_{\mathbf{m}} t'_1\{t_2\} = t'$ with $t_1 \to_{\mathbf{m}} t'_1$. We consider three further subcases, depending on whether the step $t_1\{t_2\} \rhd s$ is at the root of the wrapper, internal to t_1, or internal to t_2:

4.1.1 If the $\rhd$ step is at the root of the wrapper, then $t_1\{t_2\} \rhd t_1 = s$. Taking $s' := t'_1$ we have:

$$\begin{array}{ccc} t_1\{t_2\} & \rhd & t_1 \\ \downarrow & & \downarrow \\ t'_1\{t_2\} & \rhd & t'_1 \end{array}$$

4.1.2 If the $\rhd$ step is internal to t_1, then $s = s_1\{t_2\}$ with $t_1 \rhd s_1$, and we conclude by IH similarly as for case 3.2.1.

4.1.3 If the $\rhd$ step is internal to t_2, then $s = t_1\{s_2\}$ with $t_2 \rhd s_2$, and we conclude taking $s' := t'_1\{s_2\}$ similarly as for case 3.2.2.

4.2 If the $\to_{\mathbf{m}}$ step is internal to t_2, then $t_1\{t_2\} \to_{\mathbf{m}} t_1\{t'_2\} = t'$ with $t_2 \to_{\mathbf{m}} t'_2$. We consider three further subcases, depending on whether the step $t_1\{t_2\} \rhd s$ is at the root of the wrapper, internal to t_1, or internal to t_2:

4.2.1 If the $\rhd$ step is at the root of the wrapper, then $t_1\{t_2\} \rhd t_1 = s$. Taking $s' := t_1$ we have:

$$\begin{array}{ccc} t_1\{t_2\} & \rhd & t_1 \\ \downarrow & & \| \\ t_1\{t'_2\} & \rhd & t_1 \end{array}$$

4.2.2 If the $\rhd$ step is internal to t_1, then $t_1\{t_2\} \rhd s_1\{t_2\} = s$ with $t_1 \rhd s_1$, and we conclude taking $s' := s_1\{t'_2\}$ similarly as for case 3.2.2.

4.2.3 If the $\rhd$ step is internal to t_2, then $t_1\{t_2\} \rhd t_1\{s_2\} = s$ with $t_2 \rhd s_2$, and we conclude by IH similarly as for case 3.2.1.

◀

▶ **Proposition 46** (Forgetful reduction commutes with reduction)**.** *If $t \rhd^+ s$ and $t \to^*_{\mathbf{m}} t'$, there exists a term s' such that $t' \rhd^+ s'$ and $s \to^*_{\mathbf{m}} s'$. Graphically:*

$$\begin{array}{ccc} t & \rhd^+ & s \\ \downarrow{*} & & \vdots{*} \\ t' & \rhd^+ & s' \end{array}$$

Furthermore, if $t \rhd^+ s$ and t is a $\to_{\mathbf{m}}$-normal form, then s is also a normal form.

Proof. First we claim that if $t \rhd^+ s$ and $t \to_{\mathbf{m}} t'$, there exists a term s' such that $t' \rhd^+ s'$ and $s \to^=_{\mathbf{m}} s'$. This can be seen by induction on the number of $\rhd$ steps in a reduction sequence $t \rhd^+ s$, resorting to the local commutation lemma (Lem. 45).

The main statement of the proposition can be seen by induction on the number of steps in a reduction sequence $t \to^*_{\mathbf{m}} t'$, resorting to the claim.

For the "furthermore" part in the statement, it suffices to show that if $t \rhd s$ in one step and t is a $\to_{\mathbf{m}}$-normal form, then s is also a normal form. This is straightforward by induction on t. ◀

▶ **Lemma 47** (Reduce/forget lemma). *Let $M \to_\beta N$ be a β-step and let $M \to_{\mathbf{m}} s$ be the corresponding step in $\lambda^{\mathbf{m}}$. Then $s \rhd N$.*

Proof. We proceed by induction on M:

1. $M = x$: Impossible, as there are no steps $x \to_\beta N$.
2. $M = \lambda x.\, M_1$: Then the step must be of the form $M = \lambda x.\, M_1 \to_\beta \lambda x.\, N_1 = N$ with $M_1 \to_\beta N_1$, and the corresponding step must be of the form $M = \lambda x.\, M_1 \to_{\mathbf{m}} \lambda x.\, s_1 = s$ where $M_1 \to_{\mathbf{m}} s_1$ is the step corresponding to $M_1 \to_\beta N_1$. By IH $s_1 \rhd N_1$, so $s = \lambda x.\, s_1 \rhd \lambda x.\, N_1 = N$.
3. $M = M_1\, M_2$: We consider three subcases, depending on whether the step is at the root, internal to M_1, or internal to M_2:
   - **3.1** If the step is at the root, the step must be of the form $M = (\lambda x.\, M_{11})\, M_2 \to_\beta M_{11}[x := M_2]$ with $M_1 = \lambda x.\, M_{11}$, and the corresponding step is $M = (\lambda x.\, M_{11})\, M_2 \to_{\mathbf{m}} M_{11}[x := M_2]\{M_2\} = s$. Then:

$$\begin{array}{rcl} s & = & M_{11}[x := M_2]\{M_2\} \\ & \rhd & M_{11}[x := M_2] \\ & = & N \end{array}$$

   - **3.2** If the step is internal to M_1, the step must be of the form $M_1\, M_2 \to_\beta N_1\, M_2 = N$ with $M_1 \to_\beta N_1$, and the corresponding step is $M_1\, M_2 \to_{\mathbf{m}} s_1\, M_2 = s$ where $M_1 \to_{\mathbf{m}} s_1$ is the step corresponding to $M_1 \to_{\mathbf{m}} N_1$. By IH we have that $s_1 \rhd N_1$, so $s = s_1\, M_2 \rhd N_1\, M_2 = N$.
   - **3.3** If the step is internal to M_2, the proof is similar to the previous case.

◀

A.2 Proofs of Section 4 — Reduction by degrees

In this section we give detailed proofs of the results about reduction by degrees stated in Section 2.

▶ Remark 48. $t\mathtt{L}$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form if and only if t and $\mathtt{L}$ are in $\xrightarrow{d}_{\mathbf{m}}$-normal form.

▶ **Definition 49** (Steps and reduction sequences). *A* step of degree d *—or just* step *if clear from the context—– is formally a 5-uple $R = (\mathtt{C}, x^A, t, \mathtt{L}, s)$ where $\mathtt{C}$ is an arbitrary context and $\lambda x^A.t$ is an abstraction of degree d. The* source *of R is $R^{\mathsf{src}} \overset{\text{def}}{=} \mathtt{C}[(\lambda x.\, t)\mathtt{L}\, s]$ and its* target *is $R^{\mathsf{tgt}} \overset{\text{def}}{=} \mathtt{C}[t[x := s]\{s\}\mathtt{L}]$. We write $R : t \xrightarrow{d}_{\mathbf{m}} s$ to mean that R is a step of degree d with source t and target s.*

A forgetful step *—or just* step *if clear from the context—– is formally a triple $R = (\mathtt{C}, t, s)$ where $\mathtt{C}$ is an arbitrary context and t, s are terms. The* source *of R is $R^{\mathsf{src}} \overset{\text{def}}{=} \mathtt{C}[t\{s\}]$ and its* target *is $R^{\mathsf{tgt}} \overset{\text{def}}{=} \mathtt{C}[t]$. We write $R : t \rhd s$ to mean that R is a forgetful step of degree with source t and target s.*

Steps of degree d are generalized to reduction sequences of degree d *(and, respectively,* forgetful reduction sequences*), which are sequences of composable steps of the corresponding kind. Formally, a reduction sequence is a pair $\rho = ((t_0, \dots, t_n), (R_1, \dots, R_n))$ where $(t_0, \dots, t_n)$ is a sequence of $n+1$ terms and $(R_1, \dots, R_n)$ is a sequence of n steps $R_i^{\mathsf{src}} = t_{i-1}$ and $R_i^{\mathsf{tgt}} = t_i$ for all $i \in 1..n$. The notions of source and target are extended to reduction sequences by declaring $\rho^{\mathsf{src}} = t_0$ and $\rho^{\mathsf{tgt}} = t_n$. We write $\rho : t \xrightarrow{d}{}^{*}_{\mathbf{m}} s$ to mean that ρ is a reduction sequence of degree d with source t and target s. Similarly, we write $\rho : t \rhd^* s$ to mean that ρ is a forgetful reduction sequence with source t and target s*

A step R can be implicitly treated as the one-step reduction sequence $((R^{\mathsf{src}}, R^{\mathsf{tgt}}), R)$. If $\rho^{\mathsf{tgt}} = \sigma^{\mathsf{src}}$, we write $\rho\,\sigma$ for their composition, defined as expected.

▶ **Definition 50** (Simultaneous reduction of degree d). *We define a relation $t \overset{d}{\Longrightarrow}_{\mathbf{m}} t'$, meaning that there is a* multi-step *of degree d from t to t', inductively by the following rules:*

$$\frac{}{x \overset{d}{\Longrightarrow}_{\mathbf{m}} x}\ \texttt{p-var} \qquad \frac{t \overset{d}{\Longrightarrow}_{\mathbf{m}} t'}{\lambda x.\, t \overset{d}{\Longrightarrow}_{\mathbf{m}} \lambda x.\, t'}\ \texttt{p-abs} \qquad \frac{t \overset{d}{\Longrightarrow}_{\mathbf{m}} t' \quad s \overset{d}{\Longrightarrow}_{\mathbf{m}} s'}{t\, s \overset{d}{\Longrightarrow}_{\mathbf{m}} t'\, s'}\ \texttt{p-app}_1$$

$$\frac{t \overset{d}{\Longrightarrow}_{\mathbf{m}} t' \quad \mathtt{L} \overset{d}{\Longrightarrow}_{\mathbf{m}} \mathtt{L}' \quad s \overset{d}{\Longrightarrow}_{\mathbf{m}} s' \quad \lambda x.\, t \text{ is of degree } d}{(\lambda x.\, t)\mathtt{L}\, s \overset{d}{\Longrightarrow}_{\mathbf{m}} t'[x := s']\{s'\}\mathtt{L}'}\ \texttt{p-app}_2 \qquad \frac{t \overset{d}{\Longrightarrow}_{\mathbf{m}} t' \quad s \overset{d}{\Longrightarrow}_{\mathbf{m}} s'}{t\{s\} \overset{d}{\Longrightarrow}_{\mathbf{m}} t'\{s'\}}\ \texttt{p-wrap}$$

$$\frac{}{\Box \overset{d}{\Longrightarrow}_{\mathbf{m}} \Box}\ \texttt{p-ctx-hole} \qquad \frac{\mathtt{L} \overset{d}{\Longrightarrow}_{\mathbf{m}} \mathtt{L}' \quad t \overset{d}{\Longrightarrow}_{\mathbf{m}} t'}{\mathtt{L}\{t\} \overset{d}{\Longrightarrow}_{\mathbf{m}} \mathtt{L}'\{t'\}}\ \texttt{p-ctx-wrap}$$

If $\mathbf{R}$ is the derivation witnessing a multi-step $t \overset{d}{\Longrightarrow}^{+}_{\mathbf{m}} t'$ We say that $\mathbf{R}$ is empty if it does not use the rule $\texttt{p-app}_2$. We write $\mathbf{R} : t \overset{d}{\Longrightarrow}^{+}_{\mathbf{m}} t'$ if $\mathbf{R}$ uses the rule $\texttt{p-app}_2$ at least once.

▶ Remark 51 (Simultaneous reduction of terms with memory).

1. $t\mathtt{L} \overset{d}{\Longrightarrow}_{\mathbf{m}} s$ if and only if s is of the form $t'\mathtt{L}'$ where $t \overset{d}{\Longrightarrow}_{\mathbf{m}} t'$ and $\mathtt{L} \overset{d}{\Longrightarrow}_{\mathbf{m}} \mathtt{L}'$.
2. Furthermore, the set of derivations $\mathbf{R} : t\mathtt{L} \overset{d}{\Longrightarrow}_{\mathbf{m}} t'\mathtt{L}'$ is in bijective correspondence with the set of pairs of derivations $\mathbf{R}_1 : t \overset{d}{\Longrightarrow}_{\mathbf{m}} t'$ and $\mathbf{R}_2 : \mathtt{L} \overset{d}{\Longrightarrow}_{\mathbf{m}} \mathtt{L}'$.

▶ **Lemma 52** (Properties of simultaneous reduction by degrees).

1. *For each step $R : t \overset{d}{\to}_{\mathbf{m}} t'$ there is a multi-step $\mathtt{sim}(R) : t \overset{d}{\Longrightarrow}_{\mathbf{m}} t'$.*
2. *For each multi-step $\mathbf{R} : t \overset{d}{\Longrightarrow}_{\mathbf{m}} t'$ there is a reduction sequence $\mathtt{red}(\mathbf{R}) : t \overset{d}{\to}^{*}_{\mathbf{m}} t'$. Moreover, if $\mathbf{R}$ is non-empty, then $\mathtt{red}(\mathbf{R})$ contains at least one step.*
3. *Reflexivity: $t \overset{d}{\Longrightarrow}_{\mathbf{m}} t$.*
4. *Substitution: If $t \overset{d}{\Longrightarrow}_{\mathbf{m}} t'$ and $s \overset{d}{\Longrightarrow}_{\mathbf{m}} s'$ then $t[x := s] \overset{d}{\Longrightarrow}_{\mathbf{m}} t'[x := s']$.*

Proof. All items are straightforward by induction. ◀

▶ **Lemma 53** (Commutation of simultaneous reduction by degrees). *Let $d, D \in \mathbb{N}_0$. Given a step $R : t_1 \overset{d}{\to}_{\mathbf{m}} t_2$ and a multi-step $\mathbf{S} : t_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} t_3$, there exists a term t_4, a multi-step $\mathbf{S}/R : t_2 \overset{D}{\Longrightarrow}_{\mathbf{m}} t_4$ and a multi-step $R/\mathbf{S} : t_3 \overset{d}{\Longrightarrow}_{\mathbf{m}} t_4$. Graphically:*

$$\begin{array}{ccc} t_1 & \overset{d}{\longrightarrow} & t_2 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ t_3 & \underset{d}{\Longrightarrow} & t_4 \end{array}$$

Furthermore:

1. *If $d \neq D$ then $R/\mathbf{S}$ is non-empty,* i.e. *$R/\mathbf{S} : t_3 \overset{d}{\Longrightarrow}^{+}_{\mathbf{m}} t_4$.*
2. *If $d \neq D$, the first step of $\mathtt{red}(R/\mathbf{S})$ determines the step R. More precisely, suppose that $\mathtt{red}(R_1/\mathbf{S})$ and $\mathtt{red}(R_2/\mathbf{S})$ start with the same step. Then $R_1 = R_2$.*

Proof. We prove a more general version of the statement including memories, *i.e.* we prove that for $R : \mathtt{X}_1 \overset{d}{\to}_{\mathbf{m}} \mathtt{X}_2$ and $\mathbf{S} : \mathtt{X}_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} \mathtt{X}_3$, there exist $\mathtt{X}_4$ and $\mathbf{S}/R : \mathtt{X}_2 \overset{D}{\Longrightarrow}_{\mathbf{m}} \mathtt{X}_4$ and $R/\mathbf{S} : \mathtt{X}_3 \overset{d}{\Longrightarrow}_{\mathbf{m}} \mathtt{X}_4$, where $\mathtt{X}_1, \mathtt{X}_2, \mathtt{X}_3, \mathtt{X}_4$ stand for either terms or memories. We proceed by induction on $\mathtt{X}_1$:

1. $t_1 = x$: Impossible, as there are no reduction steps $x \xrightarrow{d}_{\mathbf{m}} t_2$.
2. $t_1 = \lambda x.\, s_1$: Then $R : t_1 = \lambda x.\, s_1 \xrightarrow{d}_{\mathbf{m}} \lambda x.\, s_2 = t_2$ with $s_1 \xrightarrow{d}_{\mathbf{m}} s_2$ and S must be derived from the `p-abs` rule, so $S : t_1 = \lambda x.\, s_1 \overset{d}{\Longrightarrow}_{\mathbf{m}} \lambda x.\, s_3 = t_3$ with $s_1 \overset{d}{\Longrightarrow}_{\mathbf{m}} s_3$. By IH we have the diagram on the left, and we can construct the one on the right:

$$\begin{array}{ccc} s_1 & \xrightarrow{d} & s_2 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ s_3 & \underset{d}{\Longrightarrow} & s_4 \end{array} \qquad \begin{array}{ccc} \lambda x.\, s_1 & \xrightarrow{d} & \lambda x.\, s_2 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ \lambda x.\, s_3 & \underset{d}{\Longrightarrow} & \lambda x.\, s_4 \end{array}$$

   Furthermore, if $d \neq D$, using the IH it is easy to show that $R/\mathbf{S}$ is non-empty and that $R/\mathbf{S}$ determines R.
3. $t_1 = s_1\, u_1$: We consider three subcases, depending on whether R is at the root, internal to s_1, or internal to u_1:

   3.1 If R is at the root: Then s_1 is a $\mathbf{m}$-abstraction of degree d, *i.e.* of the form $s_1 = (\lambda x.\, r_1)\mathtt{L}_1$, and $R : t_1 = (\lambda x.\, r_1)\mathtt{L}_1\, u_1 \xrightarrow{d}_{\mathbf{m}} r_1[x := u_1]\{u_1\}\mathtt{L}_1 = t_2$. We consider two further subcases, depending on whether S is derived using the $\mathtt{p\text{-}app}_1$ or the $\mathtt{p\text{-}app}_2$ rule:

   3.1.1 If S is derived using the $\mathtt{p\text{-}app}_1$ rule: Then by Rem. 51 we have that $S : t_1 = (\lambda x.\, r_1)\mathtt{L}_1\, u_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} (\lambda x.\, r_3)\mathtt{L}_3\, u_3 = t_3$ where $r_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} r_3$ and $\mathtt{L}_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} \mathtt{L}_3$ and $u_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} u_3$. By Lem. 52 we can construct the following diagram, using reflexivity and $\mathtt{p\text{-}app}_2$ on the bottom:

$$\begin{array}{ccc} (\lambda x.\, r_1)\mathtt{L}_1\, u_1 & \xrightarrow{d} & r_1[x := u_1]\{u_1\}\mathtt{L}_1 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ (\lambda x.\, r_3)\mathtt{L}_3\, u_3 & \underset{d}{\Longrightarrow} & r_3[x := u_3]\{u_3\}\mathtt{L}_3 \end{array}$$

   3.1.2 If S is derived using the $\mathtt{p\text{-}app}_2$ rule: Then note that $d = D$ and we have that $S : t_1 = (\lambda x.\, r_1)\mathtt{L}_1\, u_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} r_3[x := u_3]\{u_3\}\mathtt{L}_3 = t_3$ where $r_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} r_3$ and $\mathtt{L}_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} \mathtt{L}_3$ and $u_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} u_3$. By Lem. 52 we can construct the following diagram, using reflexivity on the bottom:

$$\begin{array}{ccc} (\lambda x.\, r_1)\mathtt{L}_1\, u_1 & \xrightarrow{d} & r_1[x := u_1]\{u_1\}\mathtt{L}_1 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ r_3[x := u_3]\{u_3\}\mathtt{L}_3 & \underset{d}{\Longrightarrow} & r_3[x := u_3]\{u_3\}\mathtt{L}_3 \end{array}$$

   3.2 If R is internal to s_1: Then $R : t_1 = s_1\, u_1 \xrightarrow{d}_{\mathbf{m}} s_2\, u_1$ with $s_1 \xrightarrow{d}_{\mathbf{m}} s_2$. We consider two further subcases, depending on whether S is derived using the $\mathtt{p\text{-}app}_1$ or the $\mathtt{p\text{-}app}_2$ rule:

   3.2.1 If S is derived using the $\mathtt{p\text{-}app}_1$ rule: Then $S : t_1 = s_1\, u_1 \overset{d}{\Longrightarrow}_{\mathbf{m}} s_3\, u_3 = t_3$ with $s_1 \overset{d}{\Longrightarrow}_{\mathbf{m}} s_3$ and $u_1 \overset{d}{\Longrightarrow}_{\mathbf{m}} u_3$. By IH we have the diagram on the left, and we can construct the one on the right:

$$\begin{array}{ccc} s_1 & \xrightarrow{d} & s_2 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ s_3 & \underset{d}{\Longrightarrow} & s_4 \end{array} \qquad \begin{array}{ccc} s_1\, u_1 & \xrightarrow{d} & s_2\, u_1 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ s_3\, u_3 & \underset{d}{\Longrightarrow} & s_4\, u_3 \end{array}$$

3.2.2 If S is derived using the $\texttt{p-app}_2$ rule: Then s_1 is a $\mathbf{m}$-abstraction of degree D, *i.e.* of the form $s_1 = (\lambda x. r_1)\mathtt{L}_1$, and by Rem. 51 we have that $S : t_1 = (\lambda x. r_1)\mathtt{L}_1\, u_1 \stackrel{D}{\Longrightarrow}_{\mathbf{m}} r_3[x := u_3]\{\!\!\{u_3\}\!\!\}\mathtt{L}_3$ where $r_1 \stackrel{D}{\Longrightarrow}_{\mathbf{m}} r_3$ and $\mathtt{L}_1 \stackrel{D}{\Longrightarrow}_{\mathbf{m}} \mathtt{L}_3$ and $u_1 \stackrel{D}{\Longrightarrow}_{\mathbf{m}} u_3$. Moreover, since we know $s_1 = (\lambda x. r_1)\mathtt{L}_1 \stackrel{d}{\to}_{\mathbf{m}} s_2$ we consider two further subcases, depending on whether the step $s_1 \stackrel{d}{\to}_{\mathbf{m}} s_2$ is internal to r_1 or internal to $\mathtt{L}_1$. These subcases are similar; we only give the proof for the case in which the step is internal to r_1. In such case $s_1 = (\lambda x. r_1)\mathtt{L}_1 \stackrel{d}{\to}_{\mathbf{m}} (\lambda x. r_2)\mathtt{L}_1$ with $r_1 \stackrel{d}{\to}_{\mathbf{m}} r_2$. By IH we have the diagram on the left, and we can construct the one on the right, using Lem. 52. On the right of the diagram, use $\texttt{p-app}_2$. On the bottom of the diagram, note that $u_3 \stackrel{d}{\Longrightarrow}_{\mathbf{m}} u_3$ by reflexivity so $r_3[x := u_3] \stackrel{d}{\Longrightarrow}_{\mathbf{m}} r_4[x := u_3]$:

$$\begin{array}{ccc} r_1 & \xrightarrow{d} & r_2 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ r_3 & \underset{d}{\Longrightarrow} & r_4 \end{array} \qquad \begin{array}{ccc} (\lambda x. r_1)\mathtt{L}_1\, u_1 & \xrightarrow{d} & (\lambda x. r_2)\mathtt{L}_1\, u_1 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ r_3[x := u_3]\{\!\!\{u_3\}\!\!\}\mathtt{L}_3 & \underset{d}{\Longrightarrow} & r_4[x := u_3]\{\!\!\{u_3\}\!\!\}\mathtt{L}_3 \end{array}$$

3.3 If R is internal to u_1: Then $R : t_1 = s_1\, u_1 \stackrel{d}{\to}_{\mathbf{m}} s_1\, u_2$ with $u_1 \stackrel{d}{\to}_{\mathbf{m}} u_2$. We consider two further subcases, depending on whether S is derived using the $\texttt{p-app}_1$ or the $\texttt{p-app}_2$ rule:

3.3.1 If S is derived using the $\texttt{p-app}_1$ rule: Then $S : t_1 = s_1\, u_1 \stackrel{d}{\Longrightarrow}_{\mathbf{m}} s_3\, u_3 = t_3$ with $s_1 \stackrel{d}{\Longrightarrow}_{\mathbf{m}} s_3$ and $u_1 \stackrel{d}{\Longrightarrow}_{\mathbf{m}} u_3$. By IH we have the diagram on the left, and we can construct the one on the right:

$$\begin{array}{ccc} u_1 & \xrightarrow{d} & u_2 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ u_3 & \underset{d}{\Longrightarrow} & u_4 \end{array} \qquad \begin{array}{ccc} s_1\, u_1 & \xrightarrow{d} & s_1\, u_2 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ s_3\, u_3 & \underset{d}{\Longrightarrow} & s_3\, u_4 \end{array}$$

3.3.2 If S is derived using the $\texttt{p-app}_2$ rule: Then s_1 is a $\mathbf{m}$-abstraction of degree D, *i.e.* of the form $s_1 = (\lambda x. r_1)\mathtt{L}_1$, and by Rem. 51 we have that $S : t_1 = (\lambda x. r_1)\mathtt{L}_1\, u_1 \stackrel{D}{\Longrightarrow}_{\mathbf{m}} r_3[x := u_3]\{\!\!\{u_3\}\!\!\}\mathtt{L}_3$ where $r_1 \stackrel{D}{\Longrightarrow}_{\mathbf{m}} r_3$ and $\mathtt{L}_1 \stackrel{D}{\Longrightarrow}_{\mathbf{m}} \mathtt{L}_3$ and $u_1 \stackrel{D}{\Longrightarrow}_{\mathbf{m}} u_3$. By IH we have the diagram on the left, and we can construct the one on the right. On the right of the diagram, use $\texttt{p-app}_2$. On the bottom of the diagram, note that $r_3 \stackrel{d}{\Longrightarrow}_{\mathbf{m}} r_3$ by reflexivity so $r_3[x := u_3] \stackrel{d}{\Longrightarrow}_{\mathbf{m}} r_3[x := u_4]$:

$$\begin{array}{ccc} u_1 & \xrightarrow{d} & u_2 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ u_3 & \underset{d}{\Longrightarrow} & u_4 \end{array} \qquad \begin{array}{ccc} (\lambda x. r_1)\mathtt{L}_1\, u_1 & \xrightarrow{d} & (\lambda x. r_1)\mathtt{L}_1\, u_2 \\ {\scriptstyle D}\Downarrow & & \Downarrow{\scriptstyle D} \\ r_3[x := u_3]\{\!\!\{u_3\}\!\!\}\mathtt{L}_3 & \underset{d}{\Longrightarrow} & r_3[x := u_4]\{\!\!\{u_4\}\!\!\}\mathtt{L}_3 \end{array}$$

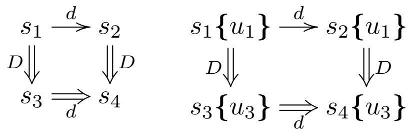
Furthermore, note that if $d \neq D$, then the multi-step $R/\mathbf{S} : t_3 \stackrel{d}{\Longrightarrow}_{\mathbf{m}} t_4$ at the bottom of the diagram must be non-empty. Indeed, case 3.1.1, uses exactly one occurrence of the $\texttt{p-app}_2$ rule to construct $R/\mathbf{S}$. Case 3.1.2 is impossible, because in such case $d = D$. In the remaining cases, the bottom of the diagram is constructed by resorting to the IH, which means that $R/\mathbf{S}$ is non-empty. An important observation is that in case 3.3.2 the argument is not erased, because it is always kept as a memorized term.

Furthermore, if $d \neq D$, to see that the first step of $\texttt{red}(R/\mathbf{S})$ determines the step R, consider the first step T of $\texttt{red}(R/\mathbf{S})$ and note that it its λ-abstraction can be uniquely traced back to the λ-abstraction of R (*i.e.* it has a unique ancestor). Indeed, in case 3.1.1

the step at the bottom has R as its unique ancestor. Case 3.1.2 is impossible, because in such case $d = D$. In the remaining cases, it suffices to resort to the IH.

4. $t_1 = s_1\{u_1\}$: We consider two subcases, depending on whether R is internal to s_1 or internal to u_1:

4.1 If R is internal to s_1: Then $R : t_1 = s_1\{u_1\} \xrightarrow{d}_{\mathbf{m}} s_2\{u_1\} = t_2$. Note that S must be derived using the `p-wrap` rule, so $S : t_1 = s_1\{u_1\} \overset{d}{\Longrightarrow}_{\mathbf{m}} s_3\{u_3\} = t_3$. By IH we have the diagram on the left, and we can construct the one on the right:



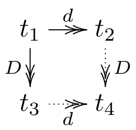
4.2 If R is internal to u_1: Similar to the previous case.
Furthermore, if $d \neq D$, using the IH it is easy to show that $R/\mathbf{S}$ is non-empty and that $R/\mathbf{S}$ determines R.

5. $\mathtt{L}_1 = \Box$: Impossible, as there are no steps $\Box \xrightarrow{d}_{\mathbf{m}} \mathtt{L}_2$.

6. $\mathtt{L}_1 = \mathtt{L}'_1\{t_1\}$: Similar to case 4.

◀

▶ **Proposition 54** (Commutation of reduction by degrees). *Let $d, D \in \mathbb{N}_0$. Then $\xrightarrow{d}_{\mathbf{m}}$ and $\xrightarrow{D}_{\mathbf{m}}$ commute. More precisely, given reduction sequences $\rho : t_1 \xrightarrow{d}{}^{*}_{\mathbf{m}} t_2$ and $\sigma : t_1 \xrightarrow{D}{}^{*}_{\mathbf{m}} t_3$, there exists a term t_4 and reduction sequences $\sigma/\rho : t_2 \xrightarrow{D}{}^{*}_{\mathbf{m}} t_4$ and $\rho/\sigma : t_3 \xrightarrow{d}{}^{*}_{\mathbf{m}} t_4$. Graphically:*



The reduction sequence ρ/σ is called the projection of ρ after σ *and symmetrically for σ/ρ. Furthermore:*

1. *If $d \neq D$, then ρ/σ contains at least as many steps as ρ.*

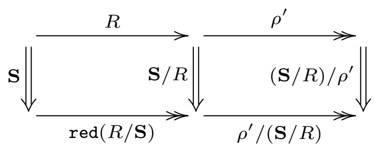
2. *If $d \neq D$, then ρ/σ determines ρ. More precisely, if $\rho_1/\sigma = \rho_2/\sigma$ then $\rho_1 = \rho_2$.*

Proof. Recall that $\xrightarrow{d}_{\mathbf{m}} \subseteq \overset{d}{\Longrightarrow}_{\mathbf{m}} \subseteq \xrightarrow{d}{}^{*}_{\mathbf{m}}$. by Lem. 52. We prove this in two stages.

First, given a reduction sequence $\rho : t_1 \xrightarrow{d}{}^{*}_{\mathbf{m}} t_2$ and a multi-step $\mathbf{S} : t_1 \overset{D}{\Longrightarrow}_{\mathbf{m}} t_3$, we claim that there exists a term t_4 and constructing $\rho/\mathbf{S} : t_3 \xrightarrow{d}{}^{*}_{\mathbf{m}} t_4$ and $\mathbf{S}/\rho : t_2 \overset{D}{\Longrightarrow}_{\mathbf{m}} t_4$ as follows, by induction on ρ, resorting to Lem. 53 for the constructions of $R/\mathbf{S}$ and $\mathbf{S}/R$.

$$\begin{array}{rclcrcl} \epsilon/\mathbf{S} & \overset{\text{def}}{=} & \epsilon & \quad & \mathbf{S}/\epsilon & \overset{\text{def}}{=} & \mathbf{S} \\ (R\,\rho')/\mathbf{S} & \overset{\text{def}}{=} & \mathtt{red}(R/\mathbf{S})(\rho'/(\mathbf{S}/R)) & \quad & \mathbf{S}/(R\,\rho') & \overset{\text{def}}{=} & (\mathbf{S}/R)/\rho' \end{array}$$

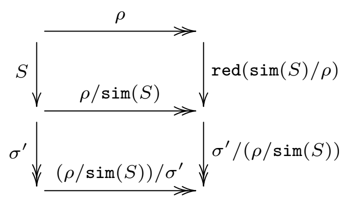
Recall from Lem. 52 that if $\mathbf{R} : u \overset{d}{\Longrightarrow}_{\mathbf{m}} u'$ is a multi-step, then $\mathtt{red}(\mathbf{R}) : u \xrightarrow{d}{}^{*}_{\mathbf{m}} u'$ denotes a reduction sequence. The inductive cases correspond to the following diagram:



For the general case, we proceed by induction on σ resorting to the previous construction for the constructions of $\mathtt{sim}(S)/\rho$ and $\rho/\mathtt{sim}(S)$:

$$\begin{array}{rclrcl} \rho/\epsilon & \stackrel{\text{def}}{=} & \rho & \epsilon/\rho & \stackrel{\text{def}}{=} & \epsilon \\ \rho/(S\,\sigma') & \stackrel{\text{def}}{=} & (\rho/\mathtt{sim}(S))/\sigma' & (S\,\sigma')/\rho & \stackrel{\text{def}}{=} & \mathtt{red}(\mathtt{sim}(S)/\rho)\,(\sigma'/(\rho/\mathtt{sim}(S)) \end{array}$$

Recall from Lem. 52 that if $R : u \xrightarrow{d}_{\mathbf{m}} u'$ is a step, then $\mathtt{sim}(R) : u \overset{d}{\Longrightarrow}_{\mathbf{m}} u'$ denotes a multi-step. The inductive cases correspond to the following diagram:



Furthermore, if $d \neq D$, note that $R/\mathbf{S} : t \overset{d}{\Longrightarrow}{}^{+}_{\mathbf{m}} s$ by Lem. 53, so $\mathtt{red}(R/\mathbf{S}) : t \xrightarrow{d}{}^{+}_{\mathbf{m}} s$ by Lem. 52. Then by induction on ρ, we can show that $\rho/\mathbf{S}$ contains at least as many steps as ρ. Finally, by induction on σ, we can show that ρ/σ contains at least as many steps as ρ.

Furthermore, to see that ρ/σ determines ρ, we proceed in stages:

1. First, if $\mathtt{red}(R_1/\mathbf{S})$ and $\mathtt{red}(R_2/\mathbf{S})$ start with the same step, then $R_1 = R_2$ by Lem. 53.
2. Second, we can see that if $\rho_1/\mathbf{S} = \rho_2/\mathbf{S}$ then $\rho_1 = \rho_2$ by induction on ρ_1. Note that if $\rho_1/\mathbf{S} = \rho_2/\mathbf{S}$ then ρ_1 and ρ_2 are either both empty or both non-empty, because if $\rho_1 = R_1\,\rho_1'$ then by definition $\rho_1/\mathbf{S} = \mathtt{red}(R_1/\mathbf{S})\,(\rho_1'/(\mathbf{S}/R_1))$ and Lem. 53 ensures that $R_1/\mathbf{S}$ is non-empty whenever $d \neq D$, so $\rho_2/\mathbf{S}$ is non-empty, and hence ρ_2 is non-empty. The base case is immediate. For the induction step, when $\rho_1 = R_1\,\rho_1'$ and $\rho_2 = R_2\,\rho_2'$ we have that $\rho_1/\mathbf{S} = \rho_2/\mathbf{S}$, so by definition $\mathtt{red}(R_1/\mathbf{S})\,(\rho_1'/(\mathbf{S}/R_1)) = \mathtt{red}(R_2/\mathbf{S})\,(\rho_2'/(\mathbf{S}/R_2))$. As before Lem. 53 ensures that $\mathtt{red}(R_1/\mathbf{S})$ and $\mathtt{red}(R_2/\mathbf{S})$ are non-empty, so they must start with the same step. Hence by Lem. 53 we have that $R_1 = R_2$. This in turn implies that $\rho_1'/(\mathbf{S}/R_1) = \rho_2'/(\mathbf{S}/R_2)$, so by IH $\rho_1' = \rho_2'$.
3. Finally, by induction on σ we can see that if $\rho_1/\sigma = \rho_2/\sigma$ then $\rho_1 = \rho_2$, resorting to the previous item.

◀

▶ **Lemma 55** (A term reduces to its simplification, by degrees)**.** *For every term t and for all $d \geq 1$ we have that $t \xrightarrow{d}{}^{*}_{\mathbf{m}} \mathtt{S}_d(t)$.*

Proof. The proof is essentially the same proof as that of Lem. 38, noting that whenever a redex is contracted, its degree is exactly d. ◀

▶ **Lemma 56** (Substitution of $\xrightarrow{d}_{\mathbf{m}}$-normal forms)**.**

1. *If t and s are not $\mathbf{m}$-abstractions of degree d, then $t[x := s]$ is not a $\mathbf{m}$-abstraction of degree d.*
2. *Let t and s be terms in $\xrightarrow{d}_{\mathbf{m}}$-normal form such that s is not an abstraction of degree d. Then $t[x := s]$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form.*

Proof. We prove the two items separately:

1. By induction on t:
   - **1.1** $t = x$: Then $t[x := s] = s$, which is not a $\mathbf{m}$-abstraction of degree d by hypothesis.
   - **1.2** $t = y \neq x$: Then $t[x := s] = y$ is not a $\mathbf{m}$-abstraction.

1.3 $t = \lambda y. t'$: By α-conversion we may assume that $y \notin \{x\} \cup \mathsf{fv}(s)$. Note that t is a $\mathbf{m}$-abstraction but, by hypothesis, it cannot be of degree d. By the substitution lemma (Lem. 34) we have that $\mathsf{type}(t[x := s]) = \mathsf{type}(\lambda y. t'[x := s]) = \mathsf{type}(\lambda y. t')$, so $t[x := s] = \lambda y. t'[x := s]$ is a $\mathbf{m}$-abstraction, but it is not of degree d.

1.4 $t = t_1\, t_2$: Then $t[x := s] = t_1[x := s]\, t_2[x := s]$ is an application, hence not a $\mathbf{m}$-abstraction.

1.5 $t = t_1\boldsymbol{\{}t_2\boldsymbol{\}}$: Since t is not a $\mathbf{m}$-abstraction of degree d, we have that t_1 is also not a $\mathbf{m}$-abstraction of degree d. By IH, $t_1[x := s]$ is not a $\mathbf{m}$-abstraction of degree d, so $t[x := s] = t_1[x := s]\boldsymbol{\{}t_2[x := s]\boldsymbol{\}}$ is not a $\mathbf{m}$-abstraction of degree d.

2. By induction t:

2.1 $t = x$: Then $t[x := s] = s$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form.

2.2 $t = y \neq x$: Then $t[x := s] = y$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form.

2.3 $t = \lambda y. t'$: By α-conversion we may assume that $y \notin \{x\} \cup \mathsf{fv}(s)$. By IH, $t'[x := s]$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form, so $t[x := s] = \lambda y. t'[x := s]$ is also in $\xrightarrow{d}_{\mathbf{m}}$-normal form.

2.4 $t = t_1\, t_2$: By IH, $t_1[x := s]$ and $t_2[x := s]$ are in $\xrightarrow{d}_{\mathbf{m}}$-normal form. To show that the whole term $t[x := s] = t_1[x := s]\, t_2[x := s]$ is a normal form, we are only left to show that the term does not have a $\xrightarrow{d}_{\mathbf{m}}$-redex at the root, *i.e.* that $t_1[x := s]$ is not a $\mathbf{m}$-abstraction of degree d. Note that t_1 cannot be a $\mathbf{m}$-abstraction of degree d, for otherwise $t = t_1\, t_2$ would be a redex of degree d, but we know by hypothesis that t is in $\xrightarrow{d}_{\mathbf{m}}$-normal form. Hence by item 1 of this lemma, $t_1[x := s]$ is not a $\mathbf{m}$-abstraction of degree d, as required.

2.5 $t = t_1\boldsymbol{\{}t_2\boldsymbol{\}}$: By IH, $t_1[x := s]$ and $t_2[x := s]$ are in $\xrightarrow{d}_{\mathbf{m}}$-normal form, so $t[x := s] = t_1[x := s]\boldsymbol{\{}t_2[x := s]\boldsymbol{\}}$ is also in $\xrightarrow{d}_{\mathbf{m}}$-normal form.

◀

▶ **Lemma 57** (Simplification does not create abstractions, by degrees)**.** *If t is not a $\mathbf{m}$-abstraction of degree d, then $\mathtt{S}_d(t)$ is not a $\mathbf{m}$-abstraction of degree d.*

Proof. By induction on t:

1. $t = x$: Then $\mathtt{S}_d(t) = x$ is not a $\mathbf{m}$-abstraction of degree d.
2. $t = \lambda y. s$: Note that t is an abstraction but, by hypothesis, it cannot be of degree d. By the fact that a term reduces to its simplification (Lem. 38) and by the substitution lemma (Lem. 34) we know that $\mathsf{type}(\mathtt{S}_d(t))$ is not of degree d, so in particular it cannot be a $\mathbf{m}$-abstraction of degree d.
3. $t = (\lambda y. s)\mathtt{L}\, u$, where $(\lambda y. s)\mathtt{L}$ is a $\mathbf{m}$-abstraction of degree d: Then $\mathtt{S}_d(t) = \mathtt{S}_d(s)[x := \mathtt{S}_d(u)]\boldsymbol{\{}\mathtt{S}_d(u)\boldsymbol{\}}\mathtt{S}_d(\mathtt{L})$. To show that this term is not a $\mathbf{m}$-abstraction of degree d, it suffices to show that $\mathtt{S}_d(s)[x := \mathtt{S}_d(u)]$ is not a $\mathbf{m}$-abstraction of degree d. Note that the abstraction $\lambda y. s$ is of type $A \to B$ where $\mathsf{type}(s) = B$ and $\mathsf{type}(u) = A$. In particular, since the abstraction $\lambda y. s$ is of degree d, we have that $\mathsf{h}(A \to B) = d$. Furthermore, by the fact that a term reduces to its simplification (Lem. 38) and by subject reduction (Prop. 3), we know that $\mathsf{h}(\mathsf{type}(\mathtt{S}_d(s))) = \mathsf{h}(\mathsf{type}(s)) = \mathsf{h}(B) < d$ and $\mathsf{h}(\mathsf{type}(\mathtt{S}_d(u))) = \mathsf{h}(\mathsf{type}(u)) = \mathsf{h}(A) < d$. In particular, $\mathtt{S}_d(s)$ and $\mathtt{S}_d(u)$ cannot be $\mathbf{m}$-abstractions of degree d. Finally by Lem. 56(1) this means that $\mathtt{S}_d(s)[x := \mathtt{S}_d(u)]$ is not a $\mathbf{m}$-abstraction of degree d, as required.
4. $t = s\, u$, where s is not a $\mathbf{m}$-abstraction of degree d: Then $\mathtt{S}_d(t) = \mathtt{S}_d(s)\, \mathtt{S}_d(u)$ is an application, hence not a $\mathbf{m}$-abstraction of degree d.
5. $t = s\boldsymbol{\{}u\boldsymbol{\}}$: Since t is not a $\mathbf{m}$-abstraction of degree d, we know that s is also not a $\mathbf{m}$-abstraction of degree d. By IH, $\mathtt{S}_d(s)$ is not a $\mathbf{m}$-abstraction of degree d, so $\mathtt{S}_d(t) = \mathtt{S}_d(s)\boldsymbol{\{}\mathtt{S}_d(u)\boldsymbol{\}}$ is not a $\mathbf{m}$-abstraction of degree d.

◀

▶ **Lemma 58** (The simplification of a term is normal, by degrees). *$\mathtt{S}_d(t)$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form.*

Proof. By induction on t, generalizing the statement also for memories, *i.e.* showing that $\mathtt{S}_d(\mathtt{L})$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form:

1. $t = x$: Then $\mathtt{S}_d(t) = x$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form.
2. $t = \lambda x.\, s$: Then $\mathtt{S}_d(t) = \lambda x.\, \mathtt{S}_d(s)$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form because $\mathtt{S}_d(s)$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form by IH.
3. $t = (\lambda x.\, s)\mathtt{L}\, u$, where $(\lambda x.\, s)\mathtt{L}$ is a $\mathbf{m}$-abstraction of degree d: Then $\mathtt{S}_d(t) = \mathtt{S}_d(s)[x := \mathtt{S}_d(u)]\{\mathtt{S}_d(u)\}\mathtt{S}_d(\mathtt{L})$. Note that, by IH, $\mathtt{S}_d(s)$, $\mathtt{S}_d(u)$, and $\mathtt{S}_d(\mathtt{L})$ are in $\xrightarrow{d}_{\mathbf{m}}$-normal form. Since $(\lambda x.\, s)\mathtt{L}$ is an abstraction of degree d, we know that $\mathsf{h}(\mathsf{type}(u)) < d$. Hence by the fact that a term reduces to its simplification (Lem. 38) and by subject reduction (Prop. 3) we know that $\mathsf{h}(\mathsf{type}(\mathtt{S}_d(u))) < d$. In particular, $\mathtt{S}_d(u)$ is not an abstraction of degree d. This allows us to apply Lem. 56(2) to conclude that $\mathtt{S}_d(s)[x := \mathtt{S}_d(u)]$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form. This lets us conclude that $\mathtt{S}_d(s)[x := \mathtt{S}_d(u)]\{\mathtt{S}_d(u)\}\mathtt{S}_d(\mathtt{L})$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form.
4. $t = s\, u$, where s is not a $\mathbf{m}$-abstraction of degree d. Then $\mathtt{S}_d(t) = \mathtt{S}_d(s)\, \mathtt{S}_d(u)$, where by IH we have that $\mathtt{S}_d(s)$ and $\mathtt{S}_d(u)$ are in $\xrightarrow{d}_{\mathbf{m}}$-normal form, and by Lem. 57 we have that $\mathtt{S}_d(s)$ is not a $\mathbf{m}$-abstraction of degree d. Hence $\mathtt{S}_d(t)$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form.
5. $t = s\{u\}$: Then $\mathtt{S}_d(t) = \mathtt{S}_d(s)\{\mathtt{S}_d(u)\}$ and we conclude by IH.
6. $\mathtt{L} = \Box$: Immediate, as $\mathtt{S}_d(\Box) = \Box$ is in $\xrightarrow{d}_{\mathbf{m}}$-normal form.
7. $\mathtt{L} = \mathtt{L}\{t\}$: Then $\mathtt{S}_d(\mathtt{L}) = \mathtt{S}_d(\mathtt{L})\{\mathtt{S}_d(t)\}$ and we conclude by IH.

◀

▶ **Lemma 59** (Reduction does not create redexes of higher degree). *Let $d \le D$ and suppose that $t \xrightarrow{d}_{\mathbf{m}} s$.*

1. *If t is not a $\mathbf{m}$-abstraction of degree D, then s is not a $\mathbf{m}$-abstraction of degree D.*
2. *If t is in $\xrightarrow{D}_{\mathbf{m}}$-normal form, then s is also in $\xrightarrow{D}_{\mathbf{m}}$-normal form.*

Proof. We prove the two items independently:

1. By induction on t:
   - **1.1** $t = x$: This case is impossible, as there are no reduction steps $t \xrightarrow{d}_{\mathbf{m}} s$.
   - **1.2** $t = \lambda x.\, t'$: Note that t is a $\mathbf{m}$-abstraction, so by IH it cannot be of degree D, that is, $\mathsf{h}(\mathsf{type}(t)) \neq D$. By subject reduction (Prop. 3) we have that $\mathsf{h}(\mathsf{type}(s)) = \mathsf{h}(\mathsf{type}(t)) \neq D$, so s cannot be a $\mathbf{m}$-abstraction of degree D.
   - **1.3** $t = t_1\, t_2$: We consider three subcases, depending on whether the reduction is at the root, internal to t_1, internal to t_2:
     - **1.3.1** If the reduction is at the root: Then $t_1 = (\lambda x.\, t_1')\mathtt{L}$ is an abstraction of degree d, and the step is of the form $t = (\lambda x.\, t_1')\mathtt{L}\, t_2 \xrightarrow{d}_{\mathbf{m}} t_1'[x := t_2]\{t_2\}\mathtt{L} = t'$. Note that $\lambda x.\, t_1'$ is an abstraction of degree d, so its type is of the form $A \to B$ with $\mathsf{h}(A \to B) = d$. The type of the body of the abstraction is $\mathsf{type}(t_1') = B$, so $\mathsf{h}(\mathsf{type}(t_1')) = \mathsf{h}(B) < d \le D$, and the type of the argument of the abstraction is $\mathsf{type}(t_2) = A$, so $\mathsf{h}(\mathsf{type}(t_2)) = \mathsf{h}(A) < d \le D$. This means that t_1' and t_2 cannot be $\mathbf{m}$-abstractions of degree D. Hence by Lem. 56(1) we have that $t_1'[x := t_2]$ is not a $\mathbf{m}$-abstractions of degree D. From this we conclude that $t' = t_1'[x := t_2]\{t_2\}\mathtt{L}$ is not a $\mathbf{m}$-abstractions of degree D.

1.3.2 If the reduction is internal to t_1: Then the step is of the form $t = t_1\,t_2 \xrightarrow{d}_{\mathbf{m}} s_1\,t_2 = s$ with $t_1 \xrightarrow{d}_{\mathbf{m}} s_1$. Note that s is an application, and hence not a **m**-abstraction of degree D.

1.3.3 If the reduction is internal to t_2: Then the step is of the form $t = t_1\,t_2 \xrightarrow{d}_{\mathbf{m}} t_1\,s_2 = s$ with $t_2 \xrightarrow{d}_{\mathbf{m}} s_2$. Note that s is an application, and hence not a **m**-abstraction of degree D.

1.4 $t = t_1\{t_2\}$: We consider two subcases, depending on whether the reduction is internal to t_1 or internal to t_2:

1.4.1 If the reduction is internal to t_1: Then the step is of the form $t = t_1\{t_2\} \xrightarrow{d}_{\mathbf{m}} s_1\{t_2\} = s$ with $t_1 \xrightarrow{d}_{\mathbf{m}} s_1$. By hypothesis t is not a **m**-abstraction of degree D, so t_1 is also not a **m**-abstraction of degree D. By IH s_1 is not a **m**-abstraction of degree D, so we conclude that $s = s_1\{t_2\}$ is not a **m**-abstraction of degree D.

1.4.2 If the reduction is internal to t_2: Then the step is of the form $t = t_1\{t_2\} \xrightarrow{d}_{\mathbf{m}} t_1\{s_2\} = s$ with $t_2 \xrightarrow{d}_{\mathbf{m}} s_2$. By hypothesis t is not a **m**-abstraction of degree D, so t_1 is also not a **m**-abstraction of degree D. Hence $s = t_1\{s_2\}$ is not a **m**-abstraction of degree D.

2. By induction on t:

2.1 $t = x$: This case is impossible, as there are no reduction steps $t \xrightarrow{d}_{\mathbf{m}} s$.

2.2 $t = \lambda x.\, t'$: Straightforward resorting to the IH.

2.3 $t = t_1\,t_2$: We consider three subcases, depending on whether the reduction is at the root, internal to t_1, internal to t_2:

2.3.1 If the reduction is at the root: Then $t_1 = (\lambda x.\, t_1')\mathtt{L}$ is an abstraction of degree d, and the step is of the form $t = (\lambda x.\, t_1')\mathtt{L}\,t_2 \xrightarrow{d}_{\mathbf{m}} t_1'[x := t_2]\{t_2\}\mathtt{L} = s$. Note that by hypothesis, $t = (\lambda x.\, t_1')\mathtt{L}\,t_2$ is in $\xrightarrow{D}_{\mathbf{m}}$-normal form, which means in particular that t_1', $\mathtt{L}$ and t_2 are in $\xrightarrow{D}_{\mathbf{m}}$-normal form. Moreover, since $\lambda x.\, t_1'$ is an abstraction of degree d, its type is of the form $A \to B$ with $\mathsf{h}(A \to B) = d$. Moreover, its argument t_2 is such that $\mathsf{type}(t_2) = A$, so $\mathsf{h}(\mathsf{type}(t_2)) = \mathsf{h}(A) < d \leq D$. In particular, t_2 cannot be an abstraction of degree D. By Lem. 56 this implies that $t_1'[x := t_2]$ is in $\xrightarrow{D}_{\mathbf{m}}$-normal form. Finally, this means that $t_1'[x := t_2]\{t_2\}\mathtt{L}$ must also be in $\xrightarrow{D}_{\mathbf{m}}$-normal form.

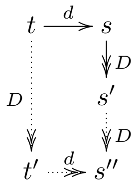
2.3.2 If the reduction is internal to t_1: Then the step is of the form $t = t_1\,t_2 \xrightarrow{d}_{\mathbf{m}} s_1\,t_2 = s$. By hypothesis $t = t_1\,t_2$ is in $\xrightarrow{D}_{\mathbf{m}}$-normal form, so we kwow that t_1 and t_2 must be in $\xrightarrow{D}_{\mathbf{m}}$-normal form and, moreover, that t_1 is not a **m**-abstraction of degree D. By IH, we have that s_1 is a $\xrightarrow{D}_{\mathbf{m}}$-normal form. Moreover, by item 1 of this lemma, we have that s_1 is not a **m**-abstraction of degree D. Hence we conclude that $s = s_1\,t_2$ is in $\xrightarrow{D}_{\mathbf{m}}$-normal form.

2.3.3 If the reduction is internal to t_2: Then the step is of the form $t = t_1\,t_2 \xrightarrow{d}_{\mathbf{m}} t_1\,s_2 = s$. By hypothesis $t = t_1\,t_2$ is in $\xrightarrow{D}_{\mathbf{m}}$-normal form, so we kwow that t_1 and t_2 must be in $\xrightarrow{D}_{\mathbf{m}}$-normal form and, moreover, that t_1 is not a **m**-abstraction of degree D. By IH, we have that s_2 is a $\xrightarrow{D}_{\mathbf{m}}$-normal form. Hence we conclude that $s = s_1\,t_2$ is in $\xrightarrow{D}_{\mathbf{m}}$-normal form.

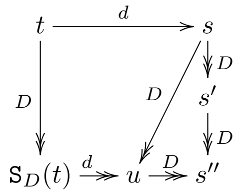
2.4 $t = t_1\{t_2\}$: Straightforward resorting to the IH.

◀

▶ **Proposition 60** (Lifting property for lower steps). *Let $d < D$ and suppose that $t \xrightarrow{d}_{\mathbf{m}} s \xrightarrow{D}{}^{*}_{\mathbf{m}} s'$. Then there exist a term t' and a term s'' such that $t \xrightarrow{D}{}^{*}_{\mathbf{m}} t'$ and $s' \xrightarrow{D}{}^{*}_{\mathbf{m}} s''$ and $t' \xrightarrow{d}{}^{+}_{\mathbf{m}} s''$ in at least one step. Graphically:*

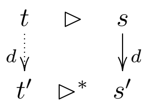


Proof. Take $t' := \mathtt{S}_D(t)$. By the fact that a term reduces to its simplification (Lem. 55) we have that $t \xrightarrow{D}{}^{*}_{\mathbf{m}} \mathtt{S}_D(t)$. Appyling commutation (Prop. 18) on the reduction sequences $t \xrightarrow{d}_{\mathbf{m}} s$ and $t \xrightarrow{d}{}^{*}_{\mathbf{m}} \mathtt{S}_D(t)$, we have that there exists a term u such that $s \xrightarrow{D}{}^{*}_{\mathbf{m}} u$ and $\mathtt{S}_D(t) \xrightarrow{d}{}^{+}_{\mathbf{m}} u$ in at least one step. Applying the commutation theorem again, this time on the reduction sequences $s \xrightarrow{D}{}^{*}_{\mathbf{m}} u$ and $s \xrightarrow{D}{}^{*}_{\mathbf{m}} s'$ we have that there exists a term s'' such that $u \xrightarrow{D}{}^{*}_{\mathbf{m}} s''$ and $s' \xrightarrow{D}{}^{*}_{\mathbf{m}} s''$. The situation is:



By Lem. 58 we know that $\mathtt{S}_D(t)$ is in $\xrightarrow{D}_{\mathbf{m}}$-normal form, and since $\mathtt{S}_D(t) \xrightarrow{d}{}^{*}_{\mathbf{m}} u$ with $d < D$, by Lem. 59 (2) we have that u is in $\xrightarrow{D}_{\mathbf{m}}$-normal form, so $u = s''$, which concludes the proof. ◀

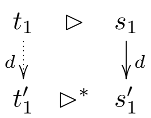
▶ **Lemma 61** (Local postponement of forgetful reduction). *If $R : t \rhd s$ is a forgetful step and $S : s \xrightarrow{d}_{\mathbf{m}} s'$ is a reduction step of degree d, there exists a term t', a forgetful reduction $R^{\frown}S : t' \rhd^{*} s'$ and a step $S^{\frown}R : t \xrightarrow{d}_{\mathbf{m}} t'$. Graphically:*

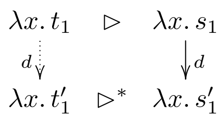


Furthermore, the step $S^{\frown}R$ determines the step S. More precisely, if $S^{\frown}R = T^{\frown}R$ then $S = T$.

Proof. By induction on t:

1. $t = x$: Impossible, as there are no reduction steps $x \rhd s$.
2. $t = \lambda x.\, t_1$: The steps must be of the form $R : t = \lambda x.\, t_1 \rhd \lambda x.\, s_1 = s$ with $t_1 \rhd s_1$, and $S : s = \lambda x.\, s_1 \xrightarrow{d}_{\mathbf{m}} \lambda x.\, s_1' = s'$ with $s_1 \xrightarrow{d}_{\mathbf{m}} s_1'$. By IH we have the diagram on the left, so we can construct the one on the right:





By IH, the step $t_1 \xrightarrow{d}_{\mathbf{m}} t_1'$ determines the step $s_1 \xrightarrow{d}_{\mathbf{m}} s_1'$, which implies that the step $\lambda x.\, t_1 \xrightarrow{d}_{\mathbf{m}} \lambda x.\, t_1'$ determines the step $\lambda x.\, s_1 \xrightarrow{d}_{\mathbf{m}} \lambda x.\, s_1'$.

3. If $t = t_1\, t_2$: We consider two subcases, depending on whether the step $R : t \triangleright s$ is internal to t_1 or internal to t_2.

3.1 If R is internal to t_1, then $R : t = t_1\, t_2 \triangleright s_1\, t_2 = s$ where $t_1 \triangleright s_1$. We consider three further subcases, depending on whether the step $S : s = s_1\, t_2 \xrightarrow{d}_{\mathbf{m}} s'$ is at the root, internal to s_1, or internal to t_2:

3.1.1 If the S is at the root of $s = s_1\, t_2$: Then s_1 is a $\mathbf{m}$-abstraction of degree d, *i.e.* of the form $s_1 = (\lambda x.\, s_{11})\mathtt{L}$, and S is of the form $S : (\lambda x.\, s_{11})\mathtt{L}\, t_2 \xrightarrow{d}_{\mathbf{m}} s_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L}$. We consider three subcases, depending on the form of the step $R_1 : t_1 \triangleright (\lambda x.\, s_{11})\mathtt{L}$:

3.1.1.1 If R_1 is of the form $t_1 = (\lambda x.\, t_{11})\mathtt{L} \triangleright (\lambda x.\, s_{11})\mathtt{L} = s_1$ where $t_{11} \triangleright s_{11}$, we can choose $t' := t_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L}$, according to the diagram:

$$\begin{array}{ccc} (\lambda x.\, t_{11})\mathtt{L}\, t_2 & \triangleright & (\lambda x.\, s_{11})\mathtt{L}\, t_2 \\ {\scriptstyle d}\,\vdots & & \downarrow{\scriptstyle d} \\ t_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L} & \triangleright & s_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L} \end{array}$$

Here we use the fact that $t_{11} \triangleright s_{11}$ implies $t_{11}[x := t_2] \triangleright s_{11}[x := t_2]$, as stated in Lem. 44.

3.1.1.2 If R_1 is of the form $t_1 = (\lambda x.\, s_{11})\mathtt{L}_1\boldsymbol{\{}t_3\boldsymbol{\}}\mathtt{L}_2 \triangleright (\lambda x.\, s_{11})\mathtt{L}_1\boldsymbol{\{}s_3\boldsymbol{\}}\mathtt{L}_2 = s_1$ with $t_3 \triangleright s_3$, we can choose $t' := t_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L}_1\boldsymbol{\{}t_3\boldsymbol{\}}\mathtt{L}_2$, according to the diagram:

$$\begin{array}{ccc} (\lambda x.\, t_{11})\mathtt{L}_1\boldsymbol{\{}t_3\boldsymbol{\}}\mathtt{L}_2\, t_2 & \triangleright & (\lambda x.\, t_{11})\mathtt{L}_1\boldsymbol{\{}s_3\boldsymbol{\}}\mathtt{L}_2\, t_2 \\ {\scriptstyle d}\,\vdots & & \downarrow{\scriptstyle d} \\ t_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L}_1\boldsymbol{\{}t_3\boldsymbol{\}}\mathtt{L}_2 & \triangleright & t_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L}_1\boldsymbol{\{}s_3\boldsymbol{\}}\mathtt{L}_2 \end{array}$$

3.1.1.3 If R_1 is of the form $t_1 = (\lambda x.\, s_{11})\mathtt{L}_1\boldsymbol{\{}t_3\boldsymbol{\}}\mathtt{L}_2 \triangleright (\lambda x.\, s_{11})\mathtt{L}_1\mathtt{L}_2 = s_1$, we can choose $t' := t_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L}_1\boldsymbol{\{}t_3\boldsymbol{\}}\mathtt{L}_2$, according to the diagram:

$$\begin{array}{ccc} (\lambda x.\, t_{11})\mathtt{L}_1\boldsymbol{\{}t_3\boldsymbol{\}}\mathtt{L}_2\, t_2 & \triangleright & (\lambda x.\, t_{11})\mathtt{L}_1\mathtt{L}_2\, t_2 \\ {\scriptstyle d}\,\vdots & & \downarrow{\scriptstyle d} \\ t_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L}_1\boldsymbol{\{}t_3\boldsymbol{\}}\mathtt{L}_2 & \triangleright & t_{11}[x := t_2]\boldsymbol{\{}t_2\boldsymbol{\}}\mathtt{L}_1\mathtt{L}_2 \end{array}$$

3.1.2 If S is internal to s_1: Then S must be of the form $S : s = s_1\, t_2 \xrightarrow{d}_{\mathbf{m}} s_1'\, t_2$ with $s_1 \xrightarrow{d}_{\mathbf{m}} s_1'$. By IH we have the diagram on the left, so we can construct the one on the right:

$$\begin{array}{ccc} t_1 & \triangleright & s_1 \\ {\scriptstyle d}\,\vdots & & \downarrow{\scriptstyle d} \\ t_1' & \triangleright^* & s_1' \end{array} \qquad \begin{array}{ccc} t_1\, t_2 & \triangleright & s_1\, t_2 \\ {\scriptstyle d}\,\vdots & & \downarrow{\scriptstyle d} \\ t_1'\, t_2 & \triangleright^* & s_1'\, t_2 \end{array}$$

3.1.3 If S is internal to t_2: Then S must be of the form $S : s = s_1\, t_2 \xrightarrow{d}_{\mathbf{m}} s_1\, t_2'$ with $t_2 \xrightarrow{d}_{\mathbf{m}} t_2'$. Then we can choose $t' := t_1\, t_2'$, according to the diagram:

$$\begin{array}{ccc} t_1\, t_2 & \triangleright & s_1\, t_2 \\ {\scriptstyle d}\,\vdots & & \downarrow{\scriptstyle d} \\ t_1\, t_2' & \triangleright & s_1\, t_2' \end{array}$$

3.2 If R is internal to t_2, then $R : t = t_1\, t_2 \rhd t_1\, s_2 = s$ where $t_2 \rhd s_2$. We consider three further subcases, depending on whether the step S is at the root, internal to t_1 or internal to s_2:

3.2.1 If S is at the root of $t_1\, s_2$: Then t_1 is a $\mathbf{m}$-abstraction of degree d, *i.e.* of the form $t_1 = (\lambda x.\, t_{11})\mathtt{L}$, and the step is of the form $S : s = (\lambda x.\, t_{11})\mathtt{L}\, s_2 \xrightarrow{d}_{\mathbf{m}} t_{11}[x := s_2]\{\!\!\{s_2\}\!\!\}\mathtt{L} = s'$. Then we can choose $t' := t_{11}[x := t_2]\{\!\!\{t_2\}\!\!\}\mathtt{L}$, according to the diagram:

$$\begin{array}{ccc} (\lambda x.\, t_{11})\mathtt{L}\, t_2 & \rhd & (\lambda x.\, t_{11})\mathtt{L}\, s_2 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ t_{11}[x := t_2]\{\!\!\{t_2\}\!\!\}\mathtt{L} & \rhd^* & t_{11}[x := s_2]\{\!\!\{s_2\}\!\!\}\mathtt{L} \end{array}$$

Here we use the fact that $t_2 \rhd s_2$ implies $t_{11}[x := t_2] \rhd^* t_{11}[x := s_2]$, as stated in Lem. 44.

3.2.2 If S is internal to t_1: Then $S : s = t_1\, s_2 \xrightarrow{d}_{\mathbf{m}} t'_1\, s_2 = s'$ with $t_1 \xrightarrow{d}_{\mathbf{m}} t'_1$, and we can choose $t' := t'_1\, t_2$, according to the diagram:

$$\begin{array}{ccc} t_1\, t_2 & \rhd & t_1\, s_2 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ t'_1\, t_2 & \rhd & t'_1\, s_2 \end{array}$$

3.2.3 If S is internal to s_2: Then $S : t_1\, s_2 \xrightarrow{d}_{\mathbf{m}} t_1\, s'_2$ with $s_2 \xrightarrow{d}_{\mathbf{m}} s'_2$. By IH we have the diagram on the left, so we can construct the one on the right:

$$\begin{array}{ccc} t_2 & \rhd & s_2 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ t'_2 & \rhd^* & s'_2 \end{array} \qquad \begin{array}{ccc} t_1\, t_2 & \rhd & t_1\, s_2 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ t_1\, t'_2 & \rhd^* & t_1\, s'_2 \end{array}$$

Furthermore, to see that the step $S^\frown R : t \xrightarrow{d}_{\mathbf{m}} t'$ determines the step $S : t \xrightarrow{d}_{\mathbf{m}} t'$, it suffices to note that there are no overlappings between the diagrams, *i.e.* if the step S and the step $S^\frown R$ are fixed, then no more than one of the cases above applies.

4. If $t = t_1\{\!\!\{t_2\}\!\!\}$: We consider three subcases, depending on whether the step $R : t \rhd s$ is at the root, internal to t_1 or internal to t_2:

4.1 If the step R is at the root: Then R is of the form $R : t = t_1\{\!\!\{t_2\}\!\!\} \rhd t_1 = s$ and S is of the form $S : s \xrightarrow{d}_{\mathbf{m}} s'$. Then we can choose $t' := s'\{\!\!\{t_2\}\!\!\}$, according to the diagram:

$$\begin{array}{ccc} s\, t_2 & \rhd & s \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ s'\{\!\!\{t_2\}\!\!\} & \rhd & s'\, t_2 \end{array}$$

4.2 If the step R is internal to t_1: Then R is of the form $R : t = t_1\{\!\!\{t_2\}\!\!\} \rhd s_1\{\!\!\{t_2\}\!\!\} = s$ with $t_1 \rhd s_1$. We consider two subcases, depending on whether the step S is internal to s_1 or internal to t_2:

4.2.1 If R is internal to s_1: Then $R : s = s_1\{\!\!\{t_2\}\!\!\} \xrightarrow{d}_{\mathbf{m}} s'_1\{\!\!\{t_2\}\!\!\} = s'$ with $s_1 \xrightarrow{d}_{\mathbf{m}} s'_1$. By IH we have the diagram on the left, so we can construct the one on the right:

$$\begin{array}{ccc} t_1 & \rhd & s_1 \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ t'_1 & \rhd^* & s'_1 \end{array} \qquad \begin{array}{ccc} t_1\{\!\!\{t_2\}\!\!\} & \rhd & t_1\{\!\!\{s_2\}\!\!\} \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ t_1\{\!\!\{t'_2\}\!\!\} & \rhd^* & t_1\{\!\!\{s'_2\}\!\!\} \end{array}$$

4.2.2 If R is internal to t_2: Then $R : s = s_1\{t_2\} \xrightarrow{d}_{\mathbf{m}} s_1\{t_2'\} = s'$ with $t_2 \xrightarrow{d}_{\mathbf{m}} t_2'$ and we can choose $t' = t_1\{t_2'\}$, according to the diagram:

$$\begin{array}{ccc} t_1\{t_2\} & \rhd & s_1\{t_2\} \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ t_1\{t_2'\} & \rhd & s_1\{t_2'\} \end{array}$$

4.3 If the step R is internal to t_2: Symmetric to the previous case.

Furthermore, to see that the step $S \curvearrowleft R : t \xrightarrow{d}_{\mathbf{m}} t'$ determines the step $S : t \xrightarrow{d}_{\mathbf{m}} t'$, it suffices to note that there are no overlappings between the diagrams, *i.e.* if the step S and the step $S \curvearrowleft R$ are fixed, then no more than one of the cases above applies.

◀

▶ **Proposition 62** (Postponement of forgetful reduction)**.** *Let $\rho : t \rhd^* t'$ be a forgetful reduction sequence and let $\sigma : t' \xrightarrow{d}{}^*_{\mathbf{m}} s'$ be a reduction sequence of degree d. Then there exist a term s and reduction sequences $\rho \curvearrowright \sigma : s \rhd^* s'$ and $\sigma \curvearrowleft \rho : t \xrightarrow{d}{}^*_{\mathbf{m}} s$. Graphically:*

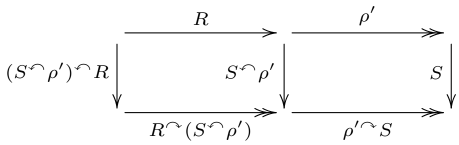
$$\begin{array}{ccc} t & \rhd^* & t' \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ s & \rhd^* & s' \end{array}$$

Furthermore, $\sigma \curvearrowleft \rho$ determines σ, that is, More precisely, $\sigma_1 \curvearrowleft \rho = \sigma_2 \curvearrowleft \rho$ then $\sigma_1 = \sigma_2$.

Proof. First, if $\rho : t \rhd^* t'$ is a forgetful reduction sequence and and $S : t' \xrightarrow{d}_{\mathbf{m}} s'$ is a single step of degree d, we can construct a forgetful reduction sequence $\rho \curvearrowright S$ and a step $S \curvearrowleft \rho$ of degree d by induction on ρ as follows, resorting to Lem. 61 for the constructions of $R \curvearrowright (S \curvearrowleft \rho')$ and $(S \curvearrowleft \rho') \curvearrowleft R$:

$$\begin{array}{rcl} \epsilon \curvearrowright S & \stackrel{\text{def}}{=} & \epsilon \\ (R\,\rho') \curvearrowright S & \stackrel{\text{def}}{=} & (R \curvearrowright (S \curvearrowleft \rho'))(\rho' \curvearrowright S) \end{array} \qquad \begin{array}{rcl} S \curvearrowleft \epsilon & \stackrel{\text{def}}{=} & S \\ S \curvearrowleft (R\,\rho') & \stackrel{\text{def}}{=} & (S \curvearrowleft \rho') \curvearrowleft R \end{array}$$

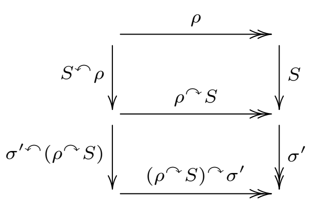
The inductive cases correspond to the following diagram:



For the general case, we proceed by induction on σ, resorting to the previous construction for the constructions of $\rho \curvearrowright S$ and $S \curvearrowleft \rho$:

$$\begin{array}{rcl} \rho \curvearrowright \epsilon & \stackrel{\text{def}}{=} & \rho \\ \rho \curvearrowright (S\,\sigma') & \stackrel{\text{def}}{=} & (\rho \curvearrowright S) \curvearrowright \sigma' \end{array} \qquad \begin{array}{rcl} \epsilon \curvearrowleft \rho & \stackrel{\text{def}}{=} & \epsilon \\ (S\,\sigma') \curvearrowleft \rho & \stackrel{\text{def}}{=} & (S \curvearrowleft \rho)(\sigma' \curvearrowleft (\rho \curvearrowright S)) \end{array}$$

The inductive cases correspond to the following diagram:



Furthermore, to see that $\sigma \curvearrowleft \rho$ determines σ, we proceed in three stages:

1. First, if $S_1^\frown R = S_2^\frown R$ then $S_1 = S_2$ by Lem. 61.
2. Second, by induction on ρ, it is easy to see that if $S_1^\frown \rho = S_2^\frown \rho$ then $S_1 = S_2$.
3. Finally, we can see that if $\sigma_1^\frown \rho = \sigma_2^\frown \rho$ then $\sigma_1 = \sigma_2$ by induction on σ_1. Note that $\sigma_1^\frown \rho = \sigma_2^\frown \rho$ then σ_1 and σ_2 are either both empty or both non-empty. The base case is immediate. For the induction step, we have that $\sigma_1 = S_1\,\sigma_1'$ and $\sigma_2 = S_2\,\sigma_2'$; then note that if $(S_1\,\sigma_1')^\frown \rho = (S_2\,\sigma_2')^\frown \rho$ then by definition $(S_1^\frown \rho)\,(\sigma_1'^\frown(\rho^\frown S)) = (S_2^\frown \rho)\,(\sigma_2'^\frown(\rho^\frown S))$ so we have that $S_1^\frown \rho = S_2^\frown \rho$ which, resorting to the previous item, means that $S_1 = S_2$, and we also have that $\sigma_1'^\frown(\rho^\frown S) = \sigma_2'^\frown(\rho^\frown S)$ which by IH implies $\sigma_1' = \sigma_2'$.

◀

A.3 Proofs of Section 4 — The $\mathcal{T}^{\mathbf{m}}$-measure

In this section we give detailed proofs of the results about reduction by degrees stated in Section 5.

▶ **Lemma 63** (Properties of the pointwise multiset order).
1. *If* $\mathfrak{m}_1 :\succ: \mathfrak{n}_1$ *and* $\mathfrak{m}_2 :\succ: \mathfrak{n}_2$ *then* $\mathfrak{m}_1 + \mathfrak{m}_2 :\succ: \mathfrak{n}_1 + \mathfrak{n}_2$.
2. *If* $\mathfrak{m} :\succ: \mathfrak{n}$ *then for all* $k \in \mathbb{N}_0$ *we have that* $\mathfrak{m} \succeq k \otimes \mathfrak{n}$. *In particular, taking* $k = 1$, $\mathfrak{m} \succeq \mathfrak{n}$.
3. *If* $\mathfrak{m} :\succ: \mathfrak{n}$ *and* $\mathfrak{m}$ *is non-empty then* $\mathfrak{m} \succ \mathfrak{n}$.

Proof. The first item is straightforward. For the second item, suppose that $\mathfrak{m} :\succ: \mathfrak{n}$ and proceed by induction on the cardinality of $\mathfrak{m}$. If $\mathfrak{m}$ is empty, then $\mathfrak{m} = [\,] = \mathfrak{n}$, so $\mathfrak{m} = [\,] \succeq [\,] = k \otimes [\,] = k \otimes \mathfrak{n}$. If $\mathfrak{m}$ is non-empty, then we can write $\mathfrak{m} = [x] + \mathfrak{m}'$ and $\mathfrak{n} = [y] + \mathfrak{n}'$ in such a way that $x > y$ and $\mathfrak{m}' :\succ: \mathfrak{n}'$. By IH we have that $\mathfrak{m}' \succeq k \otimes \mathfrak{n}'$, so $\mathfrak{m} = [x] + \mathfrak{m}' \succ k \otimes [y] + \mathfrak{m}' \succeq k \otimes [y] + k \otimes \mathfrak{n}' = k \otimes ([y] + \mathfrak{n}') = k \otimes \mathfrak{n}$. The third item is similar to the second. ◀

▶ **Lemma 64** (Higher substitution lemma). *Let t, s be typable terms and let x be a variable. Then* $\mathcal{T}^{\mathbf{m}}_d(t_0, t) \preceq \mathcal{T}^{\mathbf{m}}_d(t_0, t[x := s])$.

Proof. We generalize the lemma for the case in which t may also be a memory. That is, we prove that if $\mathtt{X}$ is a term or a memory, s is a term, and x is a variable then $\mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{X}) \preceq \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{X}[x := s])$. We proceed by induction on $\mathtt{X}$:
1. $t = x$: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, x) = [\,] \preceq \mathcal{T}^{\mathbf{m}}_d(t_0, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, x[x := s])$.
2. $t = y \neq x$: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, y) \preceq \mathcal{T}^{\mathbf{m}}_d(t_0, y) = \mathcal{T}^{\mathbf{m}}_d(t_0, y[x := s])$.
3. $t = \lambda y.\, t'$: By α-conversion, we may assume that $y \notin \{x\} \cup \mathsf{fv}(s)$. Then $\mathcal{T}^{\mathbf{m}}_d(t_0, \lambda y.\, t') = \mathcal{T}^{\mathbf{m}}_d(t_0, t') \preceq \mathcal{T}^{\mathbf{m}}_d(t_0, t'[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda y.\, t')[x := s])$ by IH.
4. If $t = (\lambda x.\, t_1)\mathtt{L}\, t_2$ is a redex of degree d: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda x.\, t_1)\mathtt{L}\, t_2) = \mathcal{T}^{\mathbf{m}}_d(t_0, t_1) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, t_2) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] \preceq \mathcal{T}^{\mathbf{m}}_d(t_0, t_1[x := s]) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}[x := s]) + \mathcal{T}^{\mathbf{m}}_d(t_0, t_2[x := s]) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] = \mathcal{T}^{\mathbf{m}}_d(t_0, ((\lambda y.\, t_1)\mathtt{L}\, t_2)[x := s])$ by IH.
5. If $t = t_1\, t_2$ is not a redex of degree d: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, t_1\, t_2) = \mathcal{T}^{\mathbf{m}}_d(t_0, t_1) + \mathcal{T}^{\mathbf{m}}_d(t_0, t_2) \preceq \mathcal{T}^{\mathbf{m}}_d(t_0, t_1[x := s]) + \mathcal{T}^{\mathbf{m}}_d(t_0, t_2[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, (t_1\, t_2)[x := s])$ by IH.
6. $t = t_1\{t_2\}$: Similar to case 5.
7. $\mathtt{L} = \Box$: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, \Box) \preceq \mathcal{T}^{\mathbf{m}}_d(t_0, \Box) = \mathcal{T}^{\mathbf{m}}_d(t_0, \Box[x := s])$.
8. $\mathtt{L} = \mathtt{L}_1\{t\}$: Similar to case 5.

◀

▶ **Proposition 65** (High/increase). *Let $D \in \mathbb{N}_0$. Then the following hold:*
1. *If* $1 \leq d < D$ *and* $t \xrightarrow{D}_{\mathbf{m}} t'$ *then* $\mathcal{R}^{\mathbf{m}}_d(t) \preceq \mathcal{R}^{\mathbf{m}}_d(t')$.
2. *If* $0 \leq d < D$ *and* $t_0 \xrightarrow{D}_{\mathbf{m}} t_0'$ *then* $\mathcal{T}^{\mathbf{m}}_d(t_0, t) \preceq \mathcal{T}^{\mathbf{m}}_d(t_0', t)$.

3. *If* $0 \leq d < D$ *and* $t_0 \xrightarrow{D}_{\mathbf{m}} t'_0$ *and* $t \xrightarrow{D}_{\mathbf{m}} t'$ *then* $\mathcal{T}^{\mathbf{m}}_d(t_0, t) \preceq \mathcal{T}^{\mathbf{m}}_d(t'_0, t')$.
4. *If* $0 \leq d < D$ *and* $t \xrightarrow{D}_{\mathbf{m}} t'$ *then* $\mathcal{T}^{\mathbf{m}}_{\leq d}(t) \preceq \mathcal{T}^{\mathbf{m}}_{\leq d}(t')$.

Proof. We prove a more general version of the statement: in items 2 and 3 we allow t to be either a term or a memory. For example, the statement of item 2 is generalized as follows: if $1 \leq d < D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t'_0$ then $\mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{X}) \preceq \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{X})$, where $\mathtt{X}$ is either a term or a memory.

We prove all items simultaneously by induction on d. Note that: item 1. resorts to the IH; item 2. resorts to item 1. (without decreasing d); item 3. resorts to items 1. and 2. (without decreasing d); item 4. resorts to item 3. (without necessarily decreasing d).

1. Let $1 \leq d < D$ and $t \xrightarrow{D}_{\mathbf{m}} t'$. We argue that $\mathcal{R}^{\mathbf{m}}_d(t) \preceq \mathcal{R}^{\mathbf{m}}_d(t')$. Let X and Y be the sets of reduction sequences $X = \{\rho \mid (\exists s)\ \rho : t \xrightarrow{d}^{*}_{\mathbf{m}} s\}$ and $Y = \{\sigma \mid (\exists s')\ \sigma : t' \xrightarrow{d}^{*}_{\mathbf{m}} s'\}$. Note that, by definition, $\mathcal{R}^{\mathbf{m}}_d(t) = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in X]$ and $\mathcal{R}^{\mathbf{m}}_d(t') = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y]$. We construct a function $\varphi : X \to Y$ as follows. Consider a reduction step $R : t \xrightarrow{D}_{\mathbf{m}} t'$; note that there may be more than one such step, but we know by hypothesis that there is at least one. By commutation (Prop. 18), given a reduction sequence $\rho \in X$, *i.e.* $\rho : t \xrightarrow{d}^{*}_{\mathbf{m}} s$ there exists a term s'_ρ and reduction sequences $\rho/R : t' \xrightarrow{d}^{*}_{\mathbf{m}} s'_\rho$ and $R/\rho : s \xrightarrow{D}^{*}_{\mathbf{m}} s'_\rho$. In particular, $\rho/R \in Y$, and we can define $\varphi(\rho) := \rho/R$. Moreover, φ is injective, because if $\rho_1, \rho_2 \in X$ are such that $\rho_1/R = \rho_2/R$ then by commutation (Prop. 18) we have that $\rho_1 = \rho_2$, given that $d < D$.
First, we claim that $\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \preceq \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}})$ for every $\rho \in X$. If $d = 1$, this is immediate since $\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) = \mathcal{T}^{\mathbf{m}}_{\leq 0}(\rho^{\mathsf{tgt}}) = [\,] = \mathcal{T}^{\mathbf{m}}_{\leq 0}(\varphi(\rho)^{\mathsf{tgt}}) = \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}})$. Assume now that $d > 1$. Then we have that:
$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) & = & \mathcal{T}^{\mathbf{m}}_{\leq d-1}(s) & \\ & \preceq & \mathcal{T}^{\mathbf{m}}_{\leq d-1}(s'_\rho) & \text{by item 4 of the IH} \\ & = & \mathcal{T}^{\mathbf{m}}_{\leq d-1}((\rho/R)^{\mathsf{tgt}}) & \\ & = & \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)) & \end{array}$$
To be able to apply item 4 of the IH, observe that $1 \leq d-1 < D$ holds because $1 \leq d < D$. We resort to the IH as many times as the length of the reduction $s \xrightarrow{D}^{*}_{\mathbf{m}} s'_\rho$. To conclude the proof, let $Z = Y \setminus \varphi(X)$, so that $Y = \varphi(X) \uplus Z$, and note that:
$$\begin{array}{rcll} \mathcal{R}^{\mathbf{m}}_d(t) & = & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in X] & \\ & \preceq & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}}) \mid\mid \rho \in X] & (\star) \\ & = & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in \varphi(X)] & (\star\star) \\ & \preceq & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in \varphi(X)] + [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Z] & \\ & = & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in \varphi(X) \uplus Z] & \\ & = & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y] & \\ & = & \mathcal{R}^{\mathbf{m}}_d(t') & \end{array}$$
To justify the step marked with $(\star)$, note that $[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in X] = \sum_{\rho \in X} [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}})] \preceq \sum_{\rho \in X} [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}})] = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}}) \mid\mid \rho \in X]$ because $\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \preceq \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\rho)^{\mathsf{tgt}})$, as we have already claimed. To justify the step marked with $(\star\star)$, note that φ is injective, so X and $\varphi(X)$ have the same cardinality.
2. Let $0 \leq d < D$ and $t_0 \xrightarrow{D}_{\mathbf{m}} t'_0$. We argue that $\mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{X}) \preceq \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{X})$, where $\mathtt{X}$ is either a term ($\mathtt{X} = t$) or a memory ($\mathtt{X} = \mathtt{L}$). We proceed by induction on $\mathtt{X}$:
   - **2.1** $t = x$: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, x) = [\,] \preceq [\,] = \mathcal{T}^{\mathbf{m}}_d(t'_0, x)$.
   - **2.2** $t = \lambda x.\, s$: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, \lambda x.\, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, s) \preceq \mathcal{T}^{\mathbf{m}}_d(t'_0, s) = \mathcal{T}^{\mathbf{m}}_d(t'_0, \lambda x.\, s)$ by the internal IH.

2.3 If $t = (\lambda x.\, s)\mathtt{L}\, u$ is a redex of degree d: Then:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_d(t_0, t) & = & \mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda x.\, s)\mathtt{L}\, u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0, s) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] & \\ & \preceq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] & \text{by the internal IH} \\ & \preceq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t'_0))] & \text{by item 1} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, (\lambda x.\, s)\mathtt{L}\, u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, t) & \end{array}$$

2.4 If $t = s\, u$ is not a redex of degree d: Then:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_d(t_0, t) & = & \mathcal{T}^{\mathbf{m}}_d(t_0, s\, u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0, s) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) & \\ & \preceq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) & \text{by the internal IH} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, s\, u) & \end{array}$$

2.5 $t = s\{u\}$: Similar to case 2.4.

2.6 $\mathtt{L} = \Box$: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, \Box) = [\,] \preceq [\,] = \mathcal{T}^{\mathbf{m}}_d(t'_0, \Box)$.

2.7 $\mathtt{L} = \mathtt{L}_1\{t\}$: Similar to case 2.4.

3. Let $0 \leq d < D$ and $t_0 \xrightarrow{D}_{\mathbf{m}} t'_0$ and let $\mathtt{X}, \mathtt{X}'$ be and $\mathtt{X} \xrightarrow{D}_{\mathbf{m}} \mathtt{X}'$ where $\mathtt{X}, \mathtt{X}'$ are either terms ($\mathtt{X} = t$ and $\mathtt{X} = t'$) or memories ($\mathtt{X} = \mathtt{L}$ and $\mathtt{X} = \mathtt{L}'$). We argue that $\mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{X}) \preceq \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{X}')$. We proceed by induction on $\mathtt{X}$:

3.1 $t = x$: Impossible, as there are no steps $x \xrightarrow{D}_{\mathbf{m}} t'$.

3.2 $t = \lambda x.\, s$: Then the step is of the form $t = \lambda x.\, s \xrightarrow{D}_{\mathbf{m}} \lambda x.\, s' = t'$ with $s \xrightarrow{D}_{\mathbf{m}} s'$, so $\mathcal{T}^{\mathbf{m}}_d(t_0, \lambda x.\, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, s) \preceq \mathcal{T}^{\mathbf{m}}_d(t'_0, s') = \mathcal{T}^{\mathbf{m}}_d(t'_0, \lambda x.\, s')$ by the internal IH.

3.3 If $t = (\lambda x.\, s)\mathtt{L}\, u$ is the redex of degree D contracted by the step $t \xrightarrow{D}_{\mathbf{m}} t'$: Then the step is of the form $t = (\lambda x.\, s)\mathtt{L}\, u \xrightarrow{D}_{\mathbf{m}} s[x := u]\{u\}\mathtt{L} = t'$. Note that t is not a redex of degree d because $d < D$, so:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_d(t_0, t) & = & \mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda x.\, s)\mathtt{L}\, u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0, s) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) & \\ & \preceq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) & \text{by item 2} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) & \\ & \preceq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s[x := u]) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) & \text{by Lem. 28} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, s[x := u]\{u\}\mathtt{L}) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, t') & \end{array}$$

3.4 If $t = (\lambda x.\, s)\mathtt{L}\, u$ is a redex of degree d: Note that t is not a redex of degree D because $d < D$. We consider three subcases, depending on whether the step $t \xrightarrow{D}_{\mathbf{m}} t'$ is internal to s, internal to $\mathtt{L}$, or internal to u. All these subcases are similar; we only give the proof for the case in which the step is internal to s. Then:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_d(t_0, t) & = & \mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda x.\, s)\mathtt{L}\, u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0, s) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] & \\ & \preceq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s') + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] & \text{by the internal IH} \\ & \preceq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s') + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] & \text{by item 2} \\ & \preceq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s') + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t'_0))] & \text{by item 1} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, (\lambda x.\, s')\mathtt{L}\, u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, t') & \end{array}$$

3.5 If $t = s\,u$ is not the redex contracted by the step $t \xrightarrow{D}_{\mathbf{m}} t'$ nor a redex of degree d: We consider two subcases, depending on whether the step $t \xrightarrow{D}_{\mathbf{m}} t'$ is internal to s or internal to u:

3.5.1 If the step is internal to s, then the step is of the form $t = s\,u \xrightarrow{D}_{\mathbf{m}} s'\,u = t'$ with $s \xrightarrow{D}_{\mathbf{m}} s'$. We know that s is not a $\mathbf{m}$-abstraction of degree d, but $d < D$, so it may be the case that s' is a $\mathbf{m}$-abstraction of degree d, *i.e.* reduction at degree D may create an abstraction of degree $d < D$. We consider two further subcases, depending on whether s' is a $\mathbf{m}$-abstraction of degree d or not:

3.5.1.1 If $s' = (\lambda x.\,s'')\mathtt{L}$ is a $\mathbf{m}$-abstraction of degree d, then:

$$
\begin{array}{rcll}
\mathcal{T}^{\mathbf{m}}_{d}(t_0, t) & = & \mathcal{T}^{\mathbf{m}}_{d}(t_0, s\,u) & \\
& = & \mathcal{T}^{\mathbf{m}}_{d}(t_0, s) + \mathcal{T}^{\mathbf{m}}_{d}(t_0, u) & \\
& \preceq & \mathcal{T}^{\mathbf{m}}_{d}(t'_0, (\lambda x.\,s'')\mathtt{L}) + \mathcal{T}^{\mathbf{m}}_{d}(t_0, u) & \text{by the internal IH} \\
& \preceq & \mathcal{T}^{\mathbf{m}}_{d}(t'_0, (\lambda x.\,s'')\mathtt{L}) + \mathcal{T}^{\mathbf{m}}_{d}(t'_0, u) & \text{by item 2} \\
& \preceq & \mathcal{T}^{\mathbf{m}}_{d}(t'_0, (\lambda x.\,s'')\mathtt{L}) + \mathcal{T}^{\mathbf{m}}_{d}(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_{d}(t'_0))] & \\
& = & \mathcal{T}^{\mathbf{m}}_{d}(t'_0, (\lambda x.\,s'')\mathtt{L}\,u) & \\
& = & \mathcal{T}^{\mathbf{m}}_{d}(t'_0, t') &
\end{array}
$$

3.5.1.2 If s' is not a $\mathbf{m}$-abstraction of degree d, then:

$$
\begin{array}{rcll}
\mathcal{T}^{\mathbf{m}}_{d}(t_0, t) & = & \mathcal{T}^{\mathbf{m}}_{d}(t_0, s\,u) & \\
& = & \mathcal{T}^{\mathbf{m}}_{d}(t_0, s) + \mathcal{T}^{\mathbf{m}}_{d}(t_0, u) & \\
& \preceq & \mathcal{T}^{\mathbf{m}}_{d}(t'_0, s') + \mathcal{T}^{\mathbf{m}}_{d}(t_0, u) & \text{by the internal IH} \\
& \preceq & \mathcal{T}^{\mathbf{m}}_{d}(t'_0, s') + \mathcal{T}^{\mathbf{m}}_{d}(t'_0, u) & \text{by item 2} \\
& = & \mathcal{T}^{\mathbf{m}}_{d}(t'_0, s'\,u) & \\
& = & \mathcal{T}^{\mathbf{m}}_{d}(t'_0, t') &
\end{array}
$$

3.5.2 If the step is internal to u: Similar to case 3.5.1.2.

3.6 $t = s\{u\}$: Similar to case 3.5.

3.7 $\mathtt{L} = \Box$: Impossible, as there are no reduction steps $\Box \xrightarrow{D}_{\mathbf{m}} \mathtt{L}'$.

3.8 $\mathtt{L} = \mathtt{L}_1\{t\}$: Similar to case 3.5.

4. Let $0 \leq d < D$ and $t \xrightarrow{D}_{\mathbf{m}} t'$. We argue that $\mathcal{T}^{\mathbf{m}}_{\leq d}(t) \preceq \mathcal{T}^{\mathbf{m}}_{\leq d}(t')$. Indeed:

$$
\begin{array}{rcll}
\mathcal{T}^{\mathbf{m}}_{\leq d}(t) & = & \sum_{i=1}^{d} \mathcal{T}^{\mathbf{m}}_{i}(t, t) & \\
& \preceq & \sum_{i=1}^{d} \mathcal{T}^{\mathbf{m}}_{i}(t', t') & \text{by item 3, resorting to the IH when } i < d \\
& = & \mathcal{T}^{\mathbf{m}}_{\leq d}(t') &
\end{array}
$$

Note that for the value $i = d$, we resort directly to item 3 and not to the IH.

◀

▶ **Lemma 66** (Substitution of degree d does not create abstractions). *If t and s are not $\mathbf{m}$-abstractions of degree d, then $t[x := s]$ is not a $\mathbf{m}$-abstraction of degree d.*

Proof. By induction on t:

1. $t = x$: Then $t[x := s] = s$ is not a $\mathbf{m}$-abstraction of degree d.
2. $t = y \neq x$: Then $t[x := s] = y$ is not a $\mathbf{m}$-abstraction of degree d.
3. $t = \lambda y.\,t'$: Then t is a $\mathbf{m}$-abstraction but, by hypothesis, we know that it cannot be of degree d. Hence $t[x := s] = \lambda y.\,t'[x := s]$. By the substitution lemma (Lem. 34) t and $t[x := s]$ have the same type, so $\lambda y.\,t'[x := s]$ is a $\mathbf{m}$-abstraction but it is not of degree d.

4. $t = t_1\, t_2$: Then $t[x := s] = t_1[x := s]\, t_2[x := s]$ is an application, hence not a $\mathbf{m}$-abstraction of degree d.
5. $t = t_1\{t_2\}$: Since t is not a $\mathbf{m}$-abstraction of degree d, we have that t_1 is also not a $\mathbf{m}$-abstraction of degree d. By IH, $t_1[x := s]$ is not a $\mathbf{m}$-abstraction of degree d. So $t[x := s] = t_1[x := s]\{t_2[x := s]\}$ is not a $\mathbf{m}$-abstraction of degree d.

◀

▶ **Lemma 67** (Lower substitution lemma). *Let t, s be typable terms and let x be a variable such that s is not a $\mathbf{m}$-abstraction of degree d. Then there exists $k \in \mathbb{N}_0$ such that $\mathcal{T}^{\mathbf{m}}_d(t_0, t[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, t) + k \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)$.*

Proof. We generalize the lemma for the case in which t may also be a memory. That is, we prove that if $\mathtt{X}$ is a term or a memory, s is a term, and x is a variable such that s is not a $\mathbf{m}$-abstraction of degree d, then there exists $k \in \mathbb{N}_0$ such that $\mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{X}[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{X}) + k \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)$. We proceed by induction on $\mathtt{X}$:

1. $t = x$: Take $k := 1$. Then $\mathcal{T}^{\mathbf{m}}_d(t_0, t[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, x[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, s) = [\,] + 1 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, x) + 1 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, t) + 1 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)$.
2. $t = y \neq x$: Take $k := 0$. Then $\mathcal{T}^{\mathbf{m}}_d(t_0, t[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, y[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, y) = \mathcal{T}^{\mathbf{m}}_d(t_0, y) + 0 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, t) + 0 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)$.
3. $t = \lambda y.\, t'$: By α-conversion we may assume that $y \notin \{x\} \cup \mathsf{fv}(s)$. Resorting to the IH, we have $\mathcal{T}^{\mathbf{m}}_d(t_0, t[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, \lambda y.\, t'[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, t'[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, t') + k \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, \lambda y.\, t') + k \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, t) + k \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)$.
4. $t = (\lambda x.\, t_1)\mathtt{L}\, t_2$ where $(\lambda x.\, t_1)\mathtt{L}$ is a $\mathbf{m}$-abstraction of degree d. Note that, by the substitution lemma (Lem. 34) we have that $(\lambda x.\, t_1[x := s])(\mathtt{L}[x := s])$ is also an abstraction of degree d. Then by IH there exist $k_1, k_2, k_3 \in \mathbb{N}_0$ such that:

$$\begin{array}{rl} & \mathcal{T}^{\mathbf{m}}_d(t_0, t[x := s]) \\ = & \mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda x.\, t_1[x := s])(\mathtt{L}[x := s])(t_2[x := s])) \\ = & \mathcal{T}^{\mathbf{m}}_d(t_0, \lambda x.\, t_1[x := s]) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}[x := s]) + \mathcal{T}^{\mathbf{m}}_d(t_0, t_2[x := s]) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] \\ = & (\mathcal{T}^{\mathbf{m}}_d(t_0, \lambda x.\, t_1) + k_1 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)) + (\mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + k_2 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)) \\ & +\ (\mathcal{T}^{\mathbf{m}}_d(t_0, t_2) + k_3 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] \\ = & \mathcal{T}^{\mathbf{m}}_d(t_0, \lambda x.\, t_1) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, t_2) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] + (k_1 + k_2 + k_3) \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s) \\ = & \mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda x.\, t_1)\mathtt{L}\, t_2) + (k_1 + k_2 + k_3) \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s) \\ = & \mathcal{T}^{\mathbf{m}}_d(t_0, t) + (k_1 + k_2 + k_3) \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s) \end{array}$$

   So taking $k := k_1 + k_2 + k_3$ we are done.
5. $t = t_1\, t_2$ where t_1 is not a $\mathbf{m}$-abstraction of degree d: Note by Lem. 66 that $t_1[x := t_2]$ is not a $\mathbf{m}$-abstraction of degree d. Then by IH there exist $k_1, k_2 \in \mathbb{N}_0$ such that:

$$\begin{array}{rcl} \mathcal{T}^{\mathbf{m}}_d(t_0, t[x := s]) & = & \mathcal{T}^{\mathbf{m}}_d(t_0, t_1[x := s]\, t_2[x := s]) \\ & & \mathcal{T}^{\mathbf{m}}_d(t_0, t_1[x := s]) + \mathcal{T}^{\mathbf{m}}_d(t_0, t_2[x := s]) \\ & = & (\mathcal{T}^{\mathbf{m}}_d(t_0, t_1) + k_1 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)) + (\mathcal{T}^{\mathbf{m}}_d(t_0, t_2) + k_2 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)) \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0, t_1) + \mathcal{T}^{\mathbf{m}}_d(t_0, t_2) + (k_1 + k_2) \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)) \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0, t_1\, t_2) + (k_1 + k_2) \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)) \end{array}$$

   So taking $k := k_1 + k_2$ we are done.
6. $t = t_1\{t_2\}$: Similar to the previous case.
7. $\mathtt{L} = \Box$: Take $k := 0$. Then $\mathcal{T}^{\mathbf{m}}_d(t_0, \Box[x := s]) = \mathcal{T}^{\mathbf{m}}_d(t_0, \Box) = \mathcal{T}^{\mathbf{m}}_d(t_0, \Box) + 0 \otimes \mathcal{T}^{\mathbf{m}}_d(t_0, s)$.
8. $\mathtt{L} = \mathtt{L}'\{t\}$: Similar to case 5.

◀

▶ **Proposition 68** (Low/decrease). *Let $D \in \mathbb{N}_0$. Then the following hold:*

1. *If $1 \leq d \leq j \leq D$ and $t \xrightarrow{d}_{\mathbf{m}} t'$ then $\mathcal{R}^{\mathbf{m}}_j(t) \succ \mathcal{R}^{\mathbf{m}}_j(t')$.*
2. *If $1 \leq d \leq j \leq D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t'_0$ then $\mathcal{T}^{\mathbf{m}}_j(t_0, t) :\succ: \mathcal{T}^{\mathbf{m}}_j(t'_0, t)$.*
3. *If $1 \leq d \leq D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t'_0$ and $t \xrightarrow{d}_{\mathbf{m}} t'$, then for all $\mathfrak{m} \in \mathbb{T}_{d-1}$ we have $\mathcal{T}^{\mathbf{m}}_d(t_0, t) \succ \mathcal{T}^{\mathbf{m}}_d(t'_0, t') + \mathfrak{m}$.*
4. *If $1 \leq d < j \leq D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t'_0$ and $t \xrightarrow{d}_{\mathbf{m}} t'$ then $\mathcal{T}^{\mathbf{m}}_j(t_0, t) \succeq \mathcal{T}^{\mathbf{m}}_j(t'_0, t')$.*
5. *If $1 \leq d \leq D$ and $t \xrightarrow{d}_{\mathbf{m}} t'$ then $\mathcal{T}^{\mathbf{m}}_{\leq D}(t) \succ \mathcal{T}^{\mathbf{m}}_{\leq D}(t')$.*

Proof. We prove a more general version of the statement: in items 2, 3, and 4 we allow t to be either a term or a memory. For example, the statement of item 2 is generalized as follows: if $1 \leq d \leq j \leq D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t'_0$ then $\mathcal{T}^{\mathbf{m}}_j(t_0, \mathtt{X}) :\succ: \mathcal{T}^{\mathbf{m}}_j(t'_0, \mathtt{X})$, where $\mathtt{X}$ is either a term or a memory.

We prove all items simultaneously by induction on D. Note that: item 1. resorts to the IH; item 2. resorts to item 1. (without decreasing D); items 3. and 4. resort to items 1. and 2. (without decreasing D); item 5. resorts to items 3. and 4. (without decreasing D).

1. Let $1 \leq d \leq j \leq D$ and $t \xrightarrow{d}_{\mathbf{m}} t'$. We argue that $\mathcal{R}^{\mathbf{m}}_j(t) \succ \mathcal{R}^{\mathbf{m}}_j(t')$. Let X and Y be the sets of reduction sequences $X := \{\rho \mid (\exists s)\ \rho : t \xrightarrow{j}^{*}_{\mathbf{m}} s\}$ and $Y := \{\sigma \mid (\exists s')\ \sigma : t' \xrightarrow{j}^{*}_{\mathbf{m}} s'\}$. Note that, by definition, $\mathcal{R}^{\mathbf{m}}_j(t) = [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in X]$ and $\mathcal{R}^{\mathbf{m}}_j(t') = [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y]$. We consider two subcases, depending on whether $d = j$ or $d < j$:

   1.1 If $d = j$, let R be the step $R : t \xrightarrow{d}_{\mathbf{m}} t'$. We construct a function $\varphi : Y \to X$ given by $\varphi(\sigma) = R\sigma$. Observe that φ is injective and that if $\sigma : t' \xrightarrow{d}^{*}_{\mathbf{m}} s'$ then $R\sigma : t \xrightarrow{d}^{*}_{\mathbf{m}} s'$, and in particular σ and $\varphi(\sigma)$ have the same target. Let $Z = X \setminus \varphi(Y)$, so that $X = \varphi(Y) \uplus Z$. Note that:

$$
\begin{array}{rcll}
\mathcal{R}^{\mathbf{m}}_j(t) & = & [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in X] & \\
& = & [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in \varphi(Y) \uplus Z] & \\
& = & [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in \varphi(Y)] + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z] & \\
& = & [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(R\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y] + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z] & \text{since } \varphi \text{ is injective} \\
& = & [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y] + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z] & \\
& = & \mathcal{R}^{\mathbf{m}}_j(t') + [\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z] &
\end{array}
$$

   By this chain of equations, in order to conclude that $\mathcal{R}^{\mathbf{m}}_j(t) \succ \mathcal{R}^{\mathbf{m}}_j(t')$, it suffices to show that Z is non-empty. Indeed, let $\epsilon : t \xrightarrow{d}^{*}_{\mathbf{m}} t$ be the empty reduction sequence. Then $\epsilon \in X \setminus \varphi(Y)$, so $\epsilon \in Z$.

   1.2 If $d < j$, we construct a function $\varphi : Y \to X$ using Prop. 20. More precisely, since $d < j$ and $t \xrightarrow{d}_{\mathbf{m}} t'$, for each reduction sequence $\sigma : t' \xrightarrow{j}^{*}_{\mathbf{m}} s'$ by Prop. 20 there exist a term s_σ and a term u_σ, such that there is a reduction sequence $\varphi(\sigma) : t \xrightarrow{j}^{*}_{\mathbf{m}} s_\sigma$ and such that $s' \xrightarrow{j}^{*}_{\mathbf{m}} u_\sigma$, and $s_\sigma \xrightarrow{d}^{+}_{\mathbf{m}} u_\sigma$ in at least one step.
   First, we claim that $\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\varphi(\sigma)^{\mathsf{tgt}}) \succ \mathcal{T}^{\mathbf{m}}_{\leq j-1}(\sigma^{\mathsf{tgt}})$ for every $\sigma \in Y$. Indeed:

$$
\begin{array}{rcll}
\mathcal{T}^{\mathbf{m}}_{\leq j-1}(\varphi(\sigma)^{\mathsf{tgt}}) & = & \mathcal{T}^{\mathbf{m}}_{\leq j-1}(s_\sigma) & \\
& \succ & \mathcal{T}^{\mathbf{m}}_{\leq j-1}(u_\sigma) & \text{by item 5 of the IH} \\
& \succeq & \mathcal{T}^{\mathbf{m}}_{\leq j-1}(s') & \text{by high/increase (Prop. 29(4))} \\
& = & \mathcal{T}^{\mathbf{m}}_{\leq j-1}(\sigma^{\mathsf{tgt}}) &
\end{array}
$$

   To be able to apply item 5 of the IH, observe that we have that $1 \leq d \leq j-1 < D$ holds because we know $d < j \leq D$. We resort to the IH as many times as the length of the reduction $s_\sigma \xrightarrow{d}{}^{+}_{\mathbf{m}} u_\sigma$. The inequality is strict because this reduction contains at least one step. To be able to apply the high/increase property, observe that $0 \le j-1 < j$. We resort to this lemma as many times as the length of the reduction $s' \xrightarrow{j}{}^{*}_{\mathbf{m}} u_\sigma$, which may be empty. To conclude the proof, let $Z = X \setminus \varphi(Y)$, so that $X = \varphi(Y) \uplus Z$, and note that:

$$\begin{array}{rcll}
\mathcal{R}^{\mathbf{m}}_j(t) & = & [\mathcal{T}^{\mathbf{m}}_{\le j-1}(\rho^{\mathtt{tgt}}) \mid\mid \rho \in X] & \\
& = & [\mathcal{T}^{\mathbf{m}}_{\le j-1}(\rho^{\mathtt{tgt}}) \mid\mid \rho \in \varphi(Y) \uplus Z] & \\
& = & [\mathcal{T}^{\mathbf{m}}_{\le j-1}(\rho^{\mathtt{tgt}}) \mid\mid \rho \in \varphi(Y)] + [\mathcal{T}^{\mathbf{m}}_{\le j-1}(\rho^{\mathtt{tgt}}) \mid\mid \rho \in Z] & \\
& = & [\mathcal{T}^{\mathbf{m}}_{\le j-1}(\varphi(\sigma)^{\mathtt{tgt}}) \mid\mid \sigma \in Y] + [\mathcal{T}^{\mathbf{m}}_{\le j-1}(\rho^{\mathtt{tgt}}) \mid\mid \rho \in Z] & \\
& \succeq & [\mathcal{T}^{\mathbf{m}}_{\le j-1}(\varphi(\sigma)^{\mathtt{tgt}}) \mid\mid \sigma \in Y] & \\
& \succ & [\mathcal{T}^{\mathbf{m}}_{\le j-1}(\sigma^{\mathtt{tgt}}) \mid\mid \sigma \in Y] & (\star) \\
& = & \mathcal{R}^{\mathbf{m}}_j(t') &
\end{array}$$

For the step marked with $(\star)$, note that $[\mathcal{T}^{\mathbf{m}}_{\le j-1}(\varphi(\sigma)^{\mathtt{tgt}}) \mid\mid \sigma \in Y] \mathrel{:\succ:} [\mathcal{T}^{\mathbf{m}}_{\le j-1}(\sigma^{\mathtt{tgt}}) \mid\mid \sigma \in Y]$ because $\mathcal{T}^{\mathbf{m}}_{\le j-1}(\varphi(\sigma)^{\mathtt{tgt}}) \succ \mathcal{T}^{\mathbf{m}}_{\le j-1}(\sigma^{\mathtt{tgt}})$ holds by the claim above. Moreover, Y is non-empty because the empty reduction sequence $\epsilon : t' \xrightarrow{j}{}^{*}_{\mathbf{m}} t'$ is in Y, so we may resort to Lem. 63.

2. Let $1 \le d \le j \le D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t_0'$. We argue that $\mathcal{T}^{\mathbf{m}}_j(t_0, \mathtt{X}) \mathrel{:\succ:} \mathcal{T}^{\mathbf{m}}_j(t_0', \mathtt{X})$, where $\mathtt{X}$ is either a term ($\mathtt{X} = t$) or a memory ($\mathtt{X} = \mathtt{L}$). We proceed by induction on $\mathtt{X}$:

2.1 $t = x$: Then $\mathcal{T}^{\mathbf{m}}_j(t_0, x) = [\,] \mathrel{:\succ:} [\,] = \mathcal{T}^{\mathbf{m}}_j(t_0', x)$.

2.2 $t = \lambda x.\, s$: Then $\mathcal{T}^{\mathbf{m}}_j(t_0, \lambda x.\, s) = \mathcal{T}^{\mathbf{m}}_j(t_0, s) = \mathcal{T}^{\mathbf{m}}_j(t_0', s) = \mathcal{T}^{\mathbf{m}}_j(t_0', \lambda x.\, s)$ by the internal IH.

2.3 If $t = (\lambda x.\, s)\mathtt{L}\, u$ is a redex of degree j: Then:

$$\begin{array}{rl}
& \mathcal{T}^{\mathbf{m}}_j(t_0, (\lambda x.\, s)\mathtt{L}\, u) \\
= & \mathcal{T}^{\mathbf{m}}_j(t_0, s) + \mathcal{T}^{\mathbf{m}}_j(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_j(t_0, u) + [(j, \mathcal{R}^{\mathbf{m}}_j(t_0))] \\
:\succ: & \mathcal{T}^{\mathbf{m}}_j(t_0', s) + \mathcal{T}^{\mathbf{m}}_j(t_0', \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_j(t_0', u) + [(j, \mathcal{R}^{\mathbf{m}}_j(t_0))] \\
& \quad \text{by the internal IH} \\
:\succ: & \mathcal{T}^{\mathbf{m}}_j(t_0', s) + \mathcal{T}^{\mathbf{m}}_j(t_0', \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_j(t_0', u) + [(j, \mathcal{R}^{\mathbf{m}}_j(t_0'))] \\
& \quad \text{since } \mathcal{R}^{\mathbf{m}}_j(t_0) \succ \mathcal{R}^{\mathbf{m}}_j(t_0') \text{ by item 1} \\
= & \mathcal{T}^{\mathbf{m}}_j(t_0', (\lambda x.\, s)\mathtt{L}\, u)
\end{array}$$

Note that we can resort to item 1, because $1 \le d \le j \le D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t_0'$.

2.4 If $t = s\, u$ is not a redex of degree j: Then $\mathcal{T}^{\mathbf{m}}_j(t_0, s\, u) = \mathcal{T}^{\mathbf{m}}_j(t_0, s) + \mathcal{T}^{\mathbf{m}}_j(t_0, u) \mathrel{:\succ:} \mathcal{T}^{\mathbf{m}}_j(t_0', s) + \mathcal{T}^{\mathbf{m}}_j(t_0', u) = \mathcal{T}^{\mathbf{m}}_j(t_0', s\, u)$ by the internal IH.

2.5 $t = s\{u\}$: Then $\mathcal{T}^{\mathbf{m}}_j(t_0, s\{u\}) = \mathcal{T}^{\mathbf{m}}_j(t_0, s) + \mathcal{T}^{\mathbf{m}}_j(t_0, u) \mathrel{:\succ:} \mathcal{T}^{\mathbf{m}}_j(t_0', s) + \mathcal{T}^{\mathbf{m}}_j(t_0', u) = \mathcal{T}^{\mathbf{m}}_j(t_0', s\{u\})$ by the internal IH.

2.6 $\mathtt{L} = \Box$: Then $\mathcal{T}^{\mathbf{m}}_j(t_0, \Box) = [\,] \mathrel{:\succ:} [\,] = \mathcal{T}^{\mathbf{m}}_j(t_0', \Box)$.

2.7 $\mathtt{L} = \mathtt{L}_1\{t\}$: $\mathcal{T}^{\mathbf{m}}_j(t_0, \mathtt{L}_1\{t\}) = \mathcal{T}^{\mathbf{m}}_j(t_0, \mathtt{L}_1) + \mathcal{T}^{\mathbf{m}}_j(t_0, t) \mathrel{:\succ:} \mathcal{T}^{\mathbf{m}}_j(t_0', \mathtt{L}_1) + \mathcal{T}^{\mathbf{m}}_j(t_0', t)$ by IH.

3. Let $1 \le d \le D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t_0'$ and $\mathtt{X} \xrightarrow{d}_{\mathbf{m}} \mathtt{X}'$, where $\mathtt{X}, \mathtt{X}'$ are either terms ($\mathtt{X} = t$ and $\mathtt{X} = t'$) or memories ($\mathtt{X} = \mathtt{L}$ and $\mathtt{X} = \mathtt{L}'$). We argue that for all $\mathtt{m} \in \mathbb{T}_{d-1}$ we have $\mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{X}) \succ \mathcal{T}^{\mathbf{m}}_d(t_0', \mathtt{X}') + \mathtt{m}$. We proceed by induction on $\mathtt{X}$:

3.1 $t = x$: Impossible, as there are no reduction steps $x \xrightarrow{d}_{\mathbf{m}} t'$.

3.2 $t = \lambda x.\, s$: Then the step is of the form $t = \lambda x.\, s \xrightarrow{d}_{\mathbf{m}} \lambda x.\, s' = t'$ with $s \xrightarrow{d}_{\mathbf{m}} s'$. Let $\mathtt{m} \in \mathbb{T}_{d-1}$. Then $\mathcal{T}^{\mathbf{m}}_d(t_0, t) = \mathcal{T}^{\mathbf{m}}_d(t_0, \lambda x.\, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, s) \succ \mathcal{T}^{\mathbf{m}}_d(t_0', s') + \mathtt{m} = \mathcal{T}^{\mathbf{m}}_d(t_0', \lambda x.\, s') + \mathtt{m} = \mathcal{T}^{\mathbf{m}}_d(t_0', t') + \mathtt{m}$ by the internal IH.

3.3 If $t = (\lambda x.\, s)\mathtt{L}\, u$ is the redex of degree d contracted by the step $t \xrightarrow{d}_{\mathbf{m}} t'$: If the reduction step is at the root, then $(\lambda x.\, s)\mathtt{L}$ is a $\mathbf{m}$-abstraction of degree d, and the step is of the form $t = (\lambda x.\, s)\mathtt{L}\, u \xrightarrow{d}_{\mathbf{m}} s[x := u]\{u\}\mathtt{L} = t'$. Given that $(\lambda x.\, s)\mathtt{L}$ is a $\mathbf{m}$-abstraction of degree d, the type of its argument u is of height strictly less than d, that is, $\mathsf{h}(\mathsf{type}(u)) < d$. In particular, u is not an abstraction of degree d so we may apply the lower substitution lemma (Lem. 28) which ensures that there exists $k \in \mathbb{N}_0$ such that $\mathcal{T}^{\mathbf{m}}_d(t'_0, s[x := u]) = \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + k \otimes \mathcal{T}^{\mathbf{m}}_d(t'_0, u)$. Furthermore, observe that $\mathcal{T}^{\mathbf{m}}_d(t_0, u) \succeq (1 + k) \otimes \mathcal{T}^{\mathbf{m}}_d(t'_0, u)$. Indeed, by item 2 we have that $\mathcal{T}^{\mathbf{m}}_d(t_0, u) :\succ: \mathcal{T}^{\mathbf{m}}_d(t'_0, u)$, so by Lem. 63 $\mathcal{T}^{\mathbf{m}}_d(t_0, u) \succeq (1 + k) \otimes \mathcal{T}^{\mathbf{m}}_d(t'_0, u)$. To conclude this case:

$$
\begin{array}{ll}
 & \mathcal{T}^{\mathbf{m}}_d(t_0, t) \\
= & \mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda x.\, s)\mathtt{L}\, u) \\
= & \mathcal{T}^{\mathbf{m}}_d(t_0, s) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] \\
\succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] \\
 & \quad \text{by item 2 and Lem. 63} \\
\succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + (1 + k) \otimes \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] \\
 & \quad \text{since } \mathcal{T}^{\mathbf{m}}_d(t_0, u) \succeq (1 + k) \otimes \mathcal{T}^{\mathbf{m}}_d(t'_0, u) \\
\succ & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + (1 + k) \otimes \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + \mathfrak{m} \\
 & \quad \text{since } \mathfrak{m} \in \mathbb{T}_{d-1}, \text{ so } [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] \succ \mathfrak{m} \\
= & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + k \otimes \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathfrak{m} \\
= & \mathcal{T}^{\mathbf{m}}_d(t'_0, s[x := u]) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathfrak{m} \\
= & \mathcal{T}^{\mathbf{m}}_d(t'_0, s[x := u]\{u\}\mathtt{L}) + \mathfrak{m} \\
= & \mathcal{T}^{\mathbf{m}}_d(t'_0, t') + \mathfrak{m}
\end{array}
$$

3.4 If $t = (\lambda x.\, s)\mathtt{L}\, u$ is a redex of degree d, but not the redex contracted by the step $t \xrightarrow{d}_{\mathbf{m}} t'$: There are three subcases, depending on whether the step $t \xrightarrow{d}_{\mathbf{m}} t'$ is internal to s, internal to $\mathtt{L}$, or internal to u. All these subcases are similar; we only give the proof for the case in which the step is internal to s. Then the step is of the form $t = (\lambda x.\, s)\mathtt{L}\, u \xrightarrow{d}_{\mathbf{m}} (\lambda x.\, s')\mathtt{L}\, u = t'$ with $s \xrightarrow{d}_{\mathbf{m}} s'$, and we have:

$$
\begin{array}{lll}
 & \mathcal{T}^{\mathbf{m}}_d(t_0, t) & \\
= & \mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda x.\, s)\mathtt{L}\, u) & \\
= & \mathcal{T}^{\mathbf{m}}_d(t_0, s) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] & \\
\succ & \mathcal{T}^{\mathbf{m}}_d(t'_0, s') + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] + \mathfrak{m} & \text{by the internal IH} \\
\succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s') + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] + \mathfrak{m} & \text{by item 2 and Lem. 63} \\
\succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s') + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t'_0))] + \mathfrak{m} & \text{by item 1} \\
= & \mathcal{T}^{\mathbf{m}}_d(t'_0, (\lambda x.\, s')\mathtt{L}\, u) + \mathfrak{m} & \\
= & \mathcal{T}^{\mathbf{m}}_d(t'_0, t') + \mathfrak{m} &
\end{array}
$$

3.5 If $t = s\, u$ is not a redex of degree d: There are two subcases, depending on whether the step $t \xrightarrow{d}_{\mathbf{m}} t'$ is internal to s or internal to u:

3.5.1 If the step is internal to s, then the step is of the form $t = s\, u \xrightarrow{d}_{\mathbf{m}} s'\, u = t'$ with $s \xrightarrow{d}_{\mathbf{m}} s'$. Note that s is not a $\mathbf{m}$-abstraction of degree d (because $s\, u$ is not a redex of degree d). Hence by Lem. 59 s' is not a $\mathbf{m}$-abstraction of degree d. Then:

$$\begin{array}{rcll} \mathcal{T}_d^{\mathbf{m}}(t_0,t) & = & \mathcal{T}_d^{\mathbf{m}}(t_0, s\,u) & \\ & = & \mathcal{T}_d^{\mathbf{m}}(t_0,s) + \mathcal{T}_d^{\mathbf{m}}(t_0,u) & \\ & \succ & \mathcal{T}_d^{\mathbf{m}}(t_0',s') + \mathcal{T}_d^{\mathbf{m}}(t_0,u) + \mathfrak{m} & \text{by the internal IH} \\ & \succeq & \mathcal{T}_d^{\mathbf{m}}(t_0',s') + \mathcal{T}_d^{\mathbf{m}}(t_0',u) + \mathfrak{m} & \text{by item 2 and Lem. 63} \\ & = & \mathcal{T}_d^{\mathbf{m}}(t_0',s'\,u) + \mathfrak{m} & \\ & = & \mathcal{T}_d^{\mathbf{m}}(t_0',t') + \mathfrak{m} & \end{array}$$

3.5.2 If the step is internal to u: Similar to the previous case.

3.6 $t = s\{u\}$: Similar to case 3.5.

3.7 $\mathtt{L} = \Box$: Impossible, as there are no reduction steps $\Box \xrightarrow{d}_{\mathbf{m}} \mathtt{L}'$.

3.8 $\mathtt{L} = \mathtt{L}_1\{t\}$: Similar to case 3.5

4. Let $1 \leq d < j \leq D$ and $t_0 \xrightarrow{d}_{\mathbf{m}} t_0'$ and $\mathtt{X} \xrightarrow{d}_{\mathbf{m}} \mathtt{X}'$, where $\mathtt{X}, \mathtt{X}'$ are either terms ($\mathtt{X} = t$ and $\mathtt{X} = t'$) or memories ($\mathtt{X} = \mathtt{L}$ and $\mathtt{X} = \mathtt{L}'$). We argue that $\mathcal{T}_j^{\mathbf{m}}(t_0, \mathtt{X}) \succeq \mathcal{T}_j^{\mathbf{m}}(t_0', \mathtt{X}')$. We proceed by induction on $\mathtt{X}$:

4.1 $t = x$: Impossible, as there are no reduction steps $x \xrightarrow{d}_{\mathbf{m}} t'$.

4.2 $t = \lambda x.\,s$: Then the step is of the form $t = \lambda x.\,s \xrightarrow{d}_{\mathbf{m}} \lambda x.\,s' = t'$ with $s \xrightarrow{d}_{\mathbf{m}} s'$. Then $\mathcal{T}_j^{\mathbf{m}}(t_0,t) = \mathcal{T}_j^{\mathbf{m}}(t_0,\lambda x.\,s) = \mathcal{T}_j^{\mathbf{m}}(t_0,s) \succeq \mathcal{T}_j^{\mathbf{m}}(t_0',s') = \mathcal{T}_j^{\mathbf{m}}(t_0',\lambda x.\,s') = \mathcal{T}_j^{\mathbf{m}}(t_0',t')$ by the internal IH.

4.3 If $t = (\lambda x.\,s)\mathtt{L}\,u$ is the redex of degree d contracted by the step $t \xrightarrow{d}_{\mathbf{m}} t'$: Then the step is of the form $t = (\lambda x.\,s)\mathtt{L}\,u \xrightarrow{d}_{\mathbf{m}} s[x := u]\{u\}\mathtt{L} = t'$. Recall that by hypothesis $d < j$, and note that the abstraction $(\lambda x.\,s)\mathtt{L}$ is of degree d, so the type of the argument u must be of height less than d, that is, $\mathsf{h}(\mathsf{type}(u)) < d < j$. In particular, the argument u cannot be a $\mathbf{m}$-abstraction of degree j, so we may apply the lower substitution lemma (Lem. 28), which ensures that there exists $k \in \mathbb{N}_0$ such that $\mathcal{T}_j^{\mathbf{m}}(t_0', s[x := u]) = \mathcal{T}_j^{\mathbf{m}}(t_0',s) + k \otimes \mathcal{T}_j^{\mathbf{m}}(t_0',u)$. Furthermore, observe that $\mathcal{T}_j^{\mathbf{m}}(t_0,u) \succeq (1+k) \otimes \mathcal{T}_j^{\mathbf{m}}(t_0',u)$. Indeed by item 2 $\mathcal{T}_j^{\mathbf{m}}(t_0,u) :\succ: \mathcal{T}_j^{\mathbf{m}}(t_0',u)$ so by Lem. 63 $\mathcal{T}_j^{\mathbf{m}}(t_0,u) \succeq (1+k) \otimes \mathcal{T}_j^{\mathbf{m}}(t_0',u)$. Moreover, note that t is not a redex of degree j, so:

$$\begin{array}{rcll} \mathcal{T}_j^{\mathbf{m}}(t_0,t) & = & \mathcal{T}_j^{\mathbf{m}}(t_0,(\lambda x.\,s)\mathtt{L}\,u) & \\ & = & \mathcal{T}_j^{\mathbf{m}}(t_0,s) + \mathcal{T}_j^{\mathbf{m}}(t_0,\mathtt{L}) + \mathcal{T}_j^{\mathbf{m}}(t_0,u) & \\ & \succeq & \mathcal{T}_j^{\mathbf{m}}(t_0',s) + \mathcal{T}_j^{\mathbf{m}}(t_0',\mathtt{L}) + \mathcal{T}_j^{\mathbf{m}}(t_0,u) & \text{by item 2 and Lem. 63} \\ & \succeq & \mathcal{T}_j^{\mathbf{m}}(t_0',s) + \mathcal{T}_j^{\mathbf{m}}(t_0',\mathtt{L}) + (1+k) \otimes \mathcal{T}_j^{\mathbf{m}}(t_0',u) & \\ & & \quad \text{since } \mathcal{T}_j^{\mathbf{m}}(t_0,u) \succeq (1+k) \otimes \mathcal{T}_j^{\mathbf{m}}(t_0',u) & \\ & = & \mathcal{T}_j^{\mathbf{m}}(t_0',s) + k \otimes \mathcal{T}_j^{\mathbf{m}}(t_0',u) + \mathcal{T}_j^{\mathbf{m}}(t_0',u) + \mathcal{T}_j^{\mathbf{m}}(t_0',\mathtt{L}) & \\ & = & \mathcal{T}_j^{\mathbf{m}}(t_0',s[x := u]) + \mathcal{T}_j^{\mathbf{m}}(t_0',u) + \mathcal{T}_j^{\mathbf{m}}(t_0',\mathtt{L}) & \\ & = & \mathcal{T}_j^{\mathbf{m}}(t_0',s[x := u]\{u\}\mathtt{L}) & \\ & = & \mathcal{T}_j^{\mathbf{m}}(t_0,t') & \end{array}$$

4.4 If $t = (\lambda x.\,s)\mathtt{L}\,u$ is a redex of degree j: Note that the step $t \xrightarrow{d}_{\mathbf{m}} t'$ cannot be at the root, because $d < j$, so the redex at the root is not of degree d. There are three subcases, depending on whether the step is internal to s, internal to $\mathtt{L}$, or internal to u. All these subcases are similar; we only give the proof for the case in which the step is internal to s. Then the step is of the form $t = (\lambda x.\,s)\mathtt{L}\,u \xrightarrow{d}_{\mathbf{m}} (\lambda x.\,s')\mathtt{L}\,u = t'$ with

$s \xrightarrow{d}_{\mathbf{m}} s'$, and we have:

$$\begin{array}{ll} & \mathcal{T}^{\mathbf{m}}_j(t_0, t) \\ = & \mathcal{T}^{\mathbf{m}}_j(t_0, (\lambda x. s)\mathtt{L}\, u) \\ = & \mathcal{T}^{\mathbf{m}}_j(t_0, s) + \mathcal{T}^{\mathbf{m}}_j(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_j(t_0, u) + [(j, \mathcal{R}^{\mathbf{m}}_j(t_0))] \\ \succeq & \mathcal{T}^{\mathbf{m}}_j(t'_0, s') + \mathcal{T}^{\mathbf{m}}_j(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_j(t_0, u) + [(j, \mathcal{R}^{\mathbf{m}}_j(t_0))] \quad \text{by the internal IH} \\ \succeq & \mathcal{T}^{\mathbf{m}}_j(t'_0, s') + \mathcal{T}^{\mathbf{m}}_j(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_j(t'_0, u) + [(j, \mathcal{R}^{\mathbf{m}}_j(t_0))] \quad \text{by item 2 and Lem. 63} \\ \succeq & \mathcal{T}^{\mathbf{m}}_j(t'_0, s') + \mathcal{T}^{\mathbf{m}}_j(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_j(t'_0, u) + [(j, \mathcal{R}^{\mathbf{m}}_j(t'_0))] \quad \text{by item 1} \\ = & \mathcal{T}^{\mathbf{m}}_j(t'_0, (\lambda x. s')\mathtt{L}\, u) \\ = & \mathcal{T}^{\mathbf{m}}_j(t'_0, t') \end{array}$$

4.5 If $t = s\,u$ is not the redex contracted by the step $t \xrightarrow{d}_{\mathbf{m}} t'$ nor a redex of degree j: There are two subcases, depending on whether the step $t \xrightarrow{d}_{\mathbf{m}} t'$ is internal to s or internal to u:

4.5.1 If the step is internal to s, then the step is of the form $t = s\,u \xrightarrow{d}_{\mathbf{m}} s'\,u = t'$ with $s \xrightarrow{d}_{\mathbf{m}} s'$. Note that s is not a **m**-abstraction of degree j, (because $s\,u$ is not a redex of degree j). Moreover $d < j$ so by Lem. 59 we have that s' is not a **m**-abstraction of degree j. Then:

$$\begin{array}{lll} \mathcal{T}^{\mathbf{m}}_j(t_0, t) & = & \mathcal{T}^{\mathbf{m}}_j(t_0, s\,u) \\ & = & \mathcal{T}^{\mathbf{m}}_j(t_0, s) + \mathcal{T}^{\mathbf{m}}_j(t_0, u) \\ & \succeq & \mathcal{T}^{\mathbf{m}}_j(t'_0, s') + \mathcal{T}^{\mathbf{m}}_j(t_0, u) \quad \text{by the internal IH} \\ & \succeq & \mathcal{T}^{\mathbf{m}}_j(t'_0, s') + \mathcal{T}^{\mathbf{m}}_j(t'_0, u) \quad \text{by item 2 and Lem. 63} \\ & = & \mathcal{T}^{\mathbf{m}}_j(t'_0, s'\,u) \\ & = & \mathcal{T}^{\mathbf{m}}_j(t'_0, t') \end{array}$$

4.5.2 If the step is internal to u: Similar to the previous case.

4.6 $t = s\{u\}$: Similar to case 4.5.

4.7 $\mathtt{L} = \square$: Impossible, as there are no reduction steps $\square \xrightarrow{d}_{\mathbf{m}} \mathtt{L}'$.

4.8 $\mathtt{L} = \mathtt{L}_1\{t\}$: Similar to case 4.5.

5. Let $1 \leq d \leq D$ and $t \xrightarrow{d}_{\mathbf{m}} t'$. We argue that $\mathcal{T}^{\mathbf{m}}_{\leq D}(t) \succ \mathcal{T}^{\mathbf{m}}_{\leq D}(t')$. Indeed:

$$\begin{array}{lll} \mathcal{T}^{\mathbf{m}}_{\leq D}(t) & = & \sum_{i=1}^{D} \mathcal{T}^{\mathbf{m}}_i(t, t) \\ & = & \mathcal{T}^{\mathbf{m}}_{\leq d-1}(t) + \mathcal{T}^{\mathbf{m}}_d(t, t) + (\sum_{j=d+1}^{D} \mathcal{T}^{\mathbf{m}}_j(t, t)) \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t, t) + (\sum_{j=d+1}^{D} \mathcal{T}^{\mathbf{m}}_j(t, t)) \\ & & \quad \text{removing the first term} \\ & \succ & \mathcal{T}^{\mathbf{m}}_{\leq d-1}(t') + \mathcal{T}^{\mathbf{m}}_d(t', t') + (\sum_{j=d+1}^{D} \mathcal{T}^{\mathbf{m}}_j(t, t)) \\ & & \quad \text{by item 3, taking } \mathtt{m} := \mathcal{T}^{\mathbf{m}}_{\leq d-1}(t') \\ & \succeq & \mathcal{T}^{\mathbf{m}}_{\leq d-1}(t') + \mathcal{T}^{\mathbf{m}}_d(t', t') + (\sum_{j=d+1}^{D} \mathcal{T}^{\mathbf{m}}_j(t', t')) \\ & & \quad \text{by item 4} \\ & = & \mathcal{T}^{\mathbf{m}}_{\leq D}(t') \end{array}$$

◀

▶ **Proposition 69** (Forget/decrease). *Let $d \in \mathbb{N}_0$. Then the following hold:*

1. *If $t \rhd t'$ then $\mathcal{R}^{\mathbf{m}}_d(t) \succeq \mathcal{R}^{\mathbf{m}}_d(t')$.*

2. *If $t_0 \rhd t'_0$ then $\mathcal{T}^{\mathbf{m}}_d(t_0, t) \succeq \mathcal{T}^{\mathbf{m}}_d(t'_0, t)$.*

3. *If $t_0 \rhd t'_0$ and $t \rhd t'$ then $\mathcal{T}^{\mathbf{m}}_d(t_0, t) \succeq \mathcal{T}^{\mathbf{m}}_d(t'_0, t')$.*

4. *If $t \rhd t'$ then $\mathcal{T}^{\mathbf{m}}_{\leq d}(t) \succeq \mathcal{T}^{\mathbf{m}}_{\leq d}(t')$.*

Proof. We prove a more general version of the statement: in items 2 and 3 we allow t to be either a term or a memory. For example, the statement of item 2 is generalized as follows: if $t_0 \rhd t'_0$ then $\mathcal{T}^{\mathbf{m}}_j(t_0, \mathtt{X}) \succeq \mathcal{T}^{\mathbf{m}}_j(t'_0, \mathtt{X})$, where $\mathtt{X}$ is either a term or a memory.

We prove all items simultaneously by induction on d. Note that: item 1. resorts to the IH; item 2. resorts to item 1. (without decreasing d); item 3. resorts to items 1. and 2. (without decreasing d); item 4. resorts to item 3. (without necessarily decreasing d).

1. Let $t \rhd t'$. We argue that $\mathcal{R}^{\mathbf{m}}_d(t) \succeq \mathcal{R}^{\mathbf{m}}_d(t')$. Let X and Y be the sets of reduction sequences $X := \{\rho \mid (\exists s)\ \rho : t \xrightarrow{d}^*_{\mathbf{m}} s\}$ and $Y := \{\sigma \mid (\exists s')\ \sigma : t' \xrightarrow{d}^*_{\mathbf{m}} s'\}$. Note that, by definition, $\mathcal{R}^{\mathbf{m}}_j(t) = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in X]$ and $\mathcal{R}^{\mathbf{m}}_j(t') = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y]$. We construct a function $\varphi : Y \to X$ as follows. Consider a forgetful step $R : t \rhd t'$; there may be more than one such step, but there is at least one by hypothesis. By postponement of forgetful reduction Prop. 21, for each reduction sequence $\sigma : t' \xrightarrow{d}^*_{\mathbf{m}} s'$ there exists a term s_σ such that $s_\sigma \rhd^* s'$ and a reduction sequence $\sigma^\frown R : t \xrightarrow{d}^*_{\mathbf{m}} s_\sigma$. In particular, $(\sigma^\frown R) \in X$, and we can define $\varphi(\sigma) := \sigma^\frown R$. Moreover, φ is injective because if $\sigma_1, \sigma_2 \in Y$ are such that $\sigma_1^\frown R = \sigma_2^\frown R$ then by Prop. 21 we have that $\sigma_1 = \sigma_2$. First, we claim that $\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}}) \succeq \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}})$ for every $\sigma \in Y$. Indeed:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}}) & = & \mathcal{T}^{\mathbf{m}}_{\leq d-1}(s_\sigma) & \\ & \succeq & \mathcal{T}^{\mathbf{m}}_{\leq d-1}(s') & \text{by item 4 of the IH} \\ & = & \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) & \end{array}$$

To be able to apply item 4 of the IH, observe that we have $d - 1 < d$. We apply the IH as many times as the number of forgetful steps in $s_\sigma \rhd^* s'$.
To conclude the proof, let $Z = X \setminus \varphi(Y)$, so that $X = \varphi(Y) \uplus Z$, and note that:

$$\begin{array}{rcll} \mathcal{R}^{\mathbf{m}}_d(t) & = & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in X] & \\ & = & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in \varphi(Y) \uplus Z] & \\ & = & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in \varphi(Y)] + [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in Z] & \\ & \succeq & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\rho^{\mathsf{tgt}}) \mid\mid \rho \in \varphi(Y)] & \\ & = & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}}) \mid\mid \sigma \in Y] & (\star) \\ & \succeq & [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y] & (\star\star) \\ & = & \mathcal{R}^{\mathbf{m}}_d(t') & \end{array}$$

To justify the step marked with $(\star)$, note that φ is injective, so Y and $\varphi(Y)$ have the same cardinality. To justify the step marked with $(\star\star)$, note that $[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}}) \mid\mid \sigma \in Y] = \sum_{\sigma \in Y}[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}})] \succeq \sum_{\sigma \in Y}[\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}})] = [\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}}) \mid\mid \sigma \in Y]$ because $\mathcal{T}^{\mathbf{m}}_{\leq d-1}(\varphi(\sigma)^{\mathsf{tgt}}) \succeq \mathcal{T}^{\mathbf{m}}_{\leq d-1}(\sigma^{\mathsf{tgt}})$, as we have already justified.

2. Let $t_0 \rhd t'_0$ and let $\mathtt{X}$ be either a term or a memory. We argue that $\mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{X}) \succeq \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{X})$. We proceed by induction on $\mathtt{X}$:
   2.1 $t = x$: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, x) = [\,] \succeq [\,] = \mathcal{T}^{\mathbf{m}}_d(t'_0, x)$.
   2.2 $t = \lambda x.\, s$: Then $\mathcal{T}^{\mathbf{m}}_d(t_0, \lambda x.\, s) = \mathcal{T}^{\mathbf{m}}_d(t_0, s) \succeq \mathcal{T}^{\mathbf{m}}_d(t'_0, s) = \mathcal{T}^{\mathbf{m}}_d(t'_0, \lambda x.\, s)$ by the internal IH.
   2.3 If $t = (\lambda x.\, s)\mathtt{L}\, u$ is a redex of degree d: Then:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_d(t_0, t) & = & \mathcal{T}^{\mathbf{m}}_d(t_0, (\lambda x.\, s)\mathtt{L}\, u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0, s) + \mathcal{T}^{\mathbf{m}}_d(t_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] & \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t_0))] & \text{by the internal IH} \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0, s) + \mathcal{T}^{\mathbf{m}}_d(t'_0, \mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0, u) + [(d, \mathcal{R}^{\mathbf{m}}_d(t'_0))] & \text{by item 1} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, (\lambda x.\, s)\mathtt{L}\, u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0, t) & \end{array}$$

2.4 If $t = s\,u$ is not a redex of degree d: Then:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_d(t_0,t) & = & \mathcal{T}^{\mathbf{m}}_d(t_0,s\,u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0,s) + \mathcal{T}^{\mathbf{m}}_d(t_0,u) & \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0,s) + \mathcal{T}^{\mathbf{m}}_d(t'_0,u) & \text{by the internal IH} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0,t) & \end{array}$$

2.5 $t = s\{u\}$: Similar to case 2.4.

2.6 $\mathtt{L} = \Box$: Then $\mathcal{T}^{\mathbf{m}}_d(t_0,\Box) = [\,] \succeq [\,] = \mathcal{T}^{\mathbf{m}}_d(t'_0,\Box)$.

2.7 $\mathtt{L} = \mathtt{L}_1\{t\}$: Similar to case 2.4.

3. Let $t_0 \rhd t'_0$ and $\mathtt{X} \rhd \mathtt{X}'$, where $\mathtt{X}$ and $\mathtt{X}'$ are either terms ($\mathtt{X} = t$ and $\mathtt{X} = t'$) or memories ($\mathtt{X} = \mathtt{L}$ and $\mathtt{X} = \mathtt{L}'$). We argue that $\mathcal{T}^{\mathbf{m}}_d(t_0,\mathtt{X}) \succeq \mathcal{T}^{\mathbf{m}}_d(t'_0,\mathtt{X}')$. We proceed by induction on $\mathtt{X}$:

3.1 $t = x$: Impossible, as there are no forgetful steps $x \rhd t'$.

3.2 $t = \lambda x.\,s$: Then $t = \lambda x.\,s \rhd \lambda x.\,s' = t'$ with $s \rhd s'$ and $\mathcal{T}^{\mathbf{m}}_d(t_0,t) = \mathcal{T}^{\mathbf{m}}_d(t_0,\lambda x.\,s) = \mathcal{T}^{\mathbf{m}}_d(t_0,s) \succeq \mathcal{T}^{\mathbf{m}}_d(t'_0,s') \succeq \mathcal{T}^{\mathbf{m}}_d(t'_0,\lambda x.\,s') \succeq \mathcal{T}^{\mathbf{m}}_d(t'_0,t')$ by the internal IH.

3.3 If $t = (\lambda x.\,s)\mathtt{L}\,u$ is a redex of degree d: There are three subcases, depending on whether the forgetful step $t \rhd t'$ is internal to s, internal to $\mathtt{L}$, or internal to u. All these subcases are similar; we only give the proof for the case in which the step is internal to s. Then $t = (\lambda x.\,s)\mathtt{L}\,u \rhd (\lambda x.\,s')\mathtt{L}\,u = t'$ with $s \rhd s'$ and:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_d(t_0,t) & = & \mathcal{T}^{\mathbf{m}}_d(t_0,(\lambda x.\,s)\mathtt{L}\,u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0,s) + \mathcal{T}^{\mathbf{m}}_d(t_0,\mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0,u) + [(d,\mathcal{R}^{\mathbf{m}}_d(t_0))] & \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0,s') + \mathcal{T}^{\mathbf{m}}_d(t_0,\mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t_0,u) + [(d,\mathcal{R}^{\mathbf{m}}_d(t_0))] & \text{by the internal IH} \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0,s') + \mathcal{T}^{\mathbf{m}}_d(t'_0,\mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0,u) + [(d,\mathcal{R}^{\mathbf{m}}_d(t_0))] & \text{by item 2} \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0,s') + \mathcal{T}^{\mathbf{m}}_d(t'_0,\mathtt{L}) + \mathcal{T}^{\mathbf{m}}_d(t'_0,u) + [(d,\mathcal{R}^{\mathbf{m}}_d(t'_0))] & \text{by item 1} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0,(\lambda x.\,s')\mathtt{L}\,u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0,t') & \end{array}$$

3.4 If $t = s\,u$ is not a redex of degree d: There are two subcases, depending on whether the forgetful step $t \rhd t'$ is internal to s or internal to u. These subcases are similar; we only give the proof for the case in which the step is internal to s. Then $t = s\,u \rhd s'\,u = t'$ with $s \rhd s'$ and:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_d(t_0,t) & = & \mathcal{T}^{\mathbf{m}}_d(t_0,s\,u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0,s) + \mathcal{T}^{\mathbf{m}}_d(t_0,u) & \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0,s') + \mathcal{T}^{\mathbf{m}}_d(t_0,u) & \text{by the internal IH} \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0,s') + \mathcal{T}^{\mathbf{m}}_d(t'_0,u) & \text{by item 2} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0,s'\,u) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0,t') & \end{array}$$

3.5 $t = s\{u\}$: There are three subcases, depending on whether the forgetful step $t \rhd t'$ is at the root, internal to s, or internal to u:

3.5.1 If the step is at the root: Then the step is of the form $t = s\{u\} \rhd s = t'$ and:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_d(t_0,t) & = & \mathcal{T}^{\mathbf{m}}_d(t_0,s\{u\}) & \\ & = & \mathcal{T}^{\mathbf{m}}_d(t_0,s) + \mathcal{T}^{\mathbf{m}}_d(t_0,u) & \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t_0,s) & \\ & \succeq & \mathcal{T}^{\mathbf{m}}_d(t'_0,s) & \text{by item 2} \\ & = & \mathcal{T}^{\mathbf{m}}_d(t'_0,t') & \end{array}$$

3.5.2 If the step is internal to s: Similar to case 3.4.

3.5.3 If the step is internal to u: Similar to case 3.4.

3.6 $\mathtt{L} = \Box$: Impossible, as there are no forgetful steps $\Box \rhd \mathtt{L}'$.

3.7 $\mathtt{L} = \mathtt{L}_1\{t\}$: Similar to case 3.5.

4. Let $t \rhd t'$. We argue that $\mathcal{T}^{\mathbf{m}}_{\leq d}(t) \succeq \mathcal{T}^{\mathbf{m}}_{\leq d}(t')$. Indeed:

$$\begin{array}{rcll} \mathcal{T}^{\mathbf{m}}_{\leq d}(t) & = & \sum_{i=1}^{d} \mathcal{T}^{\mathbf{m}}_{d}(t,t) & \\ & \succeq & \sum_{i=1}^{d} \mathcal{T}^{\mathbf{m}}_{d}(t',t') & \text{by item 3, resorting to the IH when } i < d \\ & = & \mathcal{T}^{\mathbf{m}}_{\leq d}(t') & \end{array}$$

Note that for the value $i = d$, we resort directly to item 3 and not to the IH.

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