Reductions in Higher-Order Rewriting and Their Equivalence

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Abstract

Proof terms are syntactic expressions that represent computations in term rewriting. They were introduced by Meseguer and exploited by van Oostrom and de Vrijer to study *equivalence of reductions* in (left-linear) first-order term rewriting systems. We study the problem of extending the notion of proof term to *higher-order rewriting*, which generalizes the first-order setting by allowing terms with binders and higher-order substitution. In previous works that devise proof terms for higher-order rewriting, such as Bruggink's, it has been noted that the challenge lies in reconciling composition of proof terms and higher-order substitution (β-equivalence). This led Bruggink to reject "nested" composition, other than at the outermost level. In this paper, we propose a notion of higher-order proof term we dub *rewrites* that supports nested composition. We then define *two* notions of equivalence on rewrites, namely *permutation equivalence* and *projection equivalence*, and show that they coincide.

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1 Introduction

Term rewriting systems model computation as sequences of steps between terms, *reduction sequences*, where steps are instances of term rewriting rules [15]. It is natural to consider reduction sequences up to swapping of orthogonal steps since such reductions perform the "same work". The ensuing notion of equivalence is called *permutation equivalence* and was first studied by Lévy [11] in the setting of the λ-calculus but has appeared in other guises connected with concurrency [15, Rem.8.1.1]. As an example, consider the rewrite rule $\mathbf{c}(x,\mathbf{f}(y)) \to \mathbf{d}(x,x)$ and the following reduction sequence where, in each step, the contracted redex is underlined:

$$\underline{\mathbf{c}(\mathbf{c}(z,\mathbf{f}(z)),\mathbf{f}(z))} \to \mathbf{d}(\underline{\mathbf{c}(z,\mathbf{f}(z))},\mathbf{c}(z,\mathbf{f}(z))) \to \mathbf{d}(\mathbf{d}(z,z),\underline{\mathbf{c}(z,\mathbf{f}(z))}) \to \mathbf{d}(\mathbf{d}(z,z),\mathbf{d}(z,z)) \tag{1}$$

Performing the innermost redex first, rather than the outermost one, leads to:

$$\mathbf{c}(\underline{\mathbf{c}(z,\mathbf{f}(z))},\mathbf{f}(z)) \to \underline{\mathbf{c}(\mathbf{d}(z,z),\mathbf{f}(z))} \to \mathbf{d}(\mathbf{d}(z,z),\mathbf{d}(z,z)) \tag{2}$$

The first step in (1) makes two copies of the innermost redex. It is the two steps contracting these copies that are swapped with the first one in (1) to produce (2). Such duplication (and erasure) contribute most of the complications behind permutation equivalence, both in its formulation and the study of its properties.

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Proof Terms. *Proof terms* are a natural representation for computations. They were introduced by Meseguer as a means of representing proofs in Rewriting Logic [13] and exploited by van Oostrom and de Vrijer in the setting of first-order left-linear rewriting systems, to study equivalence of reductions in [17] and [15, Chapter 9]. Rewrite rules are assigned *rule symbols* denoting the application of a rewriting rule. Proof terms are expressions built using function symbols, a binary operator "**;**" denoting sequential composition of proof terms, and rule symbols. Assuming the following rule symbol for our rewrite rule $\varrho(x, y) : \mathbf{c}(x, \mathbf{f}(y)) \rightarrow \mathbf{d}(x, x)$, reduction (1) may be represented as the proof term: $\varrho(\mathbf{c}(z, \mathbf{f}(z)), z)$ **;** $\mathbf{d}(\varrho(z, z), \mathbf{c}(z, \mathbf{f}(z)))$ **;** $\mathbf{d}(\mathbf{d}(z, z), \varrho(z, z))$ and reduction (2) as the proof term: $\mathbf{c}(\varrho(z, z), \mathbf{f}(z))$ **;** $\varrho(\mathbf{d}(z, z), z)$. One notable feature of proof terms is that they support parallel steps. For instance, both proof terms above are permutation equivalent to $\varrho(\mathbf{c}(z, \mathbf{f}(z)), z)$ **;** $\mathbf{d}(\varrho(z, z), \varrho(z, z))$, which performs the two last steps in parallel, as well as to $\varrho(\varrho(z, z), z)$, which performs all steps simultaneously. Permutation equivalence now can be studied in terms of equational theories on proof terms.

Equivalence of Reductions via Proof Terms for First-Order Rewriting. In [17], van Oostrom and de Vrijer characterize permutation equivalence of proof terms in four alternative ways. First, they formulate an equational theory of permutation equivalence $\rho \approx \sigma$ between proof terms, such that for example $\varrho(\mathbf{c}(z, \mathbf{f}(z)), z)$ **;** $\mathbf{d}(\varrho(z, z), \varrho(z, z)) \approx \varrho(\varrho(z, z), z)$ holds. These equations account for the behavior of proof term composition, which has a monoidal structure, in the sense that composition is associative and *empty* steps act as identities. Second, they define an operation of *projection* ρ/σ, denoting the computational work that is left of ρ after σ. For example, $\mathbf{c}(\varrho(z, z), \mathbf{f}(z))/\varrho(\mathbf{c}(z, \mathbf{f}(z)), z) = \mathbf{d}(\varrho(z, z), \varrho(z, z))$. This induces a notion of *projection equivalence* between proof terms ρ and σ, declared to hold when both ρ/σ and σ/ρ are empty, *i.e.* they contain no rule symbols. Third, they define a *standardization procedure* to reorder the steps of a reduction in outside-in order, mapping each proof term ρ to a proof term ρ^* in *standard form.* For example, the (parallel) standard form of $\mathbf{c}(\varrho(z, z), \mathbf{f}(z))$ **;** $\varrho(\mathbf{d}(z, z), z)$ is $\varrho(\mathbf{c}(z, \mathbf{f}(z)), z)$ **;** $\mathbf{d}(\varrho(z, z), \varrho(z, z))$. This induces a notion of *standardization equivalence* between proof terms ρ and σ, declared to hold when $\rho^* = \sigma^*$. Fourth, they define a notion of *labelling equivalence*, based on lifting computational steps to labelled terms. Although these notions of equivalence were known prior to [17], the main result of that paper is that they are systematically studied using proof terms and, moreover, shown to coincide.

Higher-Order Rewriting. Higher-order term rewriting (HOR) generalizes first-order term rewriting by allowing binders. Function symbols are generalized to constants of any given simple type, and first-order terms are generalized to simply-typed λ-terms, including constants and up to $\beta\eta$-equivalence. The paradigmatic example of a higher-order rewriting system is the λ-calculus. It includes a base type ι and two constants $\mathbf{app} : \iota \to \iota \to \iota$ and $\mathbf{lam} : (\iota \to \iota) \to \iota$; β-reduction may be expressed as the higher-order rewrite rule $\mathbf{app}\,(\mathbf{lam}\,(\lambda z.x\,z))\,y \rightarrow x\,y$. A sample reduction sequence is:

$$\mathbf{lam}(\lambda v.\mathbf{app}(\mathbf{lam}(\lambda x.x), \underline{\mathbf{app}(\mathbf{lam}(\lambda w.w), v)})) \rightarrow \mathbf{lam}(\lambda v.\underline{\mathbf{app}(\mathbf{lam}(\lambda x.x), v)}) \rightarrow \mathbf{lam}(\lambda v.v) \quad (3)$$

Generalizing proof terms to the setting of higher-order rewriting is a natural goal. Just like in the first-order case, we assign rule symbols to rewrite rules. One would then expect to obtain proof terms by adding these rule symbols and the ";" composition operator to the simply typed λ-calculus. If we assume the following rule symbol for our rewrite rule $\varrho\,x\,y : \mathbf{app}\,(\mathbf{lam}\,(\lambda z.x\,z))\,y \rightarrow x\,y$, then an example of a higher-order proof term for (3) is:

$$\mathbf{lam}\Big(\lambda v.\big(\mathbf{app}(\mathbf{lam}(\lambda x.x), \varrho\,(\lambda w.w)\,v)\ ;\ \varrho\,(\lambda u.u)\,v\big)\Big)$$

However, higher-order substitution and proof term composition seem not to be in consonance, an issue already observed by Bruggink [4]. Consider a variable x. This variable itself denotes an empty computation $x \rightharpoonup x$, so the composition $(x \,;\, x)$ also denotes an empty computation $x \rightharpoonup x$. If σ is an arbitrary proof term $s \rightharpoonup t$, the proof term $(\lambda x.(x \,;\, x))\,\sigma$ should, in principle, represent a computation $(\lambda x.x)\,s \rightharpoonup (\lambda x.x)\,t$. This is the same as $s \rightharpoonup t$, because terms are regarded up to $\beta\eta$-equivalence. The challenge lies in lifting $\beta\eta$-equivalence to the level of proof terms: if β-reduction is naively extended to operate on proof terms, the well-formed proof term $(\lambda x.(x \,;\, x))\,\sigma$ becomes equal to $(\sigma \,;\, \sigma)$, which is ill-formed because σ is not composable with itself if $s \neq_{\beta\eta} t$. Rather than simply disallowing the use of "**;**" under applications and abstractions (the route taken in [4]), our aim is to integrate it with $\beta\eta$-reduction.

Contribution. We propose a **syntax for higher-order proof terms**, called **rewrites**, that includes $\beta\eta$-equivalence and allows rewrites to be freely composed. We then define a relation $\rho \approx \sigma$ of **permutation equivalence** between rewrites, the central notion of our work. The issue mentioned above is avoided by *disallowing* the ill-behaved substitution of a rewrite in a rewrite "$\rho\{x\backslash\sigma\}$", and by only allowing notions of substitution of a term in a rewrite $\rho\{x\backslash s\}$, and of a rewrite in a term $s\{x\backslash\!\!\backslash\rho\}$. From these, a well-behaved notion of substitution of a rewrite in a rewrite $\rho\{x\backslash\!\!\backslash\sigma\}$ can be shown to be *derivable*. We also define a notion of **projection** $\rho /\!\!/ \sigma$. The induced notion of **projection equivalence coincides with permutation equivalence**, in the sense that $\rho \approx \sigma$ iff $\rho /\!\!/ \sigma \approx \sigma^{\mathsf{tgt}}$ and $\sigma /\!\!/ \rho \approx \rho^{\mathsf{tgt}}$, where ρ^{tgt} stands for the *target* term of ρ. The equivalence is established by means of **flattening**, a method to convert an arbitrary rewrite ρ into a (*flat*) representative $\rho^\flat$ that only uses the composition operator "**;**" at the top level and a notion of **flat permutation equivalence** $\rho \sim \sigma$. Flattening is achieved by means of a rewriting system whose objects are themselves rewrites. This system is shown to be confluent and strongly normalizing. We also show that **permutation equivalence is sound and complete with respect to flat permutation equivalence** in the sense that $\rho \approx \sigma$ if and only if $\rho^\flat \sim \sigma^\flat$.

Structure of the Paper. In Section 2 we review Nipkow's Higher-Order Rewriting Systems. Section 3 proposes our notion of rewrite and Section 4 introduces permutation equivalence for them. Flattening is presented in Section 5. In this section, we also formulate an equational theory defining the relation $\rho \sim \sigma$ of flat permutation equivalence between flat rewrites. It relies crucially on a ternary relation between *multisteps*, called *splitting* and written $\mu \Leftrightarrow \mu_1 \,;\, \mu_2$, meaning that μ and $\mu_1 \,;\, \mu_2$ perform the same computational work. In Section 6 we first define a projection operator for flat rewrites ρ/σ, and we lift it to a projection operator for arbitrary rewrites $\rho /\!\!/ \sigma \stackrel{\text{def}}{=} \rho^\flat/\sigma^\flat$. Then we show that the induced notion of projection equivalence coincides with permutation equivalence. Finally, we conclude and discuss related and future work. Detailed proofs can be found in the accompanying technical report [2].

2 Higher-Order Rewriting

There are various approaches to HOR in the literature, including Klop's Combinatory Reduction Systems (CRSs) [8] and Nipkow's Higher-Order Rewriting Systems (HRSs) [14, 12]. We consider HRSs in this paper. Their use of the simply-typed lambda calculus for representing terms and substitution provides a suitable starting point for modeling our rewrites. Moreover, HRS are arguably more general than CRS in that their instantiation mechanism is more powerful [15, Sec.11.4.2]. We next introduce HRS. Assume given a denumerably infinite set of *variables* $(x, y, \ldots)$, *base types* $(\alpha, \beta, \ldots)$, and *constant symbols* $(\mathbf{c}, \mathbf{d}, \ldots)$. The sets of *terms* $(s, t, \ldots)$ and *types* $(A, B, \ldots)$ are given by:

$$s \quad ::= \quad x \mid \mathbf{c} \mid \lambda x.s \mid s\,s \qquad\qquad A \quad ::= \quad \alpha \mid A \to A$$

A term can either be a variable, a constant, an abstraction or an application. A type can either be a base type or an arrow type. We write $\mathsf{fv}(s)$ for the free variables of s. We use $\overline{X_n}$, or sometimes just $\overline{X}$ if n is clear from the context, to denote a sequence $X_1, \ldots, X_n$. Following standard conventions, $s\,\overline{t_n}$ stands for the iterated application $s\,t_1 \ldots t_n$, and $\overline{A_n} \to B$ for the type $A_1 \to \ldots A_n \to B$. We write $s\{x\backslash t\}$ for the capture-avoiding substitution of all free occurrences of x in s with t and call it a *term/term substitution*. We identify terms that differ only in the names of their bound variables. A *typing context* $(\Gamma, \Gamma', \ldots)$ is a partial function from variables to types. We write $\mathsf{dom}(\Gamma)$ for the *domain* of Γ. Given a typing context Γ and $x \notin \mathsf{dom}(\Gamma)$, we write $\Gamma, x : A$ for the typing context such that $(\Gamma, x : A)(x) = A$, and $(\Gamma, x : A)(y) = \Gamma(y)$ whenever $y \neq x$. We write $\cdot$ for the empty typing context and $x \in \Gamma$ if $x \in \mathsf{dom}(\Gamma)$. A *signature* of a HRS is a set $\mathcal{C}$ of typed constants $\mathbf{c} : A$. A sample signature is $\mathcal{C} = \{\mathbf{app} : \iota \to \iota \to \iota, \mathbf{lam} : (\iota \to \iota) \to \iota\}$ for ι a base type.

▶ **Definition 1** (Type system for terms). *Terms are typed using the usual typing rules of the simply-typed λ-calculus:*

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\,\mathsf{Var} \qquad \frac{(\mathbf{c} : A) \in \mathcal{C}}{\Gamma \vdash \mathbf{c} : A}\,\mathsf{Con} \qquad \frac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x.s : A \to B}\,\mathsf{Abs} \qquad \frac{\Gamma \vdash s : A \to B \qquad \Gamma \vdash t : A}{\Gamma \vdash s\,t : B}\,\mathsf{App}$$

Given any Γ and A such that $\Gamma \vdash s : A$ can be proved using these rules, we say s is a typed term *over $\mathcal{C}$. We typically drop $\mathcal{C}$ assuming it is implicit.*

We assume the usual definition of β and η-reduction between terms. Recall that β-reduction (resp. η-reduction) is confluent and terminating on typed terms. We write $s\downarrow^{\beta}$ (resp. $s\downarrow^{\eta}$) for the unique β-normal form (resp. η-normal form) of s. The β-normal form of a term s has the form $\lambda\overline{x_k}.a\,t_1 \ldots t_m$, for a either a constant or a variable. The η-expanded form of s is defined as:

$$s\uparrow^{\eta} \quad \overset{\text{def}}{=} \quad \lambda\overline{x_{n+k}}.a\,(\overline{t_m}\uparrow^{\eta})\,(x_{n+1}\uparrow^{\eta}) \ldots (x_{n+k}\uparrow^{\eta})$$

where s is assumed to have type $\overline{A_{n+k}} \to B$ and the $x_{n+1}, \ldots, x_{n+k}$ are fresh. We use $s\updownarrow^{\eta}_{\beta}$ to denote the term $s\downarrow^{\beta}\uparrow^{\eta}$ and call it the $\beta\overline{\eta}$-normal form of s.

A *substitution* θ is a function from variables to typed terms such that $\theta(x) \neq x$ only for finitely many x. The *domain* of a substitution is defined as $\mathsf{dom}(\theta) = \{x \mid \theta(x) \neq x\}$. The application of a substitution $\theta = \{x_1 \mapsto s_1, \ldots, x_n \mapsto s_n\}$ to a term t is defined as $\theta\,t \overset{\text{def}}{=} ((\lambda\overline{x_n}.t)\,\overline{s_n})\updownarrow^{\eta}_{\beta}$.

▶ **Definition 2.** *A* pattern *is a typed term in β-normal form such that all free occurrences of a variable x_i are in a subterm of the form $x_i\,t_1 \ldots t_k$ with $t_1, \ldots, t_k$ η-equivalent to distinct bound variables. A* rewriting rule *is a pair $\langle \ell, r\rangle$ of typed terms in $\beta\overline{\eta}$-normal form of the same base type with ℓ a pattern not η-equivalent to a variable and $\mathsf{fv}(r) \subseteq \mathsf{fv}(\ell)$. An HRS is a pair consisting of a signature and a set of rewriting rules over that signature. We typically omit the signature.*

▶ **Definition 3.** *The rewrite relation $\to_{\mathcal{R}}$ for an HRS $\mathcal{R}$ is the relation over typed terms in $\beta\overline{\eta}$-normal form defined as follows:*

$$\frac{\langle \ell, r\rangle \in \mathcal{R}}{\theta\,\ell \to_{\mathcal{R}} \theta\,r}\,\mathsf{Root} \qquad \frac{s \to_{\mathcal{R}} t}{a\,\overline{r_m}\,s\,\overline{p_n} \to_{\mathcal{R}} a\,\overline{r_m}\,t\,\overline{p_n}}\,\mathsf{App} \qquad \frac{s \to_{\mathcal{R}} t}{\lambda x.s \to_{\mathcal{R}} \lambda x.t}\,\mathsf{Abs}$$

where a is either a constant or a variable of type $\overline{A_{m+1+n}} \to B$. We write $\to^*_{\mathcal{R}}$ (resp. $\leftrightarrow^*_{\mathcal{R}}$) for the reflexive, transitive (resp. reflexive, symmetric and transtive) closure of $\to_{\mathcal{R}}$.

▶ **Example 4.** Consider a base type ι and typed constants $\mathbf{mu} : (\iota \to \iota) \to \iota$ and $\mathbf{f} : \iota \to \iota$. Two sample rewriting rules are: $\langle \mathbf{mu}(\lambda y.x\,y), x\,(\mathbf{mu}(\lambda y.x\,y))\rangle$ and $\langle \mathbf{f}\,x, \mathbf{g}\,x\rangle$. All four terms have base type ι. An example of a sequence of rewrite steps is $\mathbf{mu}\,(\lambda x.\mathbf{f}\,x) \to_{\mathcal{R}} \mathbf{f}\,(\mathbf{mu}\,(\lambda x.\mathbf{f}\,x)) \to_{\mathcal{R}} \mathbf{f}\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)) \to_{\mathcal{R}} \mathbf{g}\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x))$.

An HRS is *orthogonal* if: **1.** The rules are *left-linear*, *i.e.* if the left-hand side ℓ has $\mathsf{fv}(\ell) = \{x_1, \dots, x_n\}$, then there is *exactly* one free occurrence of x_i in ℓ, for each $1 \leq i \leq n$. **2.** There are *no critical pairs*, as defined for example in [14, Def. 4.1]. Orthogonal HRSs are deterministic in the sense that their rewrite relation is confluent. All of the examples of HRSs presented above are orthogonal. In the sequel of this paper, we assume given a fixed, orthogonal HRS $\mathcal{R}$.

3 Rewrites

In this section we propose a syntax for higher-order proof terms, called **rewrites**[1]. Rewrites for an HRS $\mathcal{R}$ are a means for denoting proofs in Higher-Order Rewriting Logic (HORL, *cf.* Def. 7) which, in turn, correspond to reduction sequences in $\mathcal{R}$ (*cf.* Thm. 9). As in the first-order case [13], HORL is simply the equational theory that results from an HRS but disregarding symmetry. Given an HRS $\mathcal{R}$, let $\mathcal{R}^c$ denote the set of pairs $\langle \lambda\overline{x_n}.\ell, \lambda\overline{x_n}.r\rangle$ such that $\langle \ell, r\rangle \in \mathcal{R}$ and $\{x_1, \dots, x_n\} = \mathsf{fv}(\ell)$. We begin by recalling the definition of equational logic (*cf.* Def. 5), the equational theory induced by an HRS. It is essentially that of [12, Def. 3.11], except that in the inference rule ERule we use $\mathcal{R}^c$ rather than $\mathcal{R}$. This equivalent formulation will be convenient when introducing rewrites since free variables in the LHS of a rewrite rule will be reflected in the rewrite too.

▶ **Definition 5** (Equational Logic). *An HRS $\mathcal{R}$ induces a relation $\doteq_{\mathcal{R}}$ on terms defined by the following rules:*

$$\frac{\Gamma, x : A \vdash s : B \qquad \Gamma \vdash t : A}{\Gamma \vdash (\lambda x.s)\,t \doteq_{\mathcal{R}} s\{x\backslash t\} : B}\,\mathsf{EBeta} \qquad \frac{\Gamma, x : A \vdash s : B \qquad x \notin \mathsf{fv}(s)}{\Gamma \vdash \lambda x.s\,x \doteq_{\mathcal{R}} s : B}\,\mathsf{EEta}$$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \doteq_{\mathcal{R}} x : A}\,\mathsf{EVar} \qquad \frac{(\mathbf{c} : A) \in \mathcal{C}}{\Gamma \vdash \mathbf{c} \doteq_{\mathcal{R}} \mathbf{c} : A}\,\mathsf{ECon} \qquad \frac{\Gamma, x : A \vdash s_0 \doteq_{\mathcal{R}} s_1 : B}{\Gamma \vdash \lambda x.s_0 \doteq_{\mathcal{R}} \lambda x.s_1 : A \to B}\,\mathsf{EAbs}$$

$$\frac{\Gamma \vdash s_0 \doteq_{\mathcal{R}} s_1 : A \to B \qquad \Gamma \vdash t_0 \doteq_{\mathcal{R}} t_1 : A}{\Gamma \vdash s_0\,t_0 \doteq_{\mathcal{R}} s_1\,t_1 : B}\,\mathsf{EApp} \qquad \frac{\langle s, t\rangle \in \mathcal{R}^c \qquad \cdot \vdash s : A \qquad \cdot \vdash t : A}{\Gamma \vdash s \doteq_{\mathcal{R}} t : A}\,\mathsf{ERule}$$

$$\frac{\Gamma \vdash s_0 \doteq_{\mathcal{R}} s_1 : A}{\Gamma \vdash s_1 \doteq_{\mathcal{R}} s_0 : A}\,\mathsf{ESymm} \qquad \frac{\Gamma \vdash s_0 \doteq_{\mathcal{R}} s_1 : A \qquad \Gamma \vdash s_1 \doteq_{\mathcal{R}} s_2 : A}{\Gamma \vdash s_0 \doteq_{\mathcal{R}} s_2 : A}\,\mathsf{ETrans}$$

▶ **Theorem 6** (Thm. 3.12 in [12]). $\Gamma \vdash s \doteq_{\mathcal{R}} t : A$ *iff* $s \updownarrow^{\eta}_{\beta} \leftrightarrow^*_{\mathcal{R}} t \updownarrow^{\eta}_{\beta}$.

The ($\Leftarrow$) direction follows from observing that $\to_{\beta,\overline{\eta}}$ and $\leftrightarrow^*_{\mathcal{R}}$ are all included in $\doteq_{\mathcal{R}}$. The ($\Rightarrow$) direction is by induction on the derivation of $\Gamma \vdash s \doteq_{\mathcal{R}} t : A$.

[1] Our notion of rewrite is unrelated to that of Def. 2.4 in [13]; it corresponds to "proof terms" as introduced in Sec. 3.1 in [13].

Higher-Order Rewriting Logic results from dropping ESymm in Def. 5 and adding a proof witness. Its judgments take the form $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ where the proof witness ρ is called a *rewrite*. Given a set of *rule symbols* $(\varrho, \vartheta, \ldots)$, the set of *rewrites* $(\rho, \sigma, \ldots)$ is given by:

$$\rho ::= x \mid \mathbf{c} \mid \varrho \mid \lambda x.\rho \mid \rho\,\rho \mid \rho\,;\rho$$

A rewrite can either be a variable, a constant, a rule symbol, an abstraction congruence, an application congruence, or a composition. Note that composition may occur anywhere inside a rewrite. For the sake of clarity we present the full system for Higher-Order Rewriting Logic next. We assume given an HRS $\mathcal{R}$ such that each rewrite rule $\langle \ell, r\rangle \in \mathcal{R}$ has been assigned a unique rule symbol ϱ and shall write $\langle \varrho, \ell, r\rangle \in \mathcal{R}$ and also use the same notation for $\mathcal{R}^c$. HORL consists of two forms of typing judgments:

1. $\Gamma \vdash s =_{\beta\eta} t : A$, meaning that s and t are $\beta\eta$-equivalent terms of type A under Γ; and
2. $\Gamma \vdash \rho : s \twoheadrightarrow_{\mathcal{R}} t : A$, meaning that ρ is a rewrite with source s and target t, which are terms of type A under Γ.

▶ **Definition 7** (Higher-Order Rewriting Logic). *Term equivalence is defined as the reflexive, symmetric, transitive, and contextual closure of:*

$$\frac{\Gamma, x : A \vdash s : B \qquad \Gamma \vdash t : A}{\Gamma \vdash (\lambda x.s)\,t =_{\beta\eta} s\{x\backslash t\} : B}\,\mathsf{EqBeta} \qquad \frac{\Gamma, x : A \vdash s : B \qquad x \notin \mathsf{fv}(s)}{\Gamma \vdash \lambda x.s\,x =_{\beta\eta} s : B}\,\mathsf{EqEta}$$

Typing rules for rewrites are as follows:

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : x \twoheadrightarrow_{\mathcal{R}} x : A}\,\mathsf{RVar} \qquad \frac{(\mathbf{c} : A) \in \mathcal{C}}{\Gamma \vdash \mathbf{c} : \mathbf{c} \twoheadrightarrow_{\mathcal{R}} \mathbf{c} : A}\,\mathsf{RCon} \qquad \frac{\Gamma, x : A \vdash \rho : s_0 \twoheadrightarrow_{\mathcal{R}} s_1 : B}{\Gamma \vdash \lambda x.\rho : \lambda x.s_0 \twoheadrightarrow_{\mathcal{R}} \lambda x.s_1 : A \to B}\,\mathsf{RAbs}$$

$$\frac{\Gamma \vdash \rho : s_0 \twoheadrightarrow_{\mathcal{R}} s_1 : A \to B \qquad \Gamma \vdash \sigma : t_0 \twoheadrightarrow_{\mathcal{R}} t_1 : A}{\Gamma \vdash \rho\,\sigma : s_0\,t_0 \twoheadrightarrow_{\mathcal{R}} s_1\,t_1 : B}\,\mathsf{RApp}$$

$$\frac{\langle \varrho, s, t\rangle \in \mathcal{R}^c \qquad \cdot \vdash s : A \qquad \cdot \vdash t : A}{\Gamma \vdash \varrho : s \twoheadrightarrow_{\mathcal{R}} t : A}\,\mathsf{RRule} \qquad \frac{\Gamma \vdash \rho : s_0 \twoheadrightarrow_{\mathcal{R}} s_1 : A \qquad \Gamma \vdash \sigma : s_1 \twoheadrightarrow_{\mathcal{R}} s_2 : A}{\Gamma \vdash \rho\,;\sigma : s_0 \twoheadrightarrow_{\mathcal{R}} s_2 : A}\,\mathsf{RTrans}$$

$$\frac{\Gamma \vdash \rho : s' \twoheadrightarrow_{\mathcal{R}} t' : A \qquad \Gamma \vdash s =_{\beta\eta} s' : A \qquad \Gamma \vdash t' =_{\beta\eta} t : A}{\Gamma \vdash \rho : s \twoheadrightarrow_{\mathcal{R}} t : A}\,\mathsf{RConv}$$

The RVar and RCon rules express that variables and constants represent identity rewrites. The RAbs and RApp rules express congruence below abstraction and application. The RRule rule allows us to use a rule symbol to stand for a rewrite between its source and its target, which must be closed terms of the same type. The RConv rule states that the source and the target of a rewrite are regarded up to $\beta\eta$-equivalence. Note that there are no rules equating rewrites; such rules are the purpose of Section 4 which introduces permutation equivalence.

▶ **Example 8.** Suppose we assign the following rule symbols to the rewriting rules of Ex. 4: $\langle \varrho, \mathbf{mu}(\lambda y.x\,y), x\,(\mathbf{mu}(\lambda y.x\,y))\rangle$ and $\langle \vartheta, \mathbf{f}\,x, \mathbf{g}\,x\rangle$. Recall that $\mathcal{C} \stackrel{\text{def}}{=} \{\mathbf{mu} : (\iota \to \iota) \to \iota, \mathbf{f} : \iota \to \iota\}$. The reduction of Ex. 4 can be represented as a rewrite:

$$\cdot \vdash \varrho\,(\lambda x.\mathbf{f}\,x)\,;\mathbf{f}\,(\mathbf{mu}\,(\lambda x.\vartheta\,x))\,;\vartheta\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)) : \mathbf{mu}\,(\lambda x.\mathbf{f}\,x) \twoheadrightarrow_{\mathcal{R}} \mathbf{g}\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)) : \iota$$

Inspection of the proof of Thm. 6 in [12] reveals that β and η are only needed for substitutions in rewrite rules. As a consequence:

▶ **Theorem 9.** *There is a rewrite ρ such that $\Gamma \vdash \rho : s \twoheadrightarrow_{\mathcal{R}} t : A$ if and only if $s \updownarrow^{\eta}_{\beta} \xrightarrow{*}_{\mathcal{R}} t \updownarrow^{\eta}_{\beta}$.*

Now that we know that rewrites over an HRS $\mathcal{R}$ are sound and complete with respect to reduction sequences in $\mathcal{R}$, we review some basic properties of rewrites and then focus, in the remaining sections, on equivalences between rewrites. In the sequel we will omit $\mathcal{R}$ in $\Gamma \vdash \rho : s \twoheadrightarrow_{\mathcal{R}} t : A$ and write $\Gamma \vdash \rho : s \twoheadrightarrow t : A$.

▶ **Definition 10** (Source and target of a rewrite). *For each rewrite ρ we define the* source ρ^{src} *and the* target ρ^{tgt} *as the following terms:*

$$\begin{array}{rcll@{\qquad}rcll}
x^{\mathsf{src}} & \stackrel{\text{def}}{=} & x & & x^{\mathsf{tgt}} & \stackrel{\text{def}}{=} & x & \\
\mathbf{c}^{\mathsf{src}} & \stackrel{\text{def}}{=} & \mathbf{c} & & \mathbf{c}^{\mathsf{tgt}} & \stackrel{\text{def}}{=} & \mathbf{c} & \\
\varrho^{\mathsf{src}} & \stackrel{\text{def}}{=} & s & \textit{if } (\varrho : s \twoheadrightarrow t : A) \in \mathcal{R} & \varrho^{\mathsf{tgt}} & \stackrel{\text{def}}{=} & t & \textit{if } (\varrho : s \twoheadrightarrow t : A) \in \mathcal{R} \\
(\lambda x.\rho)^{\mathsf{src}} & \stackrel{\text{def}}{=} & \lambda x.\rho^{\mathsf{src}} & & (\lambda x.\rho)^{\mathsf{tgt}} & \stackrel{\text{def}}{=} & \lambda x.\rho^{\mathsf{tgt}} & \\
(\rho\,\sigma)^{\mathsf{src}} & \stackrel{\text{def}}{=} & \rho^{\mathsf{src}}\,\sigma^{\mathsf{src}} & & (\rho\,\sigma)^{\mathsf{tgt}} & \stackrel{\text{def}}{=} & \rho^{\mathsf{tgt}}\,\sigma^{\mathsf{tgt}} & \\
(\rho\,;\sigma)^{\mathsf{src}} & \stackrel{\text{def}}{=} & \rho^{\mathsf{src}} & & (\rho\,;\sigma)^{\mathsf{tgt}} & \stackrel{\text{def}}{=} & \rho^{\mathsf{tgt}} &
\end{array}$$

The free variables of an expression X (which may be a term or a rewrite) are written $\mathsf{fv}(X)$, and defined as expected, with lambdas binding variables in their bodies. For any given term or rewrite X, we write $X\{x\backslash t\}$ for the capture-avoiding substitution of the variable x in X by t. The operation $\rho\{x\backslash t\}$ is called *rewrite/term substitution*.

We mention a few important syntactic properties of terms and rewrites (detailed statements and proofs can be found in Section A of [2]). First, some basic properties hold, such as weakening (*e.g.* if $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ then $\Gamma, x : B \vdash \rho : s \twoheadrightarrow t : A$) and commuting substitution with the source and target operators (*e.g.* $\rho\{x\backslash s\}^{\mathsf{src}} = \rho^{\mathsf{src}}\{x\backslash s\}$). Terms appearing in valid equality and rewriting judgments can always be shown to be typable, that is, if either $\Gamma \vdash s =_{\beta\eta} t : A$ or $\Gamma \vdash \rho : s \twoheadrightarrow t : A$, then $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$. Second, given a typable rewrite, $\Gamma \vdash \rho : s \twoheadrightarrow t : A$, the source of ρ and s are not necessarily equal, but they are interconvertible, that is $\Gamma \vdash s =_{\beta\eta} \rho^{\mathsf{src}} : A$, and similarly for the target, *i.e.* $\Gamma \vdash t =_{\beta\eta} \rho^{\mathsf{tgt}} : A$. For example, if $\varrho : \lambda x.\mathbf{c}\,x \twoheadrightarrow \lambda x.\mathbf{d} : A \to A$ then it can be shown that $\vdash \varrho\,\mathbf{d} : \mathbf{c}\,\mathbf{d} \twoheadrightarrow \mathbf{d} : A$, and indeed $\mathbf{c}\,\mathbf{d} =_{\beta\eta} (\lambda x.\mathbf{c}\,x)\,\mathbf{d} = (\varrho\,\mathbf{d})^{\mathsf{src}}$. Third, any typable term s can be understood as an empty or *unit* rewrite $\underline{s}$, without occurrences of rule symbols, between s and itself: if $\Gamma \vdash s : A$ then $\Gamma \vdash \underline{s} : s \twoheadrightarrow s : A$. We usually coerce terms to rewrites implicitly if there is little danger of confusion. Substitution of a variable for a term is functorial, that is, given a rewrite $\Gamma, x : A \vdash \rho : s \twoheadrightarrow t : B$ and a term $\Gamma \vdash r : A$, then $\Gamma \vdash \rho\{x\backslash r\} : s\{x\backslash r\} \twoheadrightarrow t\{x\backslash r\} : B$.

Term/rewrite substitution generalizes term/term substitution $s\{x\backslash t\}$ when t is a rewrite, *i.e.* $s\{x\backslash\!\!\backslash\rho\}$. Sometimes we also call this notion *lifting substitution*, as $s\{x\backslash\!\!\backslash\rho\}$ “lifts” the expression s from the level of terms to the level of rewrites.

▶ **Definition 11** (Term/rewrite substitution).

$$\begin{array}{rcll@{\qquad}rcl}
y\{x\backslash\!\!\backslash\rho\} & \stackrel{\text{def}}{=} & \begin{cases} \rho & \textit{if } x = y \\ y & \textit{if } x \neq y \end{cases} & & \mathbf{c}\{x\backslash\!\!\backslash\rho\} & \stackrel{\text{def}}{=} & \mathbf{c} \\
(\lambda y.s)\{x\backslash\!\!\backslash\rho\} & \stackrel{\text{def}}{=} & \lambda y.s\{x\backslash\!\!\backslash\rho\} & \textit{if } x \neq y & (s\,t)\{x\backslash\!\!\backslash\rho\} & \stackrel{\text{def}}{=} & s\{x\backslash\!\!\backslash\rho\}\,t\{x\backslash\!\!\backslash\rho\}
\end{array}$$

We mention some important properties of term/rewrite substitution. First, term/rewrite substitution is a kind of *horizontal composition*, in the sense that if $\Gamma, x : A \vdash s : B$ and $\Gamma \vdash \rho : t \twoheadrightarrow t' : A$ then $\Gamma \vdash s\{x\backslash\!\!\backslash\rho\} : s\{x\backslash t\} \twoheadrightarrow s\{x\backslash t'\} : B$. Second, term/rewrite and rewrite/term substitution commute according to the equation $s\{x\backslash\!\!\backslash\rho\}\{y\backslash t\} = s\{y\backslash t\}\{x\backslash\!\!\backslash\rho\{y\backslash t\}\}$, assuming that $\Gamma, x : A, y : B \vdash s : C$ and $\Gamma, y : B \vdash \rho : r \twoheadrightarrow r' : A$ and $\Gamma \vdash t : B$ (where, by convention, $x \notin \mathsf{fv}(t)$). Note that, in particular, if y does not occur free in ρ, this means that

$s\{x\backslash\!\backslash\rho\}\{y\backslash t\} = s\{y\backslash t\}\{x\backslash\!\backslash\rho\}$. Third, term/rewrite substitution commutes with reflexivity in the sense that $s\{x\backslash\!\backslash\underline{t}\} = \underline{s\{x\backslash t\}}$ holds whenever $\Gamma, x : A \vdash s : B$ and $\Gamma \vdash t : A$. It also commutes with the source and target operators, in the sense that $s\{x\backslash\!\backslash\rho\}^{\mathsf{src}} = s\{x\backslash\rho^{\mathsf{src}}\}$ and $s\{x\backslash\!\backslash\rho\}^{\mathsf{tgt}} = s\{x\backslash\rho^{\mathsf{tgt}}\}$ hold whenever $\Gamma, x : A \vdash s : B$ and $\Gamma \vdash \rho : t \twoheadrightarrow t' : A$.

4 Permutation equivalence

This section presents *permutation equivalence* (Def. 12), a relation over (typed) rewrites $\rho \approx \sigma$ that identifies any two rewrites ρ and σ denoting computations in a given HRS $\mathcal{R}$ that are equivalent up to permutation of steps.

Towards Permutation Equivalence for Rewrites. Equipped with the self-evident operations of *term/rewrite substitution* $s\{x\backslash\!\backslash\rho\}$, *rewrite/term substitution* $\rho\{x\backslash t\}$ and the fact that rewrites may be freely composed, we set out to synthesize a definition of permutation equivalence by attempting to assign a meaning for $(\lambda x.\rho)\,\sigma$, where $\Gamma \vdash \rho : s_0 \twoheadrightarrow s_1 : A$ and $\Gamma \vdash \sigma : t_0 \twoheadrightarrow t_1 : A$. We begin by assuming we have equations that allow rewrites to be post-composed with their targets ($\approx$-IdR) and pre-composed with their source ($\approx$-IdL) and reason as follows:

$$(\lambda x.\rho)\,\sigma \quad \approx^{(\mathsf{IdR})} \quad ((\lambda x.\rho)\,;(\lambda x.s_1))\,\sigma \quad \approx^{(\mathsf{IdL})} \quad ((\lambda x.\rho)\,;(\lambda x.s_1))\,(t_0\,;\sigma)$$

These rewrites are syntactically valid since we allow composition inside an application. Next, we allow application to commute with composition by introducing a rule $\approx$-App: $(\rho_1\rho_2)\,;(\sigma_1\sigma_2) \approx (\rho_1\,;\sigma_1)(\rho_2\,;\sigma_2)$. Applying this equation leads us to:

$$((\lambda x.\rho)\,;(\lambda x.s_1))\,(t_0\,;\sigma) \quad \approx^{(\mathsf{App})} \quad (\lambda x.\rho)\,t_0\,;(\lambda x.s_1)\,\sigma$$

Finally, we introduce β-equality on rewrites. Arbitrary β-reduction of rewrites is not allowed *a priori*. It is only allowed when either the abstraction or the argument are unit rewrites, for which the substitution operators mentioned above can be used. These equations take the form $(\lambda x.\underline{s})\,\rho \approx s\{x\backslash\!\backslash\rho\}$ and $(\lambda x.\rho)\,\underline{s} \approx \rho\{x\backslash s\}$ and are called, $\approx$-BetaTR and $\approx$-BetaRT.

$$(\lambda x.\rho)\,t_0\,;(\lambda x.s_1)\,\sigma \quad \approx^{(\mathsf{BetaRT})} \quad \rho\{x\backslash t_0\}\,;(\lambda x.s_1)\,\sigma \quad \approx^{(\mathsf{BetaTR})} \quad \rho\{x\backslash t_0\}\,;s_1\{x\backslash\!\backslash\sigma\}$$

In summary we have $(\lambda x.\rho)\,\sigma \approx \rho\{x\backslash t_0\}\,;s_1\{x\backslash\!\backslash\sigma\}$. We could equally well have deduced $(\lambda x.\rho)\,\sigma \approx s_0\{x\backslash\!\backslash\sigma\}\,;\rho\{x\backslash t_1\}$. As it turns out, however, $\rho\{x\backslash t_0\}\,;s_1\{x\backslash\!\backslash\sigma\}$ and $s_0\{x\backslash\!\backslash\sigma\}\,;\rho\{x\backslash t_1\}$ are permutation equivalent in our theory.

Permutation Equivalence for Rewrites: Definition and Properties. We collect the observations above in the following definition.

▶ **Definition 12** (Permutation equivalence). *Suppose $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ and $\Gamma \vdash \rho' : s' \twoheadrightarrow t' : A$ are derivable. Permutation equivalence, written $\Gamma \vdash (\rho : s \twoheadrightarrow t) \approx (\rho' : s' \twoheadrightarrow t') : A$ (or simply $\rho \approx \rho'$ if Γ, s, t, s', t', A are clear from the context), is defined as the reflexive, symmetric, transitive, and contextual closure of the following axioms:*

$\underline{\rho^{\mathsf{src}}}\,;\rho$	$\approx$	ρ		$\approx$-IdL
$\rho\,;\underline{\rho^{\mathsf{tgt}}}$	$\approx$	ρ		$\approx$-IdR
$(\rho\,;\sigma)\,;\tau$	$\approx$	$\rho\,;(\sigma\,;\tau)$		$\approx$-Assoc
$(\lambda x.\rho)\,;(\lambda x.\sigma)$	$\approx$	$\lambda x.(\rho\,;\sigma)$		$\approx$-Abs
$(\rho_1\rho_2)\,;(\sigma_1\sigma_2)$	$\approx$	$(\rho_1\,;\sigma_1)(\rho_2\,;\sigma_2)$		$\approx$-App
$(\lambda x.\underline{s})\,\rho$	$\approx$	$s\{x\backslash\!\backslash\rho\}$		$\approx$-BetaTR
$(\lambda x.\rho)\,\underline{s}$	$\approx$	$\rho\{x\backslash s\}$		$\approx$-BetaRT
$\lambda x.\rho\,x$	$\approx$	ρ	*if* $x \notin \mathsf{fv}(\rho)$	$\approx$-Eta

Rules ≈-IdL, ≈-IdR and ≈-Assoc, state that rewrites together with rewrite composition have a monoidal structure. Recall from Section 3 that ρ^{src} is a term and $\underline{\rho^{\mathsf{src}}}$ is its corresponding rewrite. Rules ≈-Abs and ≈-App state that rewrite composition commutes with abstraction and application. An important thing to be wary of is that rules may be applied only if both the left and the right-hand sides are well-typed. In particular, the right-hand side of the ≈-App rule may not be well-typed even if the left-hand side is; for example given rule symbols $\mathbf{c} : A \to B$ and $\mathbf{d} : A$, the expression $((\lambda x.x)(\mathbf{c}\,\mathbf{d}))\ ;\ (\mathbf{c}\,\mathbf{d})$ is well-typed, with source and target $\mathbf{c}\,\mathbf{d}$, while $((\lambda x.x)\ ;\ \mathbf{c})\,((\mathbf{c}\,\mathbf{d})\ ;\ \mathbf{d})$ is not well-typed.

Finally, rules ≈-BetaTR, ≈-BetaRT and ≈-Eta introduce $\beta\eta$-equivalence for rewrites. Note that ≈-BetaTR and ≈-BetaRT restrict either the body of the abstraction or the argument to a unit rewrite, thus avoiding the issue mentioned in the introduction where a naive combination of composition and $\beta\eta$-equivalence can lead to invalid rewrites.

Note that there are no explicit sequencing equations such as the I/O equations[2] defining permutation equivalence in the first-order case [15] and the corresponding equations flat-l and flat-r of [4] for the higher-order case. Nonetheless, we can derive the following coherence equation (see Lem. 68 in [2] for the proof):

$$\rho\{x\backslash s'\}\ ;\ t\{x\backslash\!\!\backslash\sigma\} \quad\approx\quad s\{x\backslash\!\!\backslash\sigma\}\ ;\ \rho\{x\backslash t'\} \qquad (\approx\text{-Perm})$$

where $\Gamma, x : A \vdash \rho : s \to t : B$ and $\Gamma \vdash \sigma : s' \to t' : A$.

▶ **Example 13.** Consider the HRS of Ex. 4 and the reduction of Ex. 8. We recall the latter below (R_2) and present a second one (R_1).

$$\begin{array}{lll} R_1 & : & \mathbf{mu}\,(\lambda x.\mathbf{f}\,x) \to \mathbf{mu}\,(\lambda x.\mathbf{g}\,x) \to \mathbf{g}\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)) \\ R_2 & : & \mathbf{mu}\,(\lambda x.\mathbf{f}\,x) \to \mathbf{f}\,(\mathbf{mu}\,(\lambda x.\mathbf{f}\,x)) \to \mathbf{f}\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)) \to \mathbf{g}\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)) \end{array}$$

Reduction sequence R_1 can be encoded as the rewrite $\mathbf{mu}\,(\lambda x.\vartheta\,x); \varrho\,(\lambda x.\mathbf{g}\,x)$ and R_2 as $\varrho\,(\lambda x.\mathbf{f}\,x)\ ;\ \mathbf{f}\,(\mathbf{mu}\,(\lambda x.\vartheta\,x))\ ;\ \vartheta\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x))$. These two rewrites are permutation equivalent:

$$\begin{array}{ll} & \mathbf{mu}\,(\lambda x.\vartheta\,x)\ ;\ \varrho\,(\lambda x.\mathbf{g}\,x) \\ \approx^{(\mathsf{Eta})} & \mathbf{mu}\,\vartheta\ ;\ \varrho\,\mathbf{g} \\ = & (\mathbf{mu}\,y)\{y\backslash\!\!\backslash\vartheta\}\ ;\ (\varrho\,y)\{y\backslash\mathbf{g}\} \\ \approx^{(\mathsf{Perm})} & (\varrho\,y)\{y\backslash\mathbf{f}\}\ ;\ (y\,(\mathbf{mu}\,y))\{y\backslash\!\!\backslash\vartheta\} \\ = & \varrho\,\mathbf{f}\ ;\ \vartheta\,(\mathbf{mu}\,\vartheta) \\ \approx^{(\mathsf{IdL})} & \varrho\,\mathbf{f}\ ;\ (\mathbf{f}\ ;\ \vartheta)\,(\mathbf{mu}\,\vartheta) \\ \approx^{(\mathsf{IdR})} & \varrho\,\mathbf{f}\ ;\ (\mathbf{f}\ ;\ \vartheta)\,((\mathbf{mu}\,\vartheta)\ ;\ (\mathbf{mu}\,\mathbf{g})) \\ \approx^{(\mathsf{App})} & \varrho\,\mathbf{f}\ ;\ \mathbf{f}\,(\mathbf{mu}\,\vartheta)\ ;\ \vartheta\,(\mathbf{mu}\,\mathbf{g}) \\ \approx^{(\mathsf{Eta})} & \varrho\,(\lambda x.\mathbf{f}\,x)\ ;\ \mathbf{f}\,(\mathbf{mu}\,(\lambda x.\vartheta\,x))\ ;\ \vartheta\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)) \end{array}$$

The ≈-Perm rule motivates the definition of *rewrite/rewrite substitution*, $\rho\{x\backslash\!\!\backslash\sigma\} \stackrel{\text{def}}{=} \rho\{x\backslash s'\}\ ;\ t\{x\backslash\!\!\backslash\sigma\}$, which defines a rewrite $s\{x\backslash s'\} \to t\{x\backslash t'\}$. Note that $\rho\{x\backslash\!\!\backslash\sigma\}$ depends on t and s', and hence on the particular typing derivations for ρ and σ. Congruence results (Lem. 63 and Lem. 64 in the appendix) ensure that the value of $\rho\{x\backslash\!\!\backslash\sigma\}$ does not depend, up to permutation equivalence, on those typing derivations. Rewrite/rewrite substitution generalizes rewrite/term and term/rewrite substitution, in the sense that $\rho\{x\backslash t\} \approx \rho\{x\backslash\!\!\backslash\underline{t}\}$ and $s\{x\backslash\!\!\backslash\rho\} \approx \underline{s}\{x\backslash\!\!\backslash\rho\}$.

Other important facts involving rewrite/rewrite substitution are the following. First, it commutes with abstraction, application, and composition, that is $(\lambda y.\rho)\{x\backslash\!\!\backslash\sigma\} \approx \lambda y.\rho\{x\backslash\!\!\backslash\sigma\}$,

[2] $I : \varrho(\sigma_1, ..., \sigma_n) \approx l(\sigma_1, ..., \sigma_n) \cdot \varrho(t_1, ..., t_n)$ and $O : \varrho(\sigma_1, ..., \sigma_n) \approx \varrho(s_1, ..., s_n) \cdot r(\sigma_1, ..., \sigma_n)$

$(\rho_1\,\rho_2)\{x\backslash\!\!\backslash\sigma\} \approx \rho_1\{x\backslash\!\!\backslash\sigma\}\,\rho_2\{x\backslash\!\!\backslash\sigma\}$, and $(\rho_1\,;\,\rho_2)\{x\backslash\!\!\backslash\sigma_1\,;\,\sigma_2\} \approx \rho_1\{x\backslash\!\!\backslash\sigma_1\}\,;\,\rho_2\{x\backslash\!\!\backslash\sigma_2\}$. Second, permutation equivalence is a congruence with respect to rewrite/rewrite substitution, that is, if $\rho \approx \rho'$ and $\sigma \approx \sigma'$ then $\rho\{x\backslash\!\!\backslash\sigma\} \approx \rho'\{x\backslash\!\!\backslash\sigma'\}$. Third, an analog of the substitution lemma holds, namely $\rho\{x\backslash\!\!\backslash\sigma\}\{y\backslash\!\!\backslash\tau\} \approx \rho\{y\backslash\!\!\backslash\tau\}\{x\backslash\!\!\backslash\sigma\{y\backslash\!\!\backslash\tau\}\}$. Finally, as discussed above, a β-rule for arbitrary rewrites holds in the form $(\lambda x.\rho)\,\sigma \approx \rho\{x\backslash\!\!\backslash\sigma\}$. The full theory of rewrite/rewrite substitution is not developed here for lack of space (but see Section B.2 in [2]).

5 Flattening

Allowing composition to be nested within application and abstraction can give rise to rewrites in which it is not obvious what reduction sequences of steps are being denoted. An example from the previous section might be the rewrite $((\lambda x.\mathbf{f}\,x)\,;\,\vartheta)\,((\mathbf{mu}\,(\lambda x.\vartheta\,x))\,;\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)))$ which denotes the reduction sequence $\mathbf{f}\,(\mathbf{mu}\,(\lambda x.\mathbf{f}\,x)) \twoheadrightarrow \mathbf{g}\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x))$ that replaces both occurrences of $\mathbf{f}$ with $\mathbf{g}$ simultaneously. This section shows how rewrites can be "flattened" so as to expose an underlying reduction sequence, expressed as a canonical (*flat*) rewrite. One additional use of flattening will be to use it to show that permutation equivalence is decidable (*cf.* end of Sec. Section 6). Before introducing flat rewrites we define *multisteps*.

A *multistep* is a rewrite without any occurrences of the composition operator. We use $\mu, \nu, \xi, \ldots$ to range over multisteps. The capture-avoiding substitution of the free occurrences of x in μ by ν is written $\mu\{x\backslash\nu\}$, which is in turn a multistep. A *flat multistep* $(\hat{\mu}, \hat{\nu}, \ldots)$, is a multistep in β-normal form, *i.e.* without subterms of the form $(\lambda x.\mu)\,\nu$. A *flat rewrite* $(\hat{\rho}, \hat{\sigma}, \ldots)$, is a rewrite given by the grammar $\hat{\rho} ::= \hat{\mu} \mid \hat{\rho}\,;\,\hat{\sigma}$. Flat rewrites use the composition operator "**;**" at the top level, that is they are of the form $\hat{\mu}_1\,;\ldots;\,\hat{\mu}_n$ (up to associativity of "**;**"), where each $\hat{\mu}_i$ is a flat multistep. Note that we do not require the $\hat{\mu}_i$ to be in $\beta\eta$-normal form nor in $\beta\overline{\eta}$-normal form. As mentioned in the introduction, flattening is achieved by means of a *rewriting system whose objects are themselves rewrites* (Def. 15) which is shown to be *confluent and terminating* (Prop. 17).

We also formulate an equational theory defining a relation $\rho \sim \sigma$ of *flat permutation equivalence* between flat rewrites (Def. 19). The main result of this section is that permutation equivalence is *sound and complete* with respect to flat permutation equivalence (Thm. 20).

▶ Remark 14. A substitution $\mu\{x\backslash\nu\}$ in which μ is a term is a term/rewrite substitution, *i.e.* $s\{x\backslash\nu\} = s\{x\backslash\!\!\backslash\nu\}$. A substitution in which ν is a term is a rewrite/term substitution, *i.e.* $\mu\{x\backslash s\} = \mu\{x\backslash s\}$.

▶ **Definition 15** (Flattening Rewrite System $\mathcal{F}$). *The flattening system $\mathcal{F}$ is given by the following rules, closed under arbitrary contexts, defined between* **typable** *rewrites:*

$$\begin{array}{rcll}
\lambda x.(\rho\,;\,\sigma) & \overset{\flat}{\mapsto} & (\lambda x.\rho)\,;\,(\lambda x.\sigma) & \mathcal{F}\text{-Abs} \\
(\rho\,;\,\sigma)\,\mu & \overset{\flat}{\mapsto} & (\rho\,\underline{\mu^{\mathsf{src}}})\,;\,(\sigma\,\mu) & \mathcal{F}\text{-App1} \\
\mu\,(\rho\,;\,\sigma) & \overset{\flat}{\mapsto} & (\mu\,\rho)\,;\,(\underline{\mu^{\mathsf{tgt}}}\,\sigma) & \mathcal{F}\text{-App2} \\
(\rho_1\,;\,\rho_2)\,(\sigma_1\,;\,\sigma_2) & \overset{\flat}{\mapsto} & ((\rho_1\,;\,\rho_2)\,\underline{\sigma_1^{\mathsf{src}}})\,;\,(\underline{\rho_2^{\mathsf{tgt}}}\,(\sigma_1\,;\,\sigma_2)) & \mathcal{F}\text{-App3} \\
(\lambda x.\mu)\,\nu & \overset{\flat}{\mapsto} & \mu\{x\backslash\nu\} & \mathcal{F}\text{-BetaM} \\
\lambda x.\mu\,x & \overset{\flat}{\mapsto} & \mu \quad \text{if } x \notin \mathsf{fv}(\mu) & \mathcal{F}\text{-EtaM}
\end{array}$$

Note that rules $\mathcal{F}$-BetaM *and* $\mathcal{F}$-EtaM *apply to multisteps only. The reduction relation* $\overset{\flat}{\mapsto}$ *is the union of all these rules, closed by compatibility under arbitrary contexts. We write* $\rho^\flat$ *for the unique* $\overset{\flat}{\mapsto}$*-normal form of* ρ.

▶ **Example 16.** Consider a rewriting rule $\varrho : \mathbf{c} \twoheadrightarrow \mathbf{d} : A$. The rewrite $(\lambda x.(x\ ;\ x))\,\varrho$, whose meaning (as previously mentioned) is not obvious, can be flattened as follows:

$$\begin{array}{llllll} (\lambda x.(x\ ;\ x))\,\varrho & \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} & ((\lambda x.x)\ ;\ (\lambda x.x))\,\varrho & \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} & (\lambda x.x)\,\mathbf{c}\ ;\ (\lambda x.x)\,\varrho \\ & \overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM}} & \mathbf{c}\ ;\ (\lambda x.x)\,\varrho & \overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM}} & \mathbf{c}\ ;\ \varrho \end{array}$$

The following result is proved by noting that $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM steps can be postponed after steps of other kinds and then providing a well-founded measure for steps in $\mathcal{F}$ without $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM to prove it is SN. Confluence of $\mathcal{F}$ follows from Newman's lemma.

▶ **Proposition 17.** *The flattening system $\mathcal{F}$ is strongly normalizing and confluent.*

Flat Permutation Equivalence. We now turn to the definition of the relation $\rho \sim \sigma$ of flat permutation equivalence. The key notion to define is the following ternary relation:

▶ **Definition 18** (Splitting). *Let $\Gamma \vdash \mu : s \twoheadrightarrow t : A$ and $\Gamma \vdash \mu_1 : s' \twoheadrightarrow r_1 : A$ and $\Gamma \vdash \mu_2 : r_2 \twoheadrightarrow t' : A$ be multisteps. We say that μ* splits *into μ_1 and μ_2 if the following inductively defined ternary relation, written $\mu \Leftrightarrow \mu_1\ ;\ \mu_2$, holds:*

$$\frac{}{x \Leftrightarrow x\ ;\ x}\,\textsf{SVar} \qquad \frac{}{\mathbf{c} \Leftrightarrow \mathbf{c}\ ;\ \mathbf{c}}\,\textsf{SCon} \qquad \frac{}{\varrho \Leftrightarrow \varrho\ ;\ \underline{\varrho^{\textsf{tgt}}}}\,\textsf{SRuleL} \qquad \frac{}{\varrho \Leftrightarrow \underline{\varrho^{\textsf{src}}}\ ;\ \varrho}\,\textsf{SRuleR}$$

$$\frac{\mu \Leftrightarrow \mu_1\ ;\ \mu_2}{\lambda x.\mu \Leftrightarrow \lambda x.\mu_1\ ;\ \lambda x.\mu_2}\,\textsf{SAbs} \qquad \frac{\mu \Leftrightarrow \mu_1\ ;\ \mu_2 \qquad \nu \Leftrightarrow \nu_1\ ;\ \nu_2}{\mu\,\nu \Leftrightarrow \mu_1\,\nu_1\ ;\ \mu_2\,\nu_2}\,\textsf{SApp}$$

▶ **Definition 19** (Flat permutation equivalence). *Flat permutation equivalence judgments are of the form: $\Gamma \vdash (\rho : s \twoheadrightarrow t) \sim (\rho' : s' \twoheadrightarrow t') : A$, meaning that ρ and ρ' are equivalent rewrites, with sources s and s' respectively, and targets t and t' respectively. The rewrites ρ and ρ' are* **assumed to be in $\overset{\flat}{\mapsto}$-normal form**, *which in particular means that they must be flat rewrites. Sometimes we write $\rho \sim \rho'$ if Γ, s, t, s', t', A are irrelevant or clear from the context. Derivability is defined by the two following axioms, which are closed by reflexivity, symmetry, transitivity, and closure under* composition contexts *(given by $\mathtt{S} ::= \Box \mid \mathtt{S}\ ;\ \rho \mid \rho\ ;\ \mathtt{S}$):*

$$\begin{array}{rcll} (\rho\ ;\ \sigma)\ ;\ \tau & \sim & \rho\ ;\ (\sigma\ ;\ \tau) & & \sim\text{-Assoc} \\ \mu & \sim & \mu_1^\flat\ ;\ \mu_2^\flat & \textit{if } \mu \Leftrightarrow \mu_1\ ;\ \mu_2 & \sim\text{-Perm} \end{array}$$

Note that in $\sim$-Perm, $-^\flat$ operates over multisteps. So the only rules of $\mathcal{F}$ that are applied here are the $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM rules.

▶ **Theorem 20** (Soundness and completeness of flat permutation equivalence). *Let $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ and $\Gamma \vdash \sigma : s' \twoheadrightarrow t' : A$. Then $\rho \approx \sigma$ if and only if $\rho^\flat \sim \sigma^\flat$.*

Proof. The ($\Leftarrow$) direction is immediate, given that reduction $\overset{\flat}{\mapsto}$ in the flattening system $\mathcal{F}$ is included in permutation equivalence ($\rho \overset{\flat}{\mapsto} \sigma$ implies $\rho \approx \sigma$) and, similarly, flat permutation equivalence is included in permutation equivalence ($\rho \sim \sigma$ implies $\rho \approx \sigma$).
The ($\Rightarrow$) direction is by induction on the derivation of $\rho \approx \sigma$. It is subtle and requires numerous auxiliary results (see Section D.8 in [2]). ◀

▶ **Example 21.** With the same notation as in Ex. 13, it can be checked that the rewrites $\mathbf{mu}\,(\lambda x.\vartheta\,x)\ ;\ \varrho\,(\lambda x.\mathbf{g}\,x)$ and $\varrho\,(\lambda x.\mathbf{f}\,x)\ ;\ \mathbf{f}\,(\mathbf{mu}\,(\lambda x.\vartheta\,x))\ ;\ \vartheta\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x))$ are permutation equivalent by means of flattening. Indeed, using the $\sim$-Perm rule three times:

$$\begin{array}{lcll} \mathbf{mu}\,\vartheta\ ;\ \varrho\,\mathbf{g} & \sim & \varrho\,\vartheta & \text{as } \varrho\,\vartheta \Leftrightarrow (\lambda x.\mathbf{mu}\,(\lambda y.x\,y))\,\vartheta\ ;\ \varrho\,(\lambda x.\mathbf{g}\,x) \\ & \sim & \varrho\,\mathbf{f}\ ;\ \vartheta\,(\mathbf{mu}\,\vartheta) & \text{as } \varrho\,\vartheta \Leftrightarrow \varrho\,(\lambda x.\mathbf{f}\,x)\ ;\ (\lambda x.x\,(\mathbf{mu}\,(\lambda y.x\,y)))\,\vartheta \\ & \sim & \varrho\,\mathbf{f}\ ;\ (\mathbf{f}(\mathbf{mu}\,\vartheta)\ ;\ \vartheta(\mu\,\mathbf{g})) & \text{as } \vartheta\,(\mathbf{mu}\,\vartheta) \Leftrightarrow (\lambda x.\mathbf{f}\,x)\,(\mathbf{mu}\,\vartheta)\ ;\ \vartheta\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)) \end{array}$$

Note that $\varrho\,\vartheta \Leftrightarrow (\lambda x.\mathbf{mu}\,(\lambda y.x\,y))\,\vartheta\,;\,\varrho\,(\lambda x.\mathbf{g}\,x)$ follows from SApp, SRuleR for the upper left hypothesis and SRuleL for the upper right one. Hence

$$\begin{array}{lcl} (\mathbf{mu}\,(\lambda x.\vartheta\,x)\,;\,\varrho\,(\lambda x.\mathbf{g}\,x))^{\flat} & = & \mathbf{mu}\,\vartheta\,;\,\varrho\,\mathbf{g} \\ & \sim & \varrho\,\mathbf{f}\,;\,(\mathbf{f}(\mathbf{mu}\,\vartheta)\,;\,\vartheta(\mu\,\mathbf{g})) \\ & = & (\varrho\,(\lambda x.\mathbf{f}\,x)\,;\,\mathbf{f}\,(\mathbf{mu}\,(\lambda x.\vartheta\,x))\,;\,\vartheta\,(\mathbf{mu}\,(\lambda x.\mathbf{g}\,x)))^{\flat} \end{array}$$

6 Projection

This section presents projection equivalence. Two rewrites ρ and σ are said to be projection equivalent if the steps performed by ρ are included in those performed by σ and vice-versa. We proceed in stages as follows. First, we define *projection* of multisteps over multisteps (Def. 25) and prove some of its properties (Prop. 26). Second, we extend projection to flat rewrites (Def. 28). Third, we extend projection to arbitrary rewrites (Def. 29) and, again, we prove some of its properties (Prop. 30). Finally, we show that the induced notion of *projection equivalence* turns out to coincide with permutation equivalence (Thm. 31).

Projection for Multisteps. Consider the rewrites $\mathbf{mu}\,\vartheta$ and $\varrho\,\mathbf{f}$, using the notation of Ex. 13, each representing one step. Since rewrites are subject to $\beta\eta$-equivalence, to define projection one must "line up" rule symbols with the left-hand side of the rewrite rules they witness[3]. For example, if the above two multisteps were rewritten as $(\lambda y.\mathbf{mu}\,(\lambda x.y\,x))\,\vartheta$ and $\varrho\,(\lambda x.\mathbf{f}\,x)$, respectively, then one can reason inductively as follows to compute the projection of the former over the latter (the inference rules themselves are introduced in Def. 22):

$$\dfrac{\dfrac{}{\lambda y.\mathbf{mu}\,(\lambda x.y\,x) /\!\!/\!\!/ \varrho \Rightarrow \lambda y.y\,(\mathbf{mu}\,(\lambda x.y\,x))}\,\mathsf{ProjRuleR} \qquad \dfrac{}{\vartheta /\!\!/\!\!/ \lambda x.\mathbf{f}\,x \Rightarrow \vartheta}\,\mathsf{ProjRuleL}}{(\lambda y.\mathbf{mu}\,(\lambda x.y\,x))\,\vartheta /\!\!/\!\!/ \varrho\,(\lambda x.\mathbf{f}\,x) \Rightarrow (\lambda y.y\,(\mathbf{mu}\,(\lambda x.y\,x)))\,\vartheta}\,\mathsf{ProjApp}$$

The flat normal form of $(\lambda y.y\,(\mathbf{mu}\,(\lambda x.y\,x)))\,\vartheta$ is the rewrite $\vartheta\,(\mathbf{mu}\,\vartheta)$. Hence we would deduce $\mathbf{mu}\,\vartheta /\!\!/\!\!/ \varrho\,\mathbf{f} \Rightarrow \vartheta\,(\mathbf{mu}\,\vartheta)$. We begin by introducing an auxiliary notion of projection on coinitial multisteps that may not be flat (*i.e.* may not be in $\mathcal{F}$-BetaM, $\mathcal{F}$-EtaM-normal form) called *weak projection*. We then make use of this notion, to define projection for flat multisteps (Def. 25).

▶ **Definition 22** (Weak projection and compatibility). *Let* $\Gamma \vdash \mu : s \twoheadrightarrow t : A$ *and* $\Gamma \vdash \nu : s' \twoheadrightarrow r : A$ *be multisteps, not necessarily in normal form, such that* $s =_{\beta\eta} s'$*. The judgment* $\mu /\!\!/\!\!/ \nu \Rightarrow \xi$ *is defined as follows:*

$$\dfrac{}{x /\!\!/\!\!/ x \Rightarrow x}\,\mathsf{ProjVar} \qquad \dfrac{}{\mathbf{c} /\!\!/\!\!/ \mathbf{c} \Rightarrow \mathbf{c}}\,\mathsf{ProjCon} \qquad \dfrac{}{\varrho /\!\!/\!\!/ \varrho \Rightarrow \varrho^{\mathsf{tgt}}}\,\mathsf{ProjRule} \qquad \dfrac{}{\varrho /\!\!/\!\!/ \varrho^{\mathsf{src}} \Rightarrow \varrho}\,\mathsf{ProjRuleL}$$

$$\dfrac{}{\varrho^{\mathsf{src}} /\!\!/\!\!/ \varrho \Rightarrow \varrho^{\mathsf{tgt}}}\,\mathsf{ProjRuleR} \qquad \dfrac{\mu /\!\!/\!\!/ \nu \Rightarrow \xi}{\lambda x.\mu /\!\!/\!\!/ \lambda x.\nu \Rightarrow \lambda x.\xi}\,\mathsf{ProjAbs} \qquad \dfrac{\mu_1 /\!\!/\!\!/ \nu_1 \Rightarrow \xi_1 \qquad \mu_2 /\!\!/\!\!/ \nu_2 \Rightarrow \xi_2}{\mu_1\,\mu_2 /\!\!/\!\!/ \nu_1\,\nu_2 \Rightarrow \xi_1\,\xi_2}\,\mathsf{ProjApp}$$

We say that μ *and* ν *are* compatible, *written* $\mu \uparrow \nu$ *if, intuitively speaking,* μ *and* ν *are coinitial, and are "almost"* η*-expanded and* β*-normal forms, with the exception that the head of the term may be the source of a rule,* i.e. *a term of the form* ϱ^{src}*. Compatibility is defined as follows:*

$$\dfrac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda \overline{x}.y\,\overline{\mu} \uparrow \lambda \overline{x}.y\,\overline{\nu}} \qquad \dfrac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda \overline{x}.\mathbf{c}\,\overline{\mu} \uparrow \lambda \overline{x}.\mathbf{c}\,\overline{\nu}} \qquad \dfrac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda \overline{x}.\varrho\,\overline{\mu} \uparrow \lambda \overline{x}.\varrho\,\overline{\nu}} \qquad \dfrac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda \overline{x}.\varrho\,\overline{\mu} \uparrow \lambda \overline{x}.\varrho^{\mathsf{src}}\,\overline{\nu}} \qquad \dfrac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda \overline{x}.\varrho^{\mathsf{src}}\,\overline{\mu} \uparrow \lambda \overline{x}.\varrho\,\overline{\nu}}$$

[3] See also the discussion on pg. 120 of [4].

The interesting cases are the two last rules, which state essentially that a rule symbol is compatible with its source term. Clearly if $\mu \uparrow \nu$, then there exists a unique ξ such that $\mu /\!\!/\!\!/ \nu \Rightarrow \xi$. Moreover, weak projection is coherent with respect to flattening:

▶ **Lemma 23** (Coherence of projection). *Let $\mu_1, \nu_1, \mu_2, \nu_2$ be multisteps such that the following are satisfied:*

1. $\mu_1 \uparrow \nu_1$ *and* $\mu_2 \uparrow \nu_2$*;*
2. $\mu_1^\flat = \mu_2^\flat$ *and* $\nu_1^\flat = \nu_2^\flat$*; and*
3. $\mu_1 /\!\!/\!\!/ \nu_1 \Rightarrow \xi_1$ *and* $\mu_2 /\!\!/\!\!/ \nu_2 \Rightarrow \xi_2$*.*

Then $\xi_1^\flat = \xi_2^\flat$.

Thus for arbitrary, coinitial multisteps μ and ν, it suffices to show that we can always find corresponding *compatible* "almost" η-expanded and β-normal forms, as mentioned above.

▶ **Proposition 24** (Existence and uniqueness of projection). *Let μ, ν be such that $\mu^{\mathsf{src}} =_{\beta\eta} \nu^{\mathsf{src}}$. Then:*

1. Existence. *There exist multisteps $\dot{\mu}, \dot{\nu}, \dot{\xi}$ such that $\dot{\mu}^\flat = \mu^\flat$ and $\dot{\nu}^\flat = \nu^\flat$ and $\dot{\mu} /\!\!/\!\!/ \dot{\nu} \Rightarrow \dot{\xi}$.*
2. Compatibility. *Furthermore, $\dot{\mu}$ and $\dot{\nu}$ can be chosen in such a way that $\dot{\mu} \uparrow \dot{\nu}$.*
3. Uniqueness. *If $(\dot{\mu}')^\flat = \mu^\flat$ and $(\dot{\nu}')^\flat = \nu^\flat$ and $\dot{\mu}' /\!\!/\!\!/ \dot{\nu}' \Rightarrow \dot{\xi}'$ then $(\dot{\xi}')^\flat = \xi^\flat$.*

Prop. 24 relies on the left-hand side of the rewrite rules of the HRS being patterns. This ensures, among other things, that flattening is injective when applied to left-hand sides of rewrite rules in the sense that if $(\varrho^{\mathsf{src}}\,\mu_1 \ldots \mu_n)^\flat = (\varrho^{\mathsf{src}}\,\nu_1 \ldots \nu_n)^\flat$ then $\mu_i^\flat = \nu_i^\flat$ for all $1 \leq i \leq n$. We can now define projection on arbitrary coinitial rewrites as follows.

▶ **Definition 25** (Projection operator for multisteps). *Let μ, ν be such that $\mu^{\mathsf{src}} =_{\beta\eta} \nu^{\mathsf{src}}$. We write μ/ν for the unique multistep of the form $\dot{\xi}^\flat$ such that there exist $\dot{\mu}, \dot{\nu}$ such that $\dot{\mu}^\flat = \mu^\flat$ and $\dot{\nu}^\flat = \nu^\flat$ and $\dot{\mu} /\!\!/\!\!/ \dot{\nu} \Rightarrow \dot{\xi}$, as guaranteed by Prop. 24. The proof is constructive (this relies on the HRS being orthogonal), thus providing an effective method to compute μ/ν.*

▶ **Proposition 26** (Properties of projection for multisteps).

1. $\mu/\nu = (\mu/\nu)^\flat = \mu^\flat/\nu^\flat$
2. *Projection commutes with abstraction and application, that is, $(\lambda x.\mu)/(\lambda x.\nu) = (\lambda x.(\mu/\nu))^\flat$ and $(\mu_1\,\mu_2)/(\nu_1\,\nu_2) = ((\mu_1/\nu_1)\,(\mu_2/\nu_2))^\flat$, provided that μ_1/ν_1 and μ_2/ν_2 are defined.*
3. *The set of multisteps with the projection operator form a residual system [15, Def. 8.7.2]:*
   - **3.1** $(\mu/\nu)/(\xi/\nu) = (\mu/\xi)/(\nu/\xi)$*, known as the* Cube Lemma*.*
   - **3.2** $\mu/\mu = (\mu^{\mathsf{tgt}})^\flat$ *and, as particular cases:* $\underline{s}/\underline{s} = \underline{s}^\flat$*,* $x/x = x$*,* $\mathbf{c}/\mathbf{c} = \mathbf{c}$*, and* $\varrho/\varrho = (\varrho^{\mathsf{tgt}})^\flat$*.*
   - **3.3** $(\mu^{\mathsf{src}})^\flat/\mu = (\mu^{\mathsf{tgt}})^\flat$ *and, as a particular case,* $(\varrho^{\mathsf{src}})^\flat/\varrho = (\varrho^{\mathsf{tgt}})^\flat$*.*
   - **3.4** $\mu/(\mu^{\mathsf{src}})^\flat = \mu^\flat$ *and, as a particular case,* $\varrho/(\varrho^{\mathsf{src}})^\flat = \varrho$*.*

▶ **Example 27.** Let $\vartheta : \lambda x.\mathbf{f}\,x \to \lambda x.\mathbf{g}\,x$. Then:

$$\begin{array}{rlll} & (\lambda x.(\lambda x.\mathbf{f}\,x)\,x)/(\lambda x.\vartheta\,x) & = (\lambda x.((\lambda x.\mathbf{f}\,x)\,x)/(\vartheta\,x))^\flat & = (\lambda x.(((\lambda x.\mathbf{f}\,x)/\vartheta)(x/x))^\flat)^\flat \\ = & (\lambda x.((\lambda x.\mathbf{g}\,x)\,x)^\flat)^\flat & = (\lambda x.\mathbf{g}\,x)^\flat & = \mathbf{g} \end{array}$$

Projection for Flat Rewrites. The projection operator from Def. 25 is extended to operate on flat rewrites. One may try to define ρ/σ using equations such as $(\rho_1\,;\,\rho_2)/\sigma = (\rho_1/\sigma)\,;\,(\rho_2/(\sigma/\rho_1))$. However, it is not *a priori* clear that this recursive definition is well-founded[4]. This is why the following definition proceeds in three stages:

[4] Another way to prove well-foundedness is by interpretation, as done in [15, Example 6.5.43].

▶ **Definition 28** (Projection operator for flat rewrites). *We define:*

1. *projection of a flat multistep over a coinitial flat rewrite ($\mu /^1 \rho$), by induction on ρ;*
2. *projection of a flat rewrite over a coinitial flat multistep ($\rho /^2 \mu$), by induction on ρ; and*
3. *projection of a flat rewrite over a coinitial flat rewrite ($\rho /^3 \sigma$) by induction on σ, as follows:*

$$
\begin{array}{lllllll}
\mu /^1 \nu & \stackrel{\text{def}}{=} & \mu/\nu & \quad & \mu /^1 (\rho_1 \mathbin{;} \rho_2) & \stackrel{\text{def}}{=} & (\mu /^1 \rho_1) /^1 \rho_2 \\
\nu /^2 \mu & \stackrel{\text{def}}{=} & \nu/\mu & \quad & (\rho_1 \mathbin{;} \rho_2) /^2 \mu & \stackrel{\text{def}}{=} & (\rho_1 /^2 \mu) \mathbin{;} (\rho_2 /^2 (\mu /^1 \rho_1)) \\
\rho /^3 \mu & \stackrel{\text{def}}{=} & \rho /^2 \mu & \quad & \rho /^3 (\sigma_1 \mathbin{;} \sigma_2) & \stackrel{\text{def}}{=} & (\rho /^3 \sigma_1) /^3 \sigma_2
\end{array}
$$

Note that $/^3$ generalizes $/^2$ and $/^1$ in the sense that $\mu /^1 \rho = \mu /^3 \rho$ and $\rho /^2 \mu = \rho /^3 \mu$. With these definitions, the key equation $(\rho_1 \mathbin{;} \rho_2) /^3 \sigma = (\rho_1 /^3 \sigma) \mathbin{;} (\rho_2 /^3 (\sigma /^3 \rho_1))$ can be shown to hold.

From this point on, we overload ρ/σ to stand for either of these projection operators. The key equation ensures that this abuse of notation is harmless. In the following, we mention some important properties of projection for flat rewrites. First, projection of a rewrite over a sequence, and of a sequence over a rewrite, obey the expected equations $\rho/(\sigma_1 \mathbin{;} \sigma_2) = (\rho/\sigma_1)/\sigma_2$ and $(\rho_1 \mathbin{;} \rho_2)/\sigma = (\rho_1/\sigma) \mathbin{;} (\rho_2/(\sigma/\rho_1))$. Second, flat permutation equivalence is a congruence with respect to projection: more precisely, if $\rho \sim \sigma$ then $\tau/\rho = \tau/\sigma$ and $\rho/\tau \sim \sigma/\tau$. Third, the projection of a rewrite over itself is always empty; specifically $\rho/\rho \sim (\rho^{\mathsf{tgt}})^\flat$. Finally, an important property is that $\rho \mathbin{;} (\sigma/\rho) \sim \sigma \mathbin{;} (\rho/\sigma)$, corresponding to a strong form of confluence. The proof of these properties is technical, by induction on the structure of the rewrites. We do not develop the full theory of projection for flat rewrites here for lack of space (but see Section E in [2] for more details).

Projection for Arbitrary Rewrites. As a final step, the projection operator of Def. 28 may be extended to arbitrary rewrites by flattening first. The proof of Prop. 30 relies crucially on the properties of projection for flat rewrites and on Thm. 20; it may be found in Section G in [2].

▶ **Definition 29** (Projection operator for arbitrary rewrites). *Let ρ, σ be arbitrary coinitial rewrites. Their projection is defined as $\rho /\!\!/ \sigma \stackrel{\text{def}}{=} \rho^\flat/\sigma^\flat$.*

▶ **Proposition 30** (Properties of projection for arbitrary rewrites).

1. *Projection of a rewrite over a sequence and of a sequence over a rewrite obey the expected equations $\rho /\!\!/ (\sigma_1 \mathbin{;} \sigma_2) = (\rho /\!\!/ \sigma_1) /\!\!/ \sigma_2$ and $(\rho_1 \mathbin{;} \rho_2) /\!\!/ \sigma = (\rho_1 /\!\!/ \sigma) \mathbin{;} (\rho_2 /\!\!/ (\sigma /\!\!/ \rho_1))$.*
2. *Projection commutes with abstraction and application, that is:*
   - **2.1** *$(\lambda x.\rho) /\!\!/ (\lambda x.\sigma) \approx \lambda x.(\rho /\!\!/ \sigma)$, and more precisely $(\lambda x.\rho) /\!\!/ (\lambda x.\sigma) \stackrel{\flat}{\leftarrow}^* \lambda x.(\rho /\!\!/ \sigma)$.*
   - **2.2** *If ρ_1, σ_1 are coinitial and ρ_2, σ_2 are coinitial, then $(\rho_1\,\rho_2) /\!\!/ (\sigma_1\,\sigma_2) \approx (\rho_1 /\!\!/ \sigma_1)\,(\rho_2 /\!\!/ \sigma_2)$, and more precisely $(\rho_1\,\rho_2) /\!\!/ (\sigma_1\,\sigma_2) \stackrel{\flat}{\leftarrow}^* (\rho_1 /\!\!/ \sigma_1)\,(\rho_2 /\!\!/ \sigma_2)$.*
3. *The projection of a rewrite over itself is always empty, $\rho /\!\!/ \rho \approx \rho^{\mathsf{tgt}}$.*
4. *Permutation equivalence is a congruence with respect to projection, namely if $\rho \approx \sigma$ then $\tau /\!\!/ \rho = \tau /\!\!/ \sigma$ and $\rho /\!\!/ \tau \approx \sigma /\!\!/ \tau$.*
5. *The key equation $\rho \mathbin{;} (\sigma /\!\!/ \rho) \approx \sigma \mathbin{;} (\rho /\!\!/ \sigma)$ holds.*

Characterization of Permutation Equivalence in Terms of Projection. Finally, we are able to characterize permutation equivalence $\rho \approx \sigma$ as the condition that the projections $\rho /\!\!/ \sigma$ and $\sigma /\!\!/ \rho$ are both empty. Indeed:

▶ **Theorem 31** (Projection equivalence). *Let ρ, σ be arbitrary coinitial rewrites. Then $\rho \approx \sigma$ if and only if $\rho /\!\!/ \sigma \approx \sigma^{\mathsf{tgt}}$ and $\sigma /\!\!/ \rho \approx \rho^{\mathsf{tgt}}$.*

Proof. ($\Rightarrow$) Suppose that $\rho \approx \sigma$. Then, by Prop. 30, $\rho /\!\!/ \sigma \approx \sigma /\!\!/ \sigma \approx \sigma^{\mathsf{tgt}}$. Symmetrically, $\sigma /\!\!/ \rho \approx \rho^{\mathsf{tgt}}$. ($\Leftarrow$) Let $\rho /\!\!/ \sigma \approx \sigma^{\mathsf{tgt}}$ and $\sigma /\!\!/ \rho \approx \rho^{\mathsf{tgt}}$. Then, by Prop. 30, $\rho \approx \rho \,;\, \rho^{\mathsf{tgt}} \approx \rho \,;\, (\sigma /\!\!/ \rho) \approx \sigma \,;\, (\rho /\!\!/ \sigma) \approx \sigma \,;\, \sigma^{\mathsf{tgt}} \approx \sigma$. ◀

Since flattening and projection are computable, Thm. 20 and Thm. 31 together provide an **effective method to decide permutation equivalence** $\rho \approx \sigma$ for arbitrary rewrites. Indeed, to test whether $\rho /\!\!/ \sigma \approx \sigma^{\mathsf{tgt}}$, note by Thm. 20 that this is equivalent to testing whether $\rho /\!\!/ \sigma \sim (\sigma^{\mathsf{tgt}})^{\flat}$, so it suffices to check that $\rho /\!\!/ \sigma$ is *empty*, *i.e.* it contains no rule symbols. This is justified by the fact that if μ has no rule symbols and $\mu \sim \rho$, then ρ has no rule symbols (See Lem. 162 in [2]).

7 Related Work and Conclusions

As mentioned in the introduction, proof terms were introduced by van Oostrom and de Vrijer for first-order left-linear rewrite systems to study equivalence of reductions in [17] and [15, Chapter 9]. They are inspired in Rewriting Logic [13]. In the setting of HORs, Hilken [6] introduces rewrites for $\beta\eta$-reduction together with a notion of permutation equivalence for those rewrites. He does not study permutation equivalence for arbitrary HORs nor formulate notions of projection. Hilken does, however, justify his equations through a categorical semantics. We have already discussed Bruggink's work extensively [4, 3]. Another attempt at devising proof terms for HOR by the authors of the present paper is [1]. The latter uses a term assignment for a minimal modal logic called Logic of Proofs (LP), to model rewrites. LP is a refinement of S4 in which the modality $\Box A$ is refined to $[s]A$, where s is said to be a witness to the proof of A. The intuition is that terms and rewrites may be seen to belong to different stages of discourse; rewrites verse about terms. Terms are typed with simple types and rewrites are typed with a modal type $[s]A$ where the term s is the source term of the rewrite. However, the notion of substitution that is required for subject reduction is arguably ad-hoc. In particular, substitution of a rewrite $\rho : s \rightarrow s' : A$ for x in another rewrite $\sigma : t \rightarrow t' : A$ is *defined* as the composed rewrite $\rho\{x\backslash t\} \,;\, s'\{x\backslash\!\!\backslash\sigma\}$, where ρ is substituted for x in $\underline{t}$ followed by σ where s' is substituted for x.

Future work. It would be of interest to develop tools based on the work presented here for reasoning about computations in higher-order rewriting, as has recently been explored for first-order rewriting [9, 10]. One downside is that our rewrites cannot be treated as terms in a higher-order rewrite system. Indeed, rewrites are not defined modulo $\beta\eta$ (for good reason since an expression such as $(\lambda x.\rho)\,\sigma$ should not be subject to β reduction).

One problem that should be addressed is that of formulating *standardization* (see *e.g.* [15, Section 8.5]) using rewrites. This amounts to giving a procedure that reorders the steps of a rewrite ρ, yielding a rewrite ρ^* in which outermost steps are performed before innermost ones. Standardization finds canonical representatives of $\approx$-equivalence classes, in the sense that $\rho \approx \sigma$ if and only if $\rho^* = \sigma^*$. The flattening rewrite system of Section 5 is a first approximation to standardization, since $\rho \approx \sigma$ if and only if $\rho^\flat \sim \sigma^\flat$. In a preliminary version of this work, we proposed a procedure to compute canonical representatives of $\approx$-equivalence classes, based on the idea of repeatedly converting $\mu \,;\, \nu$ into $\mu' \,;\, \nu'$ whenever $\nu \Leftrightarrow \xi \,;\, \nu'$ and $\mu' \Leftrightarrow \mu \,;\, \xi$, an idea reminiscent of *greedy decompositions* [5]. Unfortunately, this procedure does not always terminate, due to the fact that rewrites may have infinitely long "unfoldings"; for instance, if $\varrho : \mathbf{c} \rightarrow \mathbf{c}$ and $\vartheta : \mathbf{f}(x) \rightarrow \mathbf{d}$ then $\vartheta(\mathbf{c}) : \mathbf{f}(\mathbf{c}) \rightarrow \mathbf{d}$ is equivalent to arbitrarily long rewrites of the form $\mathbf{f}(\varrho) \,;\, \dots \,;\, \mathbf{f}(\varrho) \,;\, \vartheta(\mathbf{c})$. A terminating procedure should probably rely on a measure based on the notion of *essential development* [16, Definition 11].

Another avenue to pursue is to characterize permutation equivalence via *labelling*. The application of a rewrite step leaves a witness in the term itself, manifested as a decoration (a label). These labels thus collect and record the history of a computation. By comparing them one can determine whether two computations are equivalent. Labelling equivalence for first-order rewriting is studied by van Oostrom and de Vrijer in [17] and [15, Chapter 9].

We have given semantics to rewrites via Higher-Order Rewriting Logic. A categorical semantics for a similar notion of rewrite and permutation equivalence was presented by Hirshowitz [7] (projection equivalence and flattening are not studied though). Our $s\{x\backslash\!\!\backslash\rho\}$ is called *left whiskering* and $\rho\{x\backslash s\}$ *right whiskering*, using the terminology of 2-category theory. These are then used to define $\rho\{x\backslash\!\!\backslash\!\!\backslash\sigma\}$. A precise relation between the two notions of rewrite should be investigated.

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A Rewrites

Besides the typing rules given in Def. 7, we give explicit rules for reflexive, symmetric, transitive, and contextual closure for term equivalence:

$$\frac{\Gamma \vdash s : A}{\Gamma \vdash s =_{\beta\eta} s : A}\ \mathsf{EqRefl} \qquad \frac{\Gamma \vdash s =_{\beta\eta} t : A}{\Gamma \vdash t =_{\beta\eta} s : A}\ \mathsf{EqSym} \qquad \frac{\Gamma \vdash s =_{\beta\eta} t : A \quad \Gamma \vdash t =_{\beta\eta} r : A}{\Gamma \vdash s =_{\beta\eta} r : A}\ \mathsf{EqTrans}$$

$$\frac{\Gamma, x : A \vdash s =_{\beta\eta} t : B}{\Gamma \vdash \lambda x.s =_{\beta\eta} \lambda x.t : A \to B}\ \mathsf{EqCongLam} \qquad \frac{\Gamma \vdash s =_{\beta\eta} s' : A \to B \quad \Gamma \vdash t =_{\beta\eta} t' : A}{\Gamma \vdash s\,t =_{\beta\eta} s'\,t' : A}\ \mathsf{EqCongApp}$$

▶ **Definition 32** (Notions of contexts).

1. *A* rewrite context *is a rewrite* $\mathtt{R}$ *with a single free occurrence of a distinguished variable* $\Box$ *called the* hole*. Inductively, rewrite contexts are given by the grammar:*

   $$\mathtt{R} ::= \Box \mid \lambda x.\mathtt{R} \mid \mathtt{R}\,\rho \mid \rho\,\mathtt{R} \mid \mathtt{R} \mathbin{;} \rho \mid \rho \mathbin{;} \mathtt{R}$$

   The capturing substitution of the hole of a rewrite context $\mathtt{R}$ *by the rewrite* ρ *is a rewrite, written as* $\mathtt{R}\langle\rho\rangle$*.*
2. *The set of* composition contexts *is a subset of the set of rewrite contexts, given by the grammar:*

   $$\mathtt{S} ::= \Box \mid \mathtt{S} \mathbin{;} \rho \mid \rho \mathbin{;} \mathtt{S}$$

3. *The set of* composition trees *is given by the grammar:*

   $$\mathtt{K} ::= \Box \mid \mathtt{K} \mathbin{;} \mathtt{K}$$

   For each $n \geq 1$*, an* n-hole composition tree *is a composition tree with* n *occurrences of the* hole $\Box$*. If* $\mathtt{K}$ *is an* n*-hole composition tree, we write* $\mathtt{K}\langle\rho_1, \ldots, \rho_n\rangle$ *to stand for the rewrite that results from replacing the* i*-th hole of* $\mathtt{K}$ *for* ρ_i *for each* $1 \leq i \leq n$*.*

   For example $((\Box \mathbin{;} \Box) \mathbin{;} \Box)$ *is a 3-hole composition tree and* $((\Box \mathbin{;} \Box) \mathbin{;} \Box)\langle\rho, \sigma, \tau\rangle = (\rho \mathbin{;} \sigma) \mathbin{;} \tau$*.*
4. *A rewrite context* $\mathtt{R}$ *is* applicative *if it is of the form* $\mathtt{R} = \mathtt{R}'\langle\mathtt{S}\langle\Box\rangle\,\rho\rangle$*.*

▶ **Lemma 33** (Source/target decomposition). *Define the source and the target of a context* $\mathtt{R}$ *by declaring* $\Box^{\mathsf{src}} = \Box$ *and* $\Box^{\mathsf{tgt}} = \Box$*. Let us write* $-^{\Diamond}$ *to stand for either* $-^{\mathsf{src}}$ *or* $-^{\mathsf{tgt}}$*. If* $\rho^{\Diamond} = \mathtt{C}\langle s\rangle$ *then there are two possibilities:*

(A) $\rho = \mathtt{R}\langle\alpha\rangle$ *where* $\mathtt{R}^{\Diamond} = \mathtt{C}$ *and* α *is a rewrite such that* $\alpha^{\Diamond} = s$*.*

(B) $\rho = \mathtt{R}\langle\varrho\rangle$ *where* $\mathtt{R}^{\Diamond} = \mathtt{C}_1$ *and* ϱ *is a rule symbol such that* $\varrho^{\Diamond} = \mathtt{C}_2\langle s\rangle$ *and* $\mathtt{C} = \mathtt{C}_1\langle\mathtt{C}_2\rangle$*.*

Proof. We prove the property for the source; the proof for the target is similar. We proceed by induction on ρ:

1. **Variable,** $\rho = x$**.** Then $x^{\mathsf{src}} = x = \mathtt{C}\langle s\rangle$ so $\mathtt{C} = \Box$ and $s = x$. Taking $\mathtt{R} = \Box$ and $\alpha = x$ we are in situation (A).
2. **Constant,** $\rho = \mathbf{c}$**.** Similar to the previous case.
3. **Rule symbol,** $\rho = \varrho$**.** Then $\varrho^{\mathsf{src}} = \mathtt{C}\langle s\rangle$. Taking $\mathtt{R} = \Box$ we are in situation (B).
4. **Abstraction,** $\rho = \lambda x.\sigma$**.** Then $\lambda x.\sigma^{\mathsf{src}} = \mathtt{C}\langle s\rangle$. If $\mathtt{C}$ is empty, taking $\mathtt{R} = \Box$ and $\alpha = \lambda x.\sigma$ we are in situation (A). If $\mathtt{C}$ is non-empty, then $\mathtt{C} = \lambda x.\mathtt{C}'$ and $\sigma = \mathtt{C}'\langle s\rangle$. By IH there are two possibilities:
   (A) $\sigma = \mathtt{R}'\langle\alpha\rangle$ where $\mathtt{R}'^{\mathsf{src}} = \mathtt{C}'$ and $\alpha^{\mathsf{src}} = s$. Taking $\mathtt{R} := \lambda x.\mathtt{R}'$ we are again in situation (A).

(B) $\sigma = \mathtt{R}'\langle \varrho \rangle$ where $\mathtt{R}'^{\mathsf{src}} = \mathtt{C}'_1$ and $\varrho^{\mathsf{src}} = \mathtt{C}_2\langle s \rangle$ such that $\mathtt{C}' = \mathtt{C}'_1\langle \mathtt{C}_2 \rangle$. Taking $\mathtt{R} := \lambda x.\mathtt{R}'$ we are again in situation (B).

5. **Application,** $\rho = \rho_1\,\rho_2$**.** Then $\rho_1^{\mathsf{src}}\rho_2^{\mathsf{src}} = \mathtt{C}\langle s \rangle$. If $\mathtt{C}$ is empty, taking $\mathtt{R} = \Box$ and $\alpha = \rho_1\,\rho_2$ we are in situation (A). If $\mathtt{C}$ is non-empty, there are two cases, depending on whether the hole is to the left or to the right of the application:
   5.1 **Left of the application.** Then $\mathtt{C} = \mathtt{C}'\,\rho_2^{\mathsf{src}}$ and $\rho_1^{\mathsf{src}} = \mathtt{C}'\langle s \rangle$. Then by IH there are two possibilities:
   (A) $\rho_1 = \mathtt{R}'\langle \alpha \rangle$ with $\mathtt{R}'^{\mathsf{src}} = \mathtt{C}'$ and $\alpha^{\mathsf{src}} = s$. Taking $\mathtt{R} = \mathtt{R}'\,\rho_2$ we are again in situation (A).
   (B) $\rho_1 = \mathtt{R}'\langle \varrho \rangle$ with $\mathtt{R}'^{\mathsf{src}} = \mathtt{C}'_1$ and $\varrho^{\mathsf{src}} = \mathtt{C}_2\langle s \rangle$ such that $\mathtt{C}' = \mathtt{C}'_1\langle \mathtt{C}_2 \rangle$. Taking $\mathtt{R} = \mathtt{R}'\,\rho_2$ we are again in situation (B).
   5.2 **Right of the application.** Then $\mathtt{C} = \rho_1^{\mathsf{src}}\,\mathtt{C}'$. The proof is similar to the previous case.
6. **Composition,** $\rho = \rho_1\,;\,\rho_2$**.** Then $\rho_1^{\mathsf{src}} = \mathtt{C}\langle s \rangle$. By IH on ρ_1 there are two possibilities:
   (A) $\rho_1 = \mathtt{R}'\langle \alpha \rangle$ with $\mathtt{R}'^{\mathsf{src}} = \mathtt{C}$ and $\alpha^{\mathsf{src}} = s$. Taking $\mathtt{R} := \mathtt{R}'\,;\,\rho_2$ we are again in situation (A).
   (B) $\rho_1 = \mathtt{R}'\langle \varrho \rangle$ with $\mathtt{R}'^{\mathsf{src}} = \mathtt{C}_1$ and $\varrho^{\mathsf{src}} = \mathtt{C}_2\langle s \rangle$ such that $\mathtt{C} = \mathtt{C}_1\langle \mathtt{C}_2 \rangle$. Taking $\mathtt{R} := \mathtt{R}'\,;\,\rho_2$ we are again in situation (B).

◀

We formulate a variant of the standard substitution lemma:

▶ **Lemma 34** (Substitution Lemma). *The equation* $X\{x\backslash s\}\{y\backslash t\} = X\{y\backslash t\}\{x\backslash s\{y\backslash t\}\}$ *holds for any term or rewrite* X *as long as* $x \neq y$ *and* $x \notin \mathsf{fv}(t)$*.*

Proof. By induction on X. ◀

▶ **Lemma 35** (Weakening). *Let* $x \notin \Gamma$*. Then:*
1. *If* $\Gamma \vdash s : A$ *then* $\Gamma, x : B \vdash s : A$*.*
2. *If* $\Gamma \vdash s =_{\beta\eta} t : A$ *then* $\Gamma, x : B \vdash s =_{\beta\eta} t : A$*.*
3. *If* $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ *then* $\Gamma, x : B \vdash \rho : s \twoheadrightarrow t : A$*.*

Proof. Straightfoward by induction on the derivation of the target judgment. ◀

▶ **Lemma 36** (Endpoint coherence). *Let* $\Gamma \vdash \rho : p_0 \twoheadrightarrow p_1 : A$*. Then* $\Gamma \vdash p_0 =_{\beta\eta} \rho^{\mathsf{src}} : A$ *and* $\Gamma \vdash p_0 =_{\beta\eta} \rho^{\mathsf{tgt}} : A$*.*

Proof. By induction on the derivation of $\Gamma \vdash \rho : p_0 \twoheadrightarrow p_1 : A$.
1. RVar, RCon: Immediate by EqRefl.
2. RRule: Let $\Gamma \vdash \varrho : s \twoheadrightarrow t : A$ be derived from $\cdot \vdash s : A$ and $\cdot \vdash t : A$ where $(\varrho : s \twoheadrightarrow t : A) \in \mathcal{R}$. Then by weakening Lem. 35 we have that $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$. Applying EqRefl we have that $\Gamma \vdash s =_{\beta\eta} s : A$ and $\Gamma \vdash t =_{\beta\eta} t : A$, as required.
3. RAbs: Straightforward by IH. More precisely, let $\Gamma \vdash \lambda x.\rho : \lambda x.s_0 \twoheadrightarrow \lambda x.s_1 : A \to B$ be derived from $\Gamma, x : A \vdash \rho : s_0 \twoheadrightarrow s_1 : B$. By IH we have that $\Gamma, x : A \vdash s_0 =_{\beta\eta} \rho^{\mathsf{src}} : B$ and $\Gamma, x : A \vdash s_1 =_{\beta\eta} \rho^{\mathsf{tgt}} : B$. Applying EqCongLam we have that $\Gamma \vdash \lambda x.s_0 =_{\beta\eta} \lambda x.\rho^{\mathsf{src}} : A \to B$ and $\Gamma \vdash \lambda x.s_1 =_{\beta\eta} \lambda x.\rho^{\mathsf{tgt}} : A \to B$ as required.
4. RApp: Straightforward by IH, using EqCongApp.
5. RTrans: Let $\Gamma \vdash \rho\,;\sigma : s_0 \twoheadrightarrow s_2 : A$ be derived from $\Gamma \vdash \rho : s_0 \twoheadrightarrow s_1 : A$ and $\Gamma \vdash \sigma : s_1 \twoheadrightarrow s_2 : A$. By IH we have that $\Gamma \vdash s_0 =_{\beta\eta} \rho^{\mathsf{src}} : A$ and $\Gamma \vdash s_2 =_{\beta\eta} \sigma^{\mathsf{tgt}} : A$. Hence $\Gamma \vdash s_0 =_{\beta\eta} (\rho\,;\sigma)^{\mathsf{src}} : A$ and $\Gamma \vdash s_2 =_{\beta\eta} (\rho\,;\sigma)^{\mathsf{tgt}} : A$.

6. RConv: Let $\Gamma \vdash \rho : s \rightarrow t : A$ be derived from $\Gamma \vdash \rho : s' \rightarrow t' : A$ where $\Gamma \vdash s =_{\beta\eta} s' : A$ and $\Gamma \vdash t' =_{\beta\eta} t : A$. By IH we have that $\Gamma \vdash s' =_{\beta\eta} \rho^{\mathsf{src}} : A$ and $\Gamma \vdash t' =_{\beta\eta} \rho^{\mathsf{tgt}} : A$, so applying EqTrans we have that $\Gamma \vdash s =_{\beta\eta} \rho^{\mathsf{src}} : A$ and $\Gamma \vdash t =_{\beta\eta} \rho^{\mathsf{tgt}} : A$, as required. ◀

▶ **Lemma 37** (Strengthening). *Let $x \notin \mathsf{fv}(s)$. If $\Gamma, x : A \vdash s : B$ then $\Gamma \vdash s : B$.*

Proof. Straightfoward by induction on the derivation of the target judgment. ◀

▶ **Lemma 38** (Free variables must be typed).
1. *If $\Gamma \vdash s : A$ then $\mathsf{fv}(s) \subseteq \mathsf{dom}(\Gamma)$.*
2. *If $\Gamma \vdash \rho : s \rightarrow t : A$ then $\mathsf{fv}(\rho) \cup \mathsf{fv}(s) \cup \mathsf{fv}(t) \subseteq \mathsf{dom}(\Gamma)$.*

Proof. Straightfoward by induction on the derivation of the judgment. ◀

▶ **Lemma 39** (Free term variables of endpoints). *Let $\Gamma \vdash \rho : s \rightarrow t : A$. Then $\mathsf{fvt}(s), \mathsf{fvt}(t) \subseteq \mathsf{fvt}(\rho)$.*

Proof. Straightforward by induction on the derivation of $\Gamma \vdash \rho : s \rightarrow t : A$. In the RRule case, note that $\Gamma \vdash \varrho : s \rightarrow t : A$ is derived from $\cdot \vdash s : A$ and $\cdot \vdash t : A$ so by Lem. 38 we have that s, t are closed terms. ◀

▶ **Lemma 40** (Reflexivity). *If $\Gamma \vdash s : A$ then $\Gamma \vdash \underline{s} : s \rightarrow s : A$.*

Proof. Straightforward by induction on the derivation of the judgment. ◀

▶ **Lemma 41** (Substitution). *Let $\Gamma \vdash q : A$. Then:*
1. *If $\Gamma, x : A \vdash s : B$ then $\Gamma \vdash s\{x\backslash q\} : B$.*
2. *If $\Gamma, x : A \vdash s =_{\beta\eta} t : B$ then $\Gamma \vdash s\{x\backslash q\} =_{\beta\eta} t\{x\backslash q\} : B$.*
3. *If $\Gamma, x : A \vdash \rho : s \rightarrow t : B$ then $\Gamma \vdash \rho\{x\backslash q\} : s\{x\backslash q\} \rightarrow t\{x\backslash q\} : B$.*

Proof. Each item is by induction on the derivation of the target judgment:
1. Var, Con: immediate.
2. Abs: let $\Gamma, x : A \vdash \lambda y.s : B \to C$ be derived from $\Gamma, x : A, y : B \vdash s : C$. Note that the hypothesis $\Gamma \vdash q : A$ may be weakened to $\Gamma, y : B \vdash q : A$. using Lem. 35. Hence we may apply the IH to obtain $\Gamma, y : B \vdash s\{x\backslash q\} : C$. Applying the Abs rule $\Gamma \vdash \lambda x.s\{x\backslash q\} : B \to C$.
3. App: let $\Gamma, x : A \vdash s\,t : C$ be derived from $\Gamma, x : A \vdash s : B \to C$ and $\Gamma, x : A \vdash t : B$. By IH $\Gamma \vdash s\{x\backslash q\} : B \to C$ and $\Gamma \vdash t\{x\backslash q\} : B$. Applying the App rule, $\Gamma \vdash s\{x\backslash q\}\,t\{x\backslash q\} : C$.
4. EqBeta: let $\Gamma, x : A \vdash (\lambda y.s)\,t =_{\beta\eta} s\{y\backslash t\} : C$ be derived from $\Gamma, x : A, y : B \vdash s : C$ and $\Gamma, x : A \vdash t : B$. By the first item of this lemma, $\Gamma, y : B \vdash s\{x\backslash q\} : C$ and $\Gamma \vdash t\{x\backslash q\} : B$, so applying the EqBeta rule we have that $\Gamma \vdash ((\lambda y.s)\,t)\{x\backslash q\} =_{\beta\eta} s\{x\backslash q\}\{y\backslash t\{x\backslash q\}\} : C$. Moreover, by the Substitution Lemma (Lem. 34), $s\{x\backslash q\}\{y\backslash t\{x\backslash q\}\} = s\{y\backslash t\}\{x\backslash q\}$ so we are done.
5. EqEta: let $\Gamma, x : A \vdash \lambda y.s\,y =_{\beta\eta} s : C$ be derived from $\Gamma, x : A, y : B \vdash s : C$ with $y \notin \mathsf{fv}(s)$. By the first item of this lemma, $\Gamma, y : B \vdash s\{x\backslash q\} : C$. Note that by Barendregt's convention, we may assume that $y \notin \mathsf{fv}(q)$, hence $y \notin \mathsf{fv}(s\{x\backslash q\})$, and we may apply the EqEta rule to conclude $\Gamma, y : B \vdash \lambda y.s\{x\backslash q\}\,y =_{\beta\eta} s\{x\backslash q\} : C$.
6. EqRefl, EqSym, EqTrans, EqCongLam, EqCongApp: straightforward by IH.
7. RVar: let $\Gamma, x : A \vdash y : y \rightarrow y : B$ with $(y : B) \in (\Gamma, x : A)$. There are two cases, depending on whether $x = y$ or not. If $x = y$, then by reflexivity (Lem. 40) we have that $\Gamma \vdash q : q \rightarrow q : B$. If $x \neq y$ then $\Gamma = \Gamma', y : B$. and applying the RVar rule $\Gamma', y : B \vdash y : y \rightarrow y : B$.

8. RCon: let $\Gamma, x : A \vdash \mathbf{c} : \mathbf{c} \twoheadrightarrow \mathbf{c} : B$ with $(\mathbf{c} : B) \in \mathcal{C}$. Applying the RCon rule, we have that $\Gamma \vdash \mathbf{c} : \mathbf{c} \twoheadrightarrow \mathbf{c} : B$.
9. RRule: let $\Gamma, x : A \vdash \varrho : s \twoheadrightarrow t : B$ be derived from $\cdot \vdash s : B$ and $\cdot \vdash t : B$ where $(\varrho : s \twoheadrightarrow t : B) \in \mathcal{R}$. Note that, by Lem. 38, we have that ρ, s and t have no free occurrences of x. Applying the RRule rule on the (unchanged) premises, we have that $\Gamma \vdash \varrho : s \twoheadrightarrow t : B$, that is $\Gamma \vdash \varrho : s\{x \backslash q\} \twoheadrightarrow t\{x \backslash q\} : B$, as required.
10. RAbs, RApp: similar to the Abs and App cases respectively.
11. RTrans: Let $\Gamma, x : A \vdash \rho \,;\, \sigma : s_0 \twoheadrightarrow s_2 : B$ be derived from $\Gamma, x : A \vdash \rho : s_0 \twoheadrightarrow s_1 : B$ and $\Gamma, x : A \vdash \sigma : s_1 \twoheadrightarrow s_2 : B$. Then by IH $\Gamma \vdash \rho\{x \backslash q\} : s_0\{x \backslash q\} \twoheadrightarrow s_1\{x \backslash q\} : B$ and $\Gamma \vdash \sigma\{x \backslash q\} : s_1\{x \backslash q\} \twoheadrightarrow s_2\{x \backslash q\} : B$. Applying the RTrans rule, $\Gamma \vdash \rho\{x \backslash q\} \,;\, \sigma\{x \backslash q\} : s_0\{x \backslash q\} \twoheadrightarrow s_2\{x \backslash q\} : B$
12. RConv: Let $\Gamma, x : A \vdash \rho : s \twoheadrightarrow t : B$ be derived from $\Gamma, x : A \vdash \rho : s' \twoheadrightarrow t' : B$, where $\Gamma, x : A \vdash s =_{\beta\eta} s' : B$ and $\Gamma, x : A \vdash t' =_{\beta\eta} t : B$. Then by IH $\Gamma \vdash \rho\{x \backslash q\} : s'\{x \backslash q\} \twoheadrightarrow t'\{x \backslash q\} : B$. Moreover, by the second item of this lemma, we have that $\Gamma \vdash s\{x \backslash q\} =_{\beta\eta} s'\{x \backslash q\} : B$ and $\Gamma \vdash t'\{x \backslash q\} =_{\beta\eta} t\{x \backslash q\} : B$. Applying the RConv rule, $\Gamma \vdash \rho\{x \backslash q\} : s\{x \backslash q\} \twoheadrightarrow t\{x \backslash q\} : B$.

◀

▶ **Lemma 42** (Equal terms are typable). *If* $\Gamma \vdash s =_{\beta\eta} t : A$ *then* $\Gamma \vdash s : A$ *and* $\Gamma \vdash t : A$.

Proof. By induction on the derivation of $\Gamma \vdash s =_{\beta\eta} t : A$.

1. EqBeta: let $\Gamma \vdash (\lambda x.s)\, t =_{\beta\eta} s\{x \backslash t\} : B$ be derived from $\Gamma, x : A \vdash s : B$ and $\Gamma \vdash t : A$. Applying the Abs and App rules, we have that $\Gamma \vdash (\lambda x.s)\, t : B$. Moreover, by Lem. 41 we have that $\Gamma \vdash s\{x \backslash t\} : B$.
2. EqEta: let $\Gamma \vdash \lambda x.s\, x =_{\beta\eta} s : B$ be derived from $\Gamma, x : A \vdash s : B$ and $x \notin \mathsf{fv}(s)$. Applying the App and Abs rules, we have that $\Gamma \vdash \lambda x.s\, x : B$. Moreover, by Lem. 37, $\Gamma \vdash s : B$.
3. EqRefl, EqSym, EqTrans, EqCongLam, EqCongApp: straightforward by IH.

◀

▶ **Lemma 43** (Source and target inversion). *If* $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ *then* $\Gamma \vdash s : A$ *and* $\Gamma \vdash t : A$.

Proof. By induction on the derivation of $\Gamma \vdash \rho : s \twoheadrightarrow t : A$.

1. RVar: let $\Gamma, x : A \vdash x : x \twoheadrightarrow x : A$. Then indeed $\Gamma, x : A \vdash x : A$.
2. RCon: let $\Gamma \vdash \mathbf{c} : \mathbf{c} \twoheadrightarrow \mathbf{c} : A$ with $(\mathbf{c} : A) \in \mathcal{C}$. Then indeed $\Gamma \vdash \mathbf{c} : A$.
3. RRule: let $\Gamma \vdash \varrho : s \twoheadrightarrow t : A$ be derived from $\cdot \vdash s : A$ and $\cdot \vdash t : A$ with $(\varrho : s \twoheadrightarrow t : A) \in \mathcal{R}$. Then it suffices to apply weakening (Lem. 35) to conclude.
4. RAbs: let $\Gamma \vdash \lambda x.\rho : \lambda x.s_0 \twoheadrightarrow \lambda x.s_1 : A \to B$ be derived from $\Gamma, x : A \vdash \rho : s_0 \twoheadrightarrow s_1 : B$. Then by IH the source of the premise is typable, *i.e.* $\Gamma, x : A \vdash s_0 : B$. Appying the Abs rule we are able to type the source of the conclusion, *i.e.* $\Gamma \vdash \lambda x.s_0 : A \to B$. The proof for the target is similar.
5. RApp: let $\Gamma \vdash \rho\, \sigma : s_0\, t_0 \twoheadrightarrow s_1\, t_1 : B$ be derived from $\Gamma \vdash \rho : s_0 \twoheadrightarrow s_1 : A \to B$ and $\Gamma \vdash \sigma : t_0 \twoheadrightarrow t_1 : A$. By IH we have that the sources of the premises are typable, *i.e.* that $\Gamma \vdash s_0 : A \to B$ and $\Gamma \vdash t_0 : A$. Applying the App rule, we are able to type the source of the conclusion, *i.e.* $\Gamma \vdash s_0\, t_0 : A \to B$. The proof for the target is similar.
6. RTrans: let $\Gamma \vdash \rho \,;\, \sigma : s_0 \twoheadrightarrow s_2 : A$ be derived from $\Gamma \vdash \rho : s_0 \twoheadrightarrow s_1 : A$ and $\Gamma \vdash \sigma : s_1 \twoheadrightarrow s_2 : A$. By IH on the first premise, we have that $\Gamma \vdash s_0 : A$, and by IH on the second premise, we have that $\Gamma \vdash s_2 : A$.
7. RConv: let $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ be derived from $\Gamma \vdash \rho : s' \twoheadrightarrow t' : A$ with $\Gamma \vdash s =_{\beta\eta} s' : A$ and $\Gamma \vdash t' =_{\beta\eta} t : A$. Then by Lem. 42 we have that $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$.

◀

A.1 Term/rewrite substitution

▶ **Lemma 44** (Typing rule for term/rewrite substitution). *If* $\Gamma, x : A \vdash s : B$ *and* $\Gamma \vdash \rho : q_0 \multimap q_1 : A$ *then* $\Gamma \vdash s\{x\backslash\!\backslash\rho\} : s\{x\backslash q_0\} \multimap s\{x\backslash q_1\} : B$.

Proof. By induction on the derivation of $\Gamma, x : A \vdash s : B$:

1. Var: let $\Gamma, x : A \vdash y : B$ with $y : B \in (\Gamma, x : A)$. We consider two subcases, depending on whether $x = y$ or not:
   1.1 If $x = y$, then $\Gamma \vdash \rho : s_0 \multimap s_1 : A$ holds by hypothesis.
   1.2 If $x \neq y$, then note that $y : B \in \Gamma$ so $\Gamma \vdash y : B$ and applying the RVar rule, we have that $\Gamma \vdash y : y \multimap y : B$.
2. Con: immediate applying the RCon rule.
3. Abs: let $\Gamma, x : A \vdash \lambda y.s : B \to C$ be derived from $\Gamma, x : A, y : B \vdash s : C$. By IH we have that $\Gamma, y : B \vdash s\{x\backslash\!\backslash\rho\} : s\{x\backslash q_0\} \multimap s\{x\backslash q_1\} : C$. Applying the RAbs rule, we obtain $\Gamma \vdash \lambda y.s\{x\backslash\!\backslash\rho\} : \lambda y.s\{x\backslash q_0\} \multimap \lambda y.s\{x\backslash q_1\} : B \to C$ as required.
4. App: let $\Gamma, x : A \vdash s\,t : C$ be derived from $\Gamma, x : A \vdash s : B \to C$ and $\Gamma, x : A \vdash t : B$. By IH we have that $\Gamma \vdash s\{x\backslash\!\backslash\rho\} : s\{x\backslash q_0\} \multimap s\{x\backslash q_1\} : B \to C$ and $\Gamma \vdash t\{x\backslash\!\backslash\rho\} : t\{x\backslash q_0\} \multimap t\{x\backslash q_1\} : B$. Applying the RApp rule, we obtain $\Gamma \vdash s\{x\backslash\!\backslash\rho\}\,t\{x\backslash\!\backslash\rho\} : s\{x\backslash q_0\}\,t\{x\backslash q_0\} \multimap s\{x\backslash q_1\}\,t\{x\backslash q_1\} : C$ as required.

◀

▶ **Lemma 45** (Commutation of lifting and term substitution (I)). *If* $\Gamma, x : A, y : B \vdash s : C$ *and* $\Gamma, y : B \vdash \tau : p_0 \multimap p_1 : A$ *and* $\Gamma \vdash q : B$ *then:*

$$s\{x\backslash\!\backslash\tau\}\{y\backslash q\} = s\{y\backslash q\}\{x\backslash\!\backslash\tau\{y\backslash q\}\}$$

In particular, if y *does not occur free in* τ*, then* $s\{x\backslash\!\backslash\tau\}\{y\backslash q\} = s\{y\backslash q\}\{x\backslash\!\backslash\tau\}$.

Proof. By induction on the derivation of $\Gamma, x : A, y : B \vdash s : C$:

1. Var: let $\Gamma, x : A, y : B \vdash z : C$ with $z : C \in \Gamma$. We consider three subcases, depending on whether $z = x$, $z = y$, or $z \notin \{x, y\}$.
   1.1 **If** $z = x$**:** the left and the right-hand sides are both $\tau\{y\backslash q\}$ by definition, so we are done.
   1.2 **If** $z = y$**:** then the left-hand side is q, and the right-hand side is $q\{x\backslash\tau\}$. By Lem. 38, x does not occur free in q, so $q = q\{x\backslash\tau\}$.
   1.3 **If** $z \notin \{x, y\}$**:** then the left and the right-hand sides are both z so we are done.
2. Con: let $\Gamma, x : A, y : B \vdash \mathbf{c} : C$ with $(\mathbf{c} : C) \in \mathcal{C}$. Then the left and the right-hand sides are both $\mathbf{c}$ so we are done.
3. Abs: let $\Gamma, x : A, y : B \vdash \lambda z.s : C \to D$ be derived from $\Gamma, x : A, y : B, z : C \vdash s : D$. Then:

$$\begin{array}{rcll}(\lambda z.s)\{x\backslash\!\backslash\tau\}\{y\backslash q\} & = & \lambda z.s\{x\backslash\!\backslash\tau\}\{y\backslash q\} & \\ & = & \lambda z.s\{y\backslash q\}\{x\backslash\!\backslash\tau\{y\backslash q\}\} & \text{by IH} \\ & = & (\lambda z.s)\{y\backslash q\}\{x\backslash\!\backslash\tau\{y\backslash q\}\} & \end{array}$$

   To apply the IH we use congruence of equivalence under abstractions, and weakening (Lem. 35) on the hypotheses.
4. App: let $\Gamma, x : A, y : B \vdash s\,t : D$ be derived from $\Gamma, x : A, y : B \vdash s : C \to D$ and $\Gamma, x : A, y : B \vdash t : C$. Then:

$$\begin{array}{rcll}(s\,t)\{x\backslash\!\backslash\tau\}\{y\backslash q\} & = & s\{x\backslash\!\backslash\tau\}\{y\backslash q\}\,t\{x\backslash\!\backslash\tau\}\{y\backslash q\} & \\ & = & s\{y\backslash q\}\{x\backslash\!\backslash\tau\{y\backslash q\}\}\,t\{y\backslash q\}\{x\backslash\!\backslash\tau\{y\backslash q\}\} & \text{by IH} \\ & = & (s\,t)\{y\backslash q\}\{x\backslash\!\backslash\tau\{y\backslash q\}\} & \end{array}$$

To apply the IH we use congruence of equivalence under applications. ◀

▶ Remark 46. $\underline{s}\{x\backslash t\} = \underline{s\{x\backslash t\}}$

▶ **Lemma 47** (Lifting reflexivity). *Let $\Gamma, x : A \vdash s : B$ and $\Gamma \vdash t : A$. Then $\underline{s\{x\backslash t\}} = s\{x\backslash\!\backslash \underline{t}\}$.*

Proof. By induction on the derivation of $\Gamma, x : A \vdash s : B$. ◀

▶ **Lemma 48** (Source and target of rewrite/term substitution). *If $\Gamma, x : A \vdash \rho : q_0 \twoheadrightarrow q_1 : B$ and $\Gamma \vdash s : A$ then $\rho\{x\backslash s\}^{\mathsf{src}} = \rho^{\mathsf{src}}\{x\backslash s\}$ and $\rho\{x\backslash s\}^{\mathsf{tgt}} = \rho^{\mathsf{tgt}}\{x\backslash s\}$.*

Proof. Straightforward by induction on ρ. ◀

▶ **Lemma 49** (Source and target of term/rewrite substitution). *If $\Gamma, x : A \vdash s : B$ and $\Gamma \vdash \rho : q_0 \twoheadrightarrow q_1 : A$ then $s\{x\backslash\!\backslash \rho\}^{\mathsf{src}} = s\{x\backslash\!\backslash \rho^{\mathsf{src}}\}$ and $s\{x\backslash\!\backslash \rho\}^{\mathsf{tgt}} = s\{x\backslash\!\backslash \rho^{\mathsf{tgt}}\}$.*

Proof. Straightforward by induction on s. ◀

B Permutation equivalence

▶ **Lemma 50** (Term equivalence implies permutation equivalence). *If $\Gamma \vdash s =_{\beta\eta} s' : A$ then $\underline{s} \approx \underline{s'}$.*

Proof. By induction on the derivation of $\Gamma \vdash s =_{\beta\eta} s' : A$. Reflexivity, symmetry, transitivity, and congruence under term constructors are immediate. The interesting cases are:

1. EqBeta: Let $s = (\lambda x.t)\, r$ and $s' = t\{x\backslash r\}$. Then:

$$\begin{array}{rcll} (\lambda x.\underline{t})\, \underline{r} & \approx & t\{x\backslash\!\backslash \underline{r}\} & \text{by } \approx\text{-BetaTR} \\ & = & \underline{t\{x\backslash r\}} & \text{by Lem. 47} \end{array}$$

2. EqEta: Let $s = \lambda x.s'\, x$ with $x \notin \mathsf{fv}(s')$. Then:

$$\lambda x.\underline{s'}\, x \quad \approx \quad \underline{s'} \quad \text{by } \approx\text{-Eta}$$

◀

▶ **Lemma 51** (Generalized ≈-IdL and ≈-IdR rules). *Let $\Gamma \vdash \rho : p \twoheadrightarrow q : A$. Then the following generalized variants of ≈-IdL and ≈-IdR hold:*
1. *If $\Gamma \vdash s =_{\beta\eta} \rho^{\mathsf{src}} : A$, then $(\underline{s} \,;\, \rho) \approx \rho$.*
2. *If $\Gamma \vdash s =_{\beta\eta} \rho^{\mathsf{tgt}} : A$, then $(\rho \,;\, \underline{s}) \approx \rho$.*

Sometimes by abuse we call these generalized rules ≈-IdL and ≈-IdR, without explicit reference to this lemma.

Proof. Item 1. is immediate given that:

$$\begin{array}{rcll} (\underline{s} \,;\, \rho) & \approx & (\underline{\rho^{\mathsf{src}}} \,;\, \rho) & \text{by Lem. 50} \\ & \approx & \rho & \text{by } \approx\text{-IdL} \end{array}$$

Item 2. is similar. ◀

▶ **Lemma 52** (Equivalence of endpoints of permutation equivalent rewrites). *Let $\Gamma \vdash \rho : p_0 \twoheadrightarrow p_1 : \sigma$ and $\Gamma \vdash \sigma : q_0 \twoheadrightarrow q_1 : \sigma$ be such that $\rho \approx \sigma$. Then $\rho^{\mathsf{src}} =_{\beta\eta} \sigma^{\mathsf{src}}$ and $\rho^{\mathsf{tgt}} =_{\beta\eta} \sigma^{\mathsf{tgt}}$.*

Proof. By induction on the derivation of $\rho \approx \sigma$. Reflexivity, symmetry, transitivity, and congruence under abstraction and application are immediate. The interesting case is when $\rho \approx \sigma$ is deduced from an axiom:

1. $\approx$-IdL: Let $\underline{\rho^{\mathsf{src}}}\,;\,\rho \approx \rho$. Then $(\underline{\rho^{\mathsf{src}}})^{\mathsf{src}} = \rho^{\mathsf{src}}$, as can be easily shown by induction on ρ, and $\rho^{\mathsf{tgt}} = \rho^{\mathsf{tgt}}$, so we conclude by EqRefl.
2. $\approx$-IdR: Let $\rho\,;\,\underline{\rho^{\mathsf{tgt}}} \approx \rho$. Then $\rho^{\mathsf{src}} = \rho^{\mathsf{src}}$, and $(\underline{\rho^{\mathsf{tgt}}})^{\mathsf{tgt}} = \rho^{\mathsf{tgt}}$, as can be easily shown by induction on ρ, so we conclude by EqRefl.
3. $\approx$-Assoc: Let $(\rho\,;\,\sigma)\,;\,\tau \approx \rho\,;\,(\sigma\,;\,\tau)$. Then $((\rho\,;\,\sigma)\,;\,\tau)^{\mathsf{src}} = \rho^{\mathsf{src}} = (\rho\,;\,(\sigma\,;\,\tau))^{\mathsf{src}}$ and $((\rho\,;\,\sigma)\,;\,\tau)^{\mathsf{tgt}} = \tau^{\mathsf{tgt}} = (\rho\,;\,(\sigma\,;\,\tau))^{\mathsf{tgt}}$, so we conclude by EqRefl.
4. $\approx$-Abs: Let $(\lambda x.\rho)\,;\,(\lambda x.\sigma) \approx \lambda x.(\rho\,;\,\sigma)$. Then $((\lambda x.\rho)\,;\,(\lambda x.\sigma))^{\mathsf{src}} = \lambda x.\rho^{\mathsf{src}} = (\lambda x.(\rho\,;\,\sigma))^{\mathsf{src}}$ and $((\lambda x.\rho)\,;\,(\lambda x.\sigma))^{\mathsf{tgt}} = \lambda x.\sigma^{\mathsf{tgt}} = (\lambda x.(\rho\,;\,\sigma))^{\mathsf{tgt}}$, so we conclude by EqRefl.
5. $\approx$-App: Let $(\rho_1\,\rho_2)\,;\,(\sigma_1\,\sigma_2) \approx (\rho_1\,;\,\sigma_1)\,(\rho_2\,;\,\sigma_2)$. Then $((\rho_1\,\rho_2)\,;\,(\sigma_1\,\sigma_2))^{\mathsf{src}} = \rho_1^{\mathsf{src}}\,\rho_2^{\mathsf{src}} = ((\rho_1\,;\,\sigma_1)\,(\rho_2\,;\,\sigma_2))^{\mathsf{src}}$ and $((\rho_1\,\rho_2)\,;\,(\sigma_1\,\sigma_2))^{\mathsf{tgt}} = \sigma_1^{\mathsf{tgt}}\,\sigma_2^{\mathsf{tgt}} = ((\rho_1\,;\,\sigma_1)\,(\rho_2\,;\,\sigma_2))^{\mathsf{tgt}}$, so we conclude by EqRefl.
6. $\approx$-BetaTR: Let $(\lambda x.\underline{s})\,\rho \approx s\{x\backslash\!\backslash\rho\}$. Then $((\lambda x.\underline{s})\,\rho)^{\mathsf{src}} = (\lambda x.s)\,(\rho^{\mathsf{src}}) =_{\beta\eta} s\{x\backslash\rho^{\mathsf{src}}\} = s\{x\backslash\!\backslash\rho\}^{\mathsf{src}}$ and $((\lambda x.\underline{s})\,\rho)^{\mathsf{tgt}} = (\lambda x.s)\,(\rho^{\mathsf{tgt}}) =_{\beta\eta} s\{x\backslash\rho^{\mathsf{tgt}}\} = s\{x\backslash\!\backslash\rho\}^{\mathsf{tgt}}$ hold by Lem. 49.
7. $\approx$-BetaRT: Let $(\lambda x.\rho)\,\underline{s} \approx \rho\{x\backslash s\}$. Then $((\lambda x.\rho)\,\underline{s})^{\mathsf{src}} = (\lambda x.\rho^{\mathsf{src}})\,s =_{\beta\eta} \rho^{\mathsf{src}}\{x\backslash s\} = \rho\{x\backslash s\}^{\mathsf{src}}$ and $((\lambda x.\rho)\,\underline{s})^{\mathsf{tgt}} = (\lambda x.\rho^{\mathsf{tgt}})\,s =_{\beta\eta} \rho^{\mathsf{tgt}}\{x\backslash s\} = \rho\{x\backslash s\}^{\mathsf{tgt}}$ hold by EqBeta and Lem. 48.
8. $\approx$-Eta: Let $\lambda x.\rho\,x \approx \rho$. Then $(\lambda x.\rho\,x)^{\mathsf{src}} = \lambda x.\rho^{\mathsf{src}}\,x =_{\beta\eta} \rho^{\mathsf{src}}$ and $(\lambda x.\rho\,x)^{\mathsf{tgt}} = \lambda x.\rho^{\mathsf{tgt}}\,x =_{\beta\eta} \rho^{\mathsf{tgt}}$ hold by EqEta.

◀

▶ **Lemma 53** (Term contexts distribute over composition). $\mathtt{C}\langle\rho\,;\,\sigma\rangle \approx \mathtt{C}\langle\rho\rangle\,;\,\mathtt{C}\langle\sigma\rangle$

Proof. By induction on $\mathtt{C}$:

1. **Empty,** $\mathtt{C} = \Box$**.** Immediate.
2. **Abstraction,** $\mathtt{C} = \lambda x.\mathtt{C}'$**.**

$$\begin{array}{rcll} \lambda x.\mathtt{C}'\langle\rho\,;\,\sigma\rangle & \approx & \lambda x.(\mathtt{C}'\langle\rho\rangle\,;\,\mathtt{C}'\langle\sigma\rangle) & \text{by IH} \\ & \approx & (\lambda x.\mathtt{C}'\langle\rho\rangle)\,;\,(\lambda x.\mathtt{C}'\langle\sigma\rangle) & \text{by } \approx\text{-Abs} \end{array}$$

3. **Left of an application,** $\mathtt{C} = \mathtt{C}'\,s$**.**

$$\begin{array}{rcll} \mathtt{C}'\langle\rho\,;\,\sigma\rangle\,s & \approx & (\mathtt{C}'\langle\rho\rangle\,;\,\mathtt{C}'\langle\sigma\rangle)\,s & \text{by IH} \\ & \approx & (\mathtt{C}'\langle\rho\rangle\,;\,\mathtt{C}'\langle\sigma\rangle)\,(s\,;\,s) & \text{by } \approx\text{-IdL} \\ & \approx & (\mathtt{C}'\langle\rho\rangle\,s)\,;\,(\mathtt{C}'\langle\sigma\rangle\,s) & \text{by } \approx\text{-App} \end{array}$$

4. **Right of an application,** $\mathtt{C} = s\,\mathtt{C}'$**.** Similar to the previous case.

◀

B.1 Properties of term/rewrite substitution, up to permutation equivalence

▶ **Lemma 54** (Transitivity under lifting substitution). *Let* $\Gamma, x : A \vdash s : B$ *and* $\Gamma \vdash \rho : p_0 \twoheadrightarrow p_1 : A$ *and* $\Gamma \vdash \sigma : p_1 \twoheadrightarrow p_2 : A$*. Then:*

$$s\{x\backslash\!\backslash\rho\}\,;\,s\{x\backslash\!\backslash\sigma\} \approx s\{x\backslash\!\backslash\rho\,;\,\sigma\}$$

Proof. By induction on the derivation of $\Gamma, x : A \vdash s : B$:

1. Var: let $\Gamma, x : A \vdash y : B$ with $y : B \in (\Gamma, x : A)$. We consider two subcases, depending on whether $x = y$ or not:

1.1 **If** $x = y$: then the left and the right-hand sides are both $\rho \,;\, \sigma$, so we are done.
1.2 **If** $x \neq y$: then the left-hand side is $y \,;\, y$ and the right-hand side is y, and $y \,;\, y \approx y$ by the $\approx$-IdL rule.

2. Con: let $\Gamma, x : A \vdash \mathbf{c} : B$ with $(\mathbf{c} : B) \in \mathcal{C}$. Then the left-hand side is $\mathbf{c} \,;\, \mathbf{c}$ and the right-hand side is $\mathbf{c}$, and $\mathbf{c} \,;\, \mathbf{c} \approx \mathbf{c}$ by the $\approx$-IdL rule.
3. Abs: let $\Gamma \vdash \lambda y.s : B \to C$ be derived from $\Gamma, x : A, y : B \vdash s : C$. Then:

$$\begin{array}{rcll} (\lambda y.s)\{x\backslash\!\!\backslash\rho\} \,;\, (\lambda y.s)\{x\backslash\!\!\backslash\sigma\} & = & (\lambda y.s\{x\backslash\!\!\backslash\rho\}) \,;\, (\lambda y.s\{x\backslash\!\!\backslash\sigma\}) & \\ & \approx & \lambda y.(s\{x\backslash\!\!\backslash\rho\} \,;\, s\{x\backslash\!\!\backslash\sigma\}) & \text{by } \approx\text{-Abs} \\ & \approx & \lambda y.s\{x\backslash\!\!\backslash\rho \,;\, \sigma\} & \text{by IH} \\ & = & (\lambda y.s)\{x\backslash\!\!\backslash\rho \,;\, \sigma\} & \text{by IH} \end{array}$$

4. App: let $\Gamma, x : A \vdash s\,t : C$ be derived from $\Gamma, x : A \vdash s : B \to C$ and $\Gamma, x : A \vdash t : B$. Then:

$$\begin{array}{rcll} (s\,t)\{x\backslash\!\!\backslash\rho\} \,;\, (s\,t)\{x\backslash\!\!\backslash\sigma\} & = & (s\{x\backslash\!\!\backslash\rho\}\,t\{x\backslash\!\!\backslash\rho\}) \,;\, (s\{x\backslash\!\!\backslash\sigma\}\,t\{x\backslash\!\!\backslash\sigma\}) & \\ & \approx & (s\{x\backslash\!\!\backslash\rho\} \,;\, s\{x\backslash\!\!\backslash\sigma\})\,(t\{x\backslash\!\!\backslash\rho\} \,;\, t\{x\backslash\!\!\backslash\sigma\}) & \text{by } \approx\text{-App} \\ & \approx & s\{x\backslash\!\!\backslash\rho \,;\, \sigma\}\,t\{x\backslash\!\!\backslash\rho \,;\, \sigma\} & \text{by IH} \\ & = & (s\,t)\{x\backslash\!\!\backslash\rho \,;\, \sigma\} & \end{array}$$

◀

B.2 Rewrite/rewrite substitution

▶ Remark 55. Note that $\rho\{x\backslash\!\!\backslash\!\!\backslash\sigma\}$ depends on s' and t, and hence on the particular typing derivations for ρ and σ. These particular derivations will usually be clear from the context. If there is any confusion we may write $\rho\{x\backslash t\} \,;\, s'\{x\backslash\!\!\backslash\sigma\}$ explicitly. We shall prove congruence results (in particular, Lem. 63 and Lem. 64) which ensure that the value of $\rho\{x\backslash\!\!\backslash\!\!\backslash\sigma\}$ does not depend, up to permutation equivalence, on the particular typing derivations chosen.

▶ **Lemma 56** (Rewrite/rewrite β-reduction rule). *Let $\Gamma, x : A \vdash \rho : s_0 \twoheadrightarrow s_1 : B$ and $\Gamma \vdash \sigma : t_0 \twoheadrightarrow t_1 : A$. Then the following equivalence, called* $\approx$-BetaRR*, holds:*

$$(\lambda x.\rho)\,\sigma \approx \rho\{x\backslash\!\!\backslash\!\!\backslash\sigma\} \qquad (\approx\text{-BetaRR})$$

Proof.

$$\begin{array}{rcll} (\lambda x.\rho)\,\sigma & \approx & ((\lambda x.\rho) \,;\, (\lambda x.s_1))\,\sigma & \approx\text{-IdR} \\ & \approx & ((\lambda x.\rho) \,;\, (\lambda x.s_1))\,(t_0 \,;\, \sigma) & \approx\text{-IdL} \\ & \approx & (\lambda x.\rho)\,t_0 \,;\, (\lambda x.s_1)\,\sigma & \approx\text{-App} \\ & \approx & \rho\{x\backslash t_0\} \,;\, (\lambda x.s_1)\,\sigma & \approx\text{-BetaRT} \\ & \approx & \rho\{x\backslash t_0\} \,;\, s_1\{x\backslash\!\!\backslash\sigma\} & \approx\text{-BetaTR} \\ & = & \rho\{x\backslash\!\!\backslash\!\!\backslash\sigma\} & \end{array}$$

◀

▶ **Lemma 57** (Typing rule for rewrite/rewrite substitution). *If $\Gamma, x : A \vdash \rho : s \twoheadrightarrow s' : B$ and $\Gamma \vdash \sigma : t \twoheadrightarrow t' : A$ then:*

$$\Gamma \vdash \rho\{x\backslash\!\!\backslash\!\!\backslash\sigma\} : s\{x\backslash t\} \twoheadrightarrow s'\{x\backslash t'\} : B$$

Proof. An immediate consequence of Lem. 41 and Lem. 44. ◀

The notion of rewrite/rewrite substitution generalizes the notions of rewrite/term and term/rewrite (lifting) substitution, as noted in the two following remarks:

▶ Remark 58 (Rewrite/rewrite generalizes rewrite/term substitution). If $\Gamma, x : A \vdash \rho : s \to s' : B$ and $\Gamma \vdash t : A$ then:

$$\rho\{x\backslash t\} \approx \rho\{x\backslash\!\!\backslash \underline{t}\}$$

Indeed:

$$\begin{array}{lll} \rho\{x\backslash t\} & \approx & \rho\{x\backslash t\} \,;\, \underline{s'\{x\backslash t\}} & \text{by } \approx\text{-IdR} \\ & = & \rho\{x\backslash t\} \,;\, s'\{x\backslash\!\!\backslash \underline{t}\} & \text{by Lem. 47} \\ & = & \rho\{x\backslash\!\!\backslash \underline{t}\} \end{array}$$

▶ Remark 59 (Rewrite/rewrite generalizes term/rewrite substitution). If $\Gamma, x : A \vdash s : B$ and $\Gamma \vdash \rho : t \to t' : A$ then:

$$s\{x\backslash\!\!\backslash \rho\} \approx \underline{s}\{x\backslash\!\!\backslash \rho\}$$

Indeed:

$$\begin{array}{lll} s\{x\backslash\!\!\backslash \rho\} & \approx & \underline{s\{x\backslash t\}} \,;\, s\{x\backslash\!\!\backslash \rho\} & \text{by } \approx\text{-IdL} \\ & = & \underline{s}\{x\backslash t\} \,;\, s\{x\backslash\!\!\backslash \rho\} & \text{since } \underline{s}\{x\backslash t\} = \underline{s\{x\backslash t\}} \text{ by definition} \\ & = & \underline{s}\{x\backslash\!\!\backslash \rho\} & \text{by definition of } \underline{s}\{x\backslash\!\!\backslash \rho\} \end{array}$$

▶ **Lemma 60** (Trivial rewrite/rewrite substitution). *Let $\Gamma, x : A \vdash \rho : s_0 \to s_1 : B$ be such that $x \notin \mathsf{fv}(\rho)$, and let $\Gamma \vdash \sigma : t_0 \to t_1 : A$. Then $\rho\{x\backslash\!\!\backslash \sigma\} \approx \rho$.*

Proof. Note that $x \notin \mathsf{fv}(s_1)$ by Lem. 39.

$$\begin{array}{lll} \rho\{x\backslash\!\!\backslash \sigma\} & = & \rho\{x\backslash t_0\} \,;\, s_1\{x\backslash\!\!\backslash \sigma\} & \text{by definition} \\ & = & \rho \,;\, s_1\{x\backslash\!\!\backslash \sigma\} & \text{since } x \notin \mathsf{fv}(\rho) \\ & = & \rho \,;\, \underline{s_1} & \text{since } x \notin \mathsf{fv}(s_1) \\ & \approx & \rho & \text{by } \approx\text{-IdR} \end{array}$$

◀

▶ **Lemma 61** (Recursive equations for rewrite/rewrite substitution). *Let $\Gamma \vdash \sigma : q_0 \to q_1 : A$. The following recursive equations hold for rewrite/rewrite substitution:*

1. *If $\Gamma, x : A, y : B \vdash \rho : s_0 \to s_1 : C$, then:*

$$(\lambda y.\rho)\{x\backslash\!\!\backslash \sigma\} \approx \lambda y.\rho\{y\backslash\!\!\backslash \sigma\}$$

2. *If $\Gamma, x : A \vdash \rho_1 : s_0 \to s_1 : B \to C$ and $\Gamma, x : A \vdash \rho_2 : t_0 \to t_1 : B$, then:*

$$(\rho_1\, \rho_2)\{x\backslash\!\!\backslash \sigma\} \approx \rho_1\{x\backslash\!\!\backslash \sigma\}\, \rho_2\{x\backslash\!\!\backslash \sigma\}$$

Proof. We check each item separately:

1. Abstraction.

$$\begin{array}{lll} (\lambda y.\rho)\{x\backslash\!\!\backslash \sigma\} & = & (\lambda y.\rho)\{x\backslash q_0\} \,;\, (\lambda y.s_1)\{x\backslash\!\!\backslash \sigma\} \\ & = & (\lambda y.\rho\{x\backslash q_0\}) \,;\, (\lambda y.s_1\{x\backslash\!\!\backslash \sigma\}) \\ & \approx & \lambda y.(\rho\{x\backslash q_0\} \,;\, s_1\{x\backslash\!\!\backslash \sigma\}) & \text{by } \approx\text{-Abs} \\ & = & \lambda y.\rho\{x\backslash\!\!\backslash \sigma\} \end{array}$$

2. Application: similar to the previous case, using the $\approx$-App rule.

◀

▶ **Lemma 62** (Commutation of lifting and term substitution (II)). *If* $\Gamma, x : A, y : B \vdash s : C$ *and* $\Gamma, y : B \vdash q : A$ *and* $\Gamma \vdash \tau : p_0 \twoheadrightarrow p_1 : B$ *then:*

$$s\{x\backslash q\}\{y\backslash\!\backslash \tau\} \approx s\{y\backslash\!\backslash \tau\}\{x\backslash\!\backslash\!\backslash q\{y\backslash\!\backslash \tau\}\}$$

Proof. By induction on the derivation of $\Gamma, x : A, y : B \vdash s : C$.

1. Var: let $\Gamma, x : A, y : B \vdash z : C$ with $z : C \in (\Gamma, x : A, y : B)$. We consider three subcases, depending on whether $z = x$, $z = y$, or $z \notin \{x, y\}$:
   - **1.1** **If** $z = x$: Then the left and the right-hand sides are both $q\{y\backslash\!\backslash \tau\}$, so it is immediate to conclude.
   - **1.2** **If** $z = y$: Then the left-hand side is τ and the right-hand side is $\tau\{x\backslash\!\backslash\!\backslash q\{y\backslash\!\backslash \tau\}\}$ Note that by Lem. 38, x does not occur free in τ, so by Lem. 60 we conclude.
   - **1.3** **If** $z \notin \{x, y\}$: Then the left and the right-hand sides are both z so we are done.
2. Con: let $\Gamma, x : A, x : B \vdash \mathbf{c} : C$ with $(\mathbf{c} : C) \in \mathcal{C}$. Then the left and the right-hand sides are both $\mathbf{c}$, so we are done.
3. Abs: let $\Gamma, x : A, y : B \vdash \lambda z.s : C \to D$ be derived from $\Gamma, x : A, y : B, z : C \vdash s : D$. Then:

$$\begin{array}{rcll} (\lambda z.s)\{x\backslash q\}\{y\backslash\!\backslash \tau\} & = & \lambda z.s\{x\backslash q\}\{y\backslash\!\backslash \tau\} & \\ & \approx & \lambda z.s\{y\backslash\!\backslash \tau\}\{x\backslash\!\backslash\!\backslash q\{y\backslash\!\backslash \tau\}\} & \text{by IH} \\ & \approx & (\lambda z.s)\{y\backslash\!\backslash \tau\}\{x\backslash\!\backslash\!\backslash q\{y\backslash\!\backslash \tau\}\} & \text{by Lem. 61} \end{array}$$

4. App: let $\Gamma, x : A, y : B \vdash s\,t : D$ be derived from $\Gamma, x : A, y : B \vdash s : C \to D$ and $\Gamma, x : A, y : B \vdash t : C$ Then:

$$\begin{array}{rcll} (s\,t)\{x\backslash q\}\{y\backslash\!\backslash \tau\} & = & s\{x\backslash q\}\{y\backslash\!\backslash \tau\}\,t\{x\backslash q\}\{y\backslash\!\backslash \tau\} & \\ & \approx & s\{y\backslash\!\backslash \tau\}\{x\backslash\!\backslash\!\backslash q\{y\backslash\!\backslash \tau\}\}\,t\{y\backslash\!\backslash \tau\}\{x\backslash\!\backslash\!\backslash q\{y\backslash\!\backslash \tau\}\} & \text{by IH} \\ & \approx & (s\,t)\{y\backslash\!\backslash \tau\}\{x\backslash\!\backslash\!\backslash q\{y\backslash\!\backslash \tau\}\} & \text{by Lem. 61} \end{array}$$

◀

B.3 Congruence properties

▶ **Lemma 63** ($=_{\beta\eta}$ under rewrite/term substitution). *Let* $\Gamma, x : A \vdash \rho : s \twoheadrightarrow t : B$ *and let* $\Gamma \vdash q =_{\beta\eta} q' : A$*. Then* $\rho\{x\backslash q\} \approx \rho\{x\backslash q'\}$*.*

Proof. By induction on the derivation of ρ:

1. RVar: let $\Gamma, x : A \vdash y : y \twoheadrightarrow y : B$ with $(y : B) \in (\Gamma, x : A)$. We consider two subcases, depending on whether $x = y$ or not.
   - **1.1** If $x = y$, it suffices to note that $q \approx q'$ by Lem. 50.
   - **1.2** If $x \neq y$, then trivially $\underline{y} \approx \underline{y}$.
2. RCon: let $\Gamma, x : A \vdash \mathbf{c} : \mathbf{c} \twoheadrightarrow \mathbf{c} : B$ with $(\mathbf{c} : B) \in \mathcal{C}$. Then trivially $\underline{y} \approx \underline{y}$.
3. RRule: let $\Gamma, x : A \vdash \underline{\varrho} : s \twoheadrightarrow t : B$ be derived from $\cdot \vdash s : B$ and $\cdot \vdash t : B$ with $(\varrho : s \twoheadrightarrow t : B) \in \mathcal{R}$. Then trivially $\underline{\varrho} \approx \underline{\varrho}$.
4. RAbs: let $\Gamma, x : A \vdash \lambda y.\rho : \lambda y.s_0 \twoheadrightarrow \lambda y.s_1 : B \to C$ be derived from $\Gamma, x : A, y : B \vdash \rho : s_0 \twoheadrightarrow s_1 : C$. Then by IH $\rho\{x\backslash q\} \approx \rho\{x\backslash q'\}$ so $\lambda y.\rho\{x\backslash q\} \approx \lambda y.\rho\{x\backslash q'\}$.
5. RApp: let $\Gamma, x : A \vdash \rho\,\sigma : s_0\,t_0 \twoheadrightarrow s_1\,t_1 : C$ be derived from $\Gamma, x : A \vdash \rho : s_0 \twoheadrightarrow s_1 : B \to C$ and $\Gamma, x : A \vdash \sigma : t_0 \twoheadrightarrow t_1 : B$. Then, applying the IH:

$$\rho\{x\backslash q\}\,\sigma\{x\backslash q\} \approx \rho\{x\backslash q'\}\,\sigma\{x\backslash q\} \approx \rho\{x\backslash q'\}\,\sigma\{x\backslash q'\}$$

6. RTrans: let $\Gamma, x : A \vdash \rho \,;\, \sigma : s_0 \rightarrow s_2 : B$ be derived from $\Gamma, x : A \vdash \rho : s_0 \rightarrow s_1 : B$ and $\Gamma, x : A \vdash \sigma : s_1 \rightarrow s_2 : B$. Then applying the IH:

$$\rho\{x\backslash q\} \,;\, \sigma\{x\backslash q\} \approx \rho\{x\backslash q'\} \,;\, \sigma\{x\backslash q\} \approx \rho\{x\backslash q'\} \,;\, \sigma\{x\backslash q'\}$$

7. RConv: immediate by IH.

◀

▶ **Lemma 64** ($=_{\beta\eta}$ under term/rewrite substitution)**.** *Let $\Gamma, x : A \vdash s =_{\beta\eta} s' : B$ and let $\Gamma \vdash \rho : q_0 \rightarrow q_1 : A$. Then $s\{x\backslash\!\backslash\rho\} \approx s'\{x\backslash\!\backslash\rho\}$.*

Proof. By induction on the derivation of the judgment $\Gamma, x : A \vdash s =_{\beta\eta} s' : B$. Reflexivity, symmetry, transitivity, and congruence under term constructors are immediate by IH. The interesting cases are:

1. EqBeta: let $s = (\lambda x.t)\, r$ and $s' = t\{x\backslash r\}$. Then:

$$\begin{array}{rcll} ((\lambda x.t)\, r)\{x\backslash\!\backslash\rho\} & = & (\lambda x.t\{x\backslash\!\backslash\rho\})\, r\{x\backslash\!\backslash\rho\} & \\ & \approx & t\{x\backslash\!\backslash\rho\}\{y\backslash\!\backslash\!\backslash r\{x\backslash\!\backslash\rho\}\} & \text{by } \approx\text{-BetaRR (Lem. 56)} \\ & \approx & t\{y\backslash r\}\{x\backslash\!\backslash\rho\} & \text{by Lem. 62} \end{array}$$

2. EqEta: let $s = \lambda y.s'\, x$ with $y \notin \mathsf{fv}(s')$. Then:

$$\begin{array}{rcll} (\lambda y.s'\, y)\{x\backslash\!\backslash\rho\} & = & \lambda y.s'\{x\backslash\!\backslash\rho\}\, y & \\ & \approx & s' & \text{by } \approx\text{-Eta, as } y \notin \mathsf{fv}(s'\{x\backslash\!\backslash\rho\}) \end{array}$$

   In the last step, note that we may assume that $y \notin \mathsf{fv}(\rho)$ by Barendregt's convention.

◀

▶ **Lemma 65** (Congruence for $\approx$ below rewrite/term substitution)**.** *Let $\Gamma, x : A \vdash \rho : p_0 \rightarrow p_1 : B$ and $\Gamma, x : A \vdash \rho' : p'_0 \rightarrow p'_1 : B$ be such that $\rho \approx \rho'$, and let $\Gamma \vdash q : A$. Then $\rho\{x\backslash q\} \approx \rho'\{x\backslash q\}$.*

Proof.

$$\begin{array}{rcll} \rho\{x\backslash q\} & \approx & (\lambda x.\rho)\, q & \text{by } \approx\text{-BetaRT} \\ & \approx & (\lambda x.\rho')\, q & \\ & \approx & \rho'\{x\backslash q\} & \text{by } \approx\text{-BetaRT} \end{array}$$

◀

▶ **Lemma 66** (Congruence for $\approx$ below term/rewrite substitution)**.** *Let $\Gamma, x : A \vdash s : B$ and let $\Gamma \vdash \rho : p_0 \rightarrow p_1 : A$ and $\Gamma \vdash \rho' : p'_0 \rightarrow p'_1 : A$ such that $\rho \approx \rho'$. Then $s\{x\backslash\!\backslash\rho\} \approx s\{x\backslash\!\backslash\rho'\}$.*

Proof.

$$\begin{array}{rcll} s\{x\backslash\!\backslash\rho\} & \approx & (\lambda x.s)\, \rho & \text{by } \approx\text{-BetaTR} \\ & \approx & (\lambda x.s)\, \rho' & \\ & \approx & s'\{x\backslash\!\backslash\rho\} & \text{by } \approx\text{-BetaTR} \end{array}$$

◀

▶ **Proposition 67** (Congruence for $\approx$ below rewrite/rewrite substitution)**.** *Let $\Gamma, x : A \vdash \rho : p_0 \rightarrow p_1 : B$ and $\Gamma, x : A \vdash \rho' : p'_0 \rightarrow p'_1 : B$ be such that $\rho \approx \rho'$, and let $\Gamma \vdash \sigma : q_0 \rightarrow q_1 : A$ and $\Gamma \vdash \sigma' : q'_0 \rightarrow q'_1 : A$ be such that $\sigma \approx \sigma'$. Then $\rho\{x\backslash\!\backslash\!\backslash\sigma\} \approx \rho'\{x\backslash\!\backslash\!\backslash\sigma'\}$.*

Proof.

$$
\begin{array}{rcll}
\rho\{x\backslash\!\!\backslash\sigma\} & = & \rho\{x\backslash q_0\}\,;\,p_1\{x\backslash\!\!\backslash\sigma\} & \\
 & \approx & \rho'\{x\backslash q_0\}\,;\,p_1\{x\backslash\!\!\backslash\sigma\} & \text{by Lem. 65} \\
 & \approx & \rho'\{x\backslash q_0'\}\,;\,p_1\{x\backslash\!\!\backslash\sigma\} & \text{by Lem. 63} \\
 & \approx & \rho'\{x\backslash q_0'\}\,;\,p_1'\{x\backslash\!\!\backslash\sigma\} & \text{by Lem. 64} \\
 & \approx & \rho'\{x\backslash q_0'\}\,;\,p_1'\{x\backslash\!\!\backslash\sigma'\} & \text{by Lem. 66} \\
 & = & \rho'\{x\backslash\!\!\backslash\sigma'\} &
\end{array}
$$

◀

B.4 Permutation lemma

▶ **Lemma 68** (Coherence of term variable substitution). *Let* $\Gamma, x : A \vdash \rho : p_0 \twoheadrightarrow p_1 : B$ *and* $\Gamma \vdash \sigma : q_0 \twoheadrightarrow q_1 : A$. *Then:*

$$\rho\{x\backslash q_0\}\,;\,p_1\{x\backslash\!\!\backslash\sigma\} \approx p_0\{x\backslash\!\!\backslash\sigma\}\,;\,\rho\{x\backslash q_1\}$$

Proof. By induction on the derivation of $\Gamma, x : A \vdash \rho : p_0 \twoheadrightarrow p_1 : B$:

1. RVar: let $\Gamma, x : A \vdash y : y \twoheadrightarrow y : B$. We consider two subcases depending on whether $x = y$ or not:
   1.1 If $x = y$, then:

$$
\begin{array}{rcll}
q_0\,;\,\sigma & \approx & \sigma & \text{by } \approx\text{-IdL} \\
 & \approx & \sigma\,;\,q_1 & \text{by } \approx\text{-IdR}
\end{array}
$$

   1.2 If $x \neq y$, then both the left and the right-hand side are $y\,;\,y$, so we are done.
2. RCon: let $\Gamma \vdash \mathbf{c} : \mathbf{c} \twoheadrightarrow \mathbf{c} : B$ where $(\mathbf{c} : B) \in \mathcal{C}$. Then both the left and the right-hand side are $\mathbf{c}\,;\,\mathbf{c}$, so we are done.
3. RRule: let $\Gamma, x : A \vdash \varrho : s \twoheadrightarrow t : B$ be derived from $\cdot \vdash s : B$ and $\cdot \vdash t : B$ with $(\varrho : p_0 \twoheadrightarrow p_1 : B) \in \mathcal{R}$. Note that, by Lem. 38, p_0 and p_1 have no free occurrences of x. Hence $p_0\{x\backslash\!\!\backslash\sigma\} \approx \underline{p_0}$ and $p_1\{x\backslash\!\!\backslash\sigma\} \approx \underline{p_1}$, and we have:

$$
\begin{array}{rcll}
\varrho\,;\,t\{x\backslash\!\!\backslash\sigma\} & \approx & \varrho\,;\,\underline{t} & \\
 & \approx & \varrho & \text{by } \approx\text{-IdR} \\
 & \approx & \underline{s}\,;\,\varrho & \text{by } \approx\text{-IdL} \\
 & \approx & s\{x\backslash\!\!\backslash\sigma\}\,;\,\varrho &
\end{array}
$$

4. RAbs: let $\Gamma, x : A \vdash \lambda y.\rho : \lambda y.s_0 \twoheadrightarrow \lambda y.s_1 : B \to C$ be derived from $\Gamma, x : A, y : B \vdash \rho : s_0 \twoheadrightarrow s_1 : C$. So:

$$
\begin{array}{rcll}
(\lambda y.\rho\{x\backslash\!\!\backslash q_0\})\,;\,(\lambda y.s_1\{x\backslash\!\!\backslash\sigma\}) & \approx & \lambda y.(\rho\{x\backslash\!\!\backslash q_0\}\,;\,s_1\{x\backslash\!\!\backslash\sigma\}) & \text{by } \approx\text{-Abs} \\
 & \approx & \lambda y.(s_0\{x\backslash\!\!\backslash\sigma\}\,;\,\rho\{x\backslash\!\!\backslash q_1\}) & \text{by IH} \\
 & \approx & (\lambda y.s_0\{x\backslash\!\!\backslash\sigma\})\,;\,(\lambda y.\rho\{x\backslash\!\!\backslash q_1\}) & \text{by } \approx\text{-Abs}
\end{array}
$$

5. RApp: let $\Gamma, x : A \vdash \rho_1\,\rho_2 : s_0\,t_0 \twoheadrightarrow s_1\,t_1 : C$ be derived from $\Gamma, x : A \vdash \rho_1 : s_0 \twoheadrightarrow s_1 : B \to C$ and $\Gamma, x : A \vdash \rho_2 : t_0 \twoheadrightarrow t_1 : B$. Then:

$$
\begin{array}{cll}
 & (\rho_1\{x\backslash q_0\}\,\rho_2\{x\backslash q_0\})\,;\,(s_1\{x\backslash\!\!\backslash\sigma\}\,t_1\{x\backslash\!\!\backslash\sigma\}) & \\
\approx & (\rho_1\{x\backslash q_0\}\,;\,s_1\{x\backslash\!\!\backslash\sigma\})\,(\rho_2\{x\backslash q_0\}\,;\,t_1\{x\backslash\!\!\backslash\sigma\}) & \text{by } \approx\text{-App} \\
\approx & (s_0\{x\backslash\!\!\backslash\sigma\}\,;\,\rho_1\{x\backslash q_1\})\,(t_0\{x\backslash\!\!\backslash\sigma\}\,;\,\rho_2\{x\backslash q_1\}) & \text{by IH} \\
\approx & (s_0\{x\backslash\!\!\backslash\sigma\}\,t_0\{x\backslash\!\!\backslash\sigma\})\,;\,(\rho_1\{x\backslash q_1\}\,\rho_2\{x\backslash q_1\}) & \text{by } \approx\text{-App}
\end{array}
$$

6. RTrans: let $\Gamma, x : A \vdash \rho_1 \,;\, \rho_2 : s_0 \twoheadrightarrow s_2 : B$ be derived from $\Gamma, x : A \vdash \rho_1 : s_0 \twoheadrightarrow s_1 : B$ and $\Gamma, x : A \vdash \rho_2 : s_1 \twoheadrightarrow s_2 : B$. Then:

$$\begin{array}{rcll}
(\rho_1\{x\backslash q_0\} \,;\, \rho_2\{x\backslash q_0\}) \,;\, s_2\{x\backslash\!\!\backslash\sigma\} & \approx & \rho_1\{x\backslash q_0\} \,;\, (\rho_2\{x\backslash q_0\} \,;\, s_2\{x\backslash\!\!\backslash\sigma\}) & \text{by } \approx\text{-Assoc} \\
& \approx & \rho_1\{x\backslash q_0\} \,;\, (s_1\{x\backslash\!\!\backslash\sigma\} \,;\, \rho_2\{x\backslash q_1\}) & \text{by IH} \\
& \approx & (\rho_1\{x\backslash q_0\} \,;\, s_1\{x\backslash\!\!\backslash\sigma\}) \,;\, \rho_2\{x\backslash q_1\} & \text{by } \approx\text{-Assoc} \\
& \approx & (s_0\{x\backslash\!\!\backslash\sigma\} \,;\, \rho_1\{x\backslash q_1\}) \,;\, \rho_2\{x\backslash q_1\} & \text{by IH} \\
& \approx & s_0\{x\backslash\!\!\backslash\sigma\} \,;\, (\rho_1\{x\backslash q_1\} \,;\, \rho_2\{x\backslash q_1\}) & \text{by } \approx\text{-Assoc}
\end{array}$$

7. RConv: let $\Gamma, x : A \vdash \rho : p_0 \twoheadrightarrow p_1 : B$ be derived from $\Gamma, x : A \vdash \rho : p_0' \twoheadrightarrow p_1' : B$ where $\Gamma, x : A \vdash p_0 =_{\beta\eta} p_0' : B$ and $\Gamma, x : A \vdash p_1' =_{\beta\eta} p_1 : B$. Then:

$$\begin{array}{rcll}
\rho\{x\backslash q_0\} \,;\, p_1\{x\backslash\!\!\backslash\sigma\} & \approx & \rho\{x\backslash q_0\} \,;\, p_1'\{x\backslash\!\!\backslash\sigma\} & \text{by Lem. 64} \\
& \approx & p_0'\{x\backslash\!\!\backslash\sigma\} \,;\, \rho\{x\backslash q_1\} & \text{by IH} \\
& \approx & p_0\{x\backslash\!\!\backslash\sigma\} \,;\, \rho\{x\backslash q_1\} & \text{by Lem. 64}
\end{array}$$

◀

▶ **Proposition 69** (Transitivity under rewrite/rewrite substitution). *Suppose that:*

- $\Gamma, x : A \vdash \rho_1 : s_0 \twoheadrightarrow s_1 : B$ *and* $\Gamma, x : A \vdash \rho_2 : s_1 \twoheadrightarrow s_2 : B$
- $\Gamma \vdash \sigma_1 : t_0 \twoheadrightarrow t_1 : B$ *and* $\Gamma \vdash \sigma_2 : t_1 \twoheadrightarrow t_2 : B$

Then:

$$(\rho_1 \,;\, \rho_2)\{x\backslash\!\!\backslash\!\!\backslash\sigma_1 \,;\, \sigma_2\} \approx \rho_1\{x\backslash\!\!\backslash\!\!\backslash\sigma_1\} \,;\, \rho_2\{x\backslash\!\!\backslash\!\!\backslash\sigma_2\}$$

Proof. We work implicitly modulo associativity of composition ("$;$"), using the ≈-Assoc rule:

$$\begin{array}{rcll}
(\rho_1 \,;\, \rho_2)\{x\backslash\!\!\backslash\!\!\backslash\sigma_1 \,;\, \sigma_2\} & = & (\rho_1 \,;\, \rho_2)\{x\backslash t_0\} \,;\, s_2\{x\backslash\!\!\backslash\sigma_1 \,;\, \sigma_2\} & \\
& = & \rho_1\{x\backslash t_0\} \,;\, \rho_2\{x\backslash t_0\} \,;\, s_2\{x\backslash\!\!\backslash\sigma_1 \,;\, \sigma_2\} & \\
& \approx & \rho_1\{x\backslash t_0\} \,;\, \rho_2\{x\backslash t_0\} \,;\, s_2\{x\backslash\!\!\backslash\sigma_1\} \,;\, s_2\{x\backslash\!\!\backslash\sigma_2\} & \text{by Lem. 54} \\
& \approx & \rho_1\{x\backslash t_0\} \,;\, s_1\{x\backslash\!\!\backslash\sigma_1\} \,;\, \rho_2\{x\backslash t_1\} \,;\, s_2\{x\backslash\!\!\backslash\sigma_2\} & \text{by Lem. 68} \\
& \approx & \rho_1\{x\backslash\!\!\backslash\!\!\backslash\sigma_1\} \,;\, \rho_2\{x\backslash\!\!\backslash\!\!\backslash\sigma_2\} &
\end{array}$$

◀

▶ **Proposition 70** (Substitution property for rewrite/rewrite substitution). *Suppose that:*

- $\Gamma, x : A, y : B \vdash \rho : s_0 \twoheadrightarrow s_1 : C$
- $\Gamma, y : B \vdash \sigma : t_0 \twoheadrightarrow t_1 : A$
- $\Gamma \vdash \tau : r_0 \twoheadrightarrow r_1 : B$

Then:

$$\rho\{x\backslash\!\!\backslash\!\!\backslash\sigma\}\{y\backslash\!\!\backslash\!\!\backslash\tau\} \approx \rho\{y\backslash\!\!\backslash\!\!\backslash\tau\}\{x\backslash\!\!\backslash\!\!\backslash\sigma\{y\backslash\!\!\backslash\!\!\backslash\tau\}\}$$

Proof. We work implicitly modulo associativity of composition ("$;$"), using the ≈-Assoc rule:

$$\begin{array}{cll}
& \rho\{x\backslash\!\!\backslash\!\!\backslash\sigma\}\{y\backslash\!\!\backslash\!\!\backslash\tau\} & \\
= & (\rho\{x\backslash t_0\} \,;\, s_1\{x\backslash\!\!\backslash\sigma\})\{y\backslash\!\!\backslash\!\!\backslash\tau\} & \text{by definition of } \{x\backslash\!\!\backslash\!\!\backslash\sigma\} \\
= & (\rho\{x\backslash t_0\} \,;\, s_1\{x\backslash\!\!\backslash\sigma\})\{y\backslash r_0\} \,;\, s_1\{x\backslash t_1\}\{y\backslash\!\!\backslash\tau\} & \text{by definition of } \{y\backslash\!\!\backslash\!\!\backslash\tau\} \\
= & \rho\{x\backslash t_0\}\{y\backslash r_0\} \,;\, s_1\{x\backslash\!\!\backslash\sigma\}\{y\backslash r_0\} \,;\, s_1\{x\backslash t_1\}\{y\backslash\!\!\backslash\tau\} & \\
= & \rho\{y\backslash r_0\}\{x\backslash t_0\{y\backslash r_0\}\} \,;\, s_1\{x\backslash\!\!\backslash\sigma\}\{y\backslash r_0\} \,;\, s_1\{x\backslash t_1\}\{y\backslash\!\!\backslash\tau\} & \text{Substitution Lemma } (\star) \\
\approx & \rho\{y\backslash r_0\}\{x\backslash t_0\{y\backslash r_0\}\} \,;\, s_1\{y\backslash r_0\}\{x\backslash\!\!\backslash\sigma\{y\backslash r_0\}\} \,;\, s_1\{x\backslash t_1\}\{y\backslash\!\!\backslash\tau\} & \text{Lem. 45} \\
\approx & \rho\{y\backslash r_0\}\{x\backslash t_0\{y\backslash r_0\}\} \,;\, s_1\{y\backslash r_0\}\{x\backslash\!\!\backslash\sigma\{y\backslash r_0\}\} \,;\, s_1\{y\backslash\!\!\backslash\tau\}\{x\backslash\!\!\backslash\!\!\backslash t_1\{y\backslash\!\!\backslash\tau\}\} & \text{Lem. 62} \\
= & \rho\{y\backslash r_0\}\{x\backslash\!\!\backslash\!\!\backslash t_0\{y\backslash r_0\}\} \,;\, s_1\{y\backslash r_0\}\{x\backslash\!\!\backslash\sigma\{y\backslash r_0\}\} \,;\, s_1\{y\backslash\!\!\backslash\tau\}\{x\backslash\!\!\backslash\!\!\backslash t_1\{y\backslash\!\!\backslash\tau\}\} & \text{Rem. 58} \\
\approx & \rho\{y\backslash r_0\}\{x\backslash\!\!\backslash\!\!\backslash \underline{t_0\{y\backslash r_0\}}\} \,;\, \underline{s_1\{y\backslash r_0\}}\{x\backslash\!\!\backslash\!\!\backslash\sigma\{y\backslash r_0\}\} \,;\, s_1\{y\backslash\!\!\backslash\tau\}\{x\backslash\!\!\backslash\!\!\backslash t_1\{y\backslash\!\!\backslash\tau\}\} & \text{Rem. 59} \\
\approx & (\rho\{y\backslash r_0\} \,;\, \underline{s_1\{y\backslash r_0\}} \,;\, s_1\{y\backslash\!\!\backslash\tau\})\{x\backslash\!\!\backslash\!\!\backslash \underline{t_0\{y\backslash r_0\}} \,;\, \sigma\{y\backslash r_0\} \,;\, t_1\{y\backslash\!\!\backslash\tau\}\} & \text{Prop. 69 (twice)} \\
\approx & (\rho\{y\backslash r_0\} \,;\, s_1\{y\backslash\!\!\backslash\tau\})\{x\backslash\!\!\backslash\!\!\backslash\sigma\{y\backslash r_0\} \,;\, t_1\{y\backslash\!\!\backslash\tau\}\} & \text{by } \approx\text{-IdL } (\star\star) \\
\approx & \rho\{y\backslash\!\!\backslash\!\!\backslash\tau\}\{x\backslash\!\!\backslash\!\!\backslash\sigma\{y\backslash\!\!\backslash\!\!\backslash\tau\}\} & \text{by } \approx\text{-IdL}
\end{array}$$

In the step marked with ($\star$) we use a variant of the standard Substitution Lemma for capture-avoiding substitution of a variable for a term. In the step marked with ($\star\star$) we use the fact that permutation equivalence is compatible with rewrite/rewrite substitution (Prop. 67). ◀

▶ **Lemma 71** (Generalized $\approx$-App). *Suppose that:*
- $\Gamma \vdash \rho : s \twoheadrightarrow s' : A_1 \to \ldots \to A_n \to B$ *and* $\Gamma \vdash \rho' : s' \twoheadrightarrow s'' : A_1 \to \ldots \to A_n \to B$,
- $\Gamma \vdash \sigma_i : t_i \twoheadrightarrow t_i' : A_i$ *and* $\Gamma \vdash \sigma_i : t_i' \twoheadrightarrow t_i'' : A_i$ *for all* $1 \le i \le n$.

Then:

$$(\rho\,\sigma_1 \ldots \sigma_n) \,;\, (\rho'\,\sigma_1' \ldots \sigma_n') \approx (\rho \,;\, \rho')\,(\sigma_1 \,;\, \sigma_1') \ldots (\sigma_n \,;\, \sigma_n')$$

Proof. By induction on n. The base case when $n = 0$ is immediate. In the inductive case:

$$\begin{array}{lll} & (\rho\,\sigma_1 \ldots \sigma_n\,\sigma_{n+1}) \,;\, (\rho'\,\sigma_1' \ldots \sigma_n'\,\sigma_{n+1}) & \\ \approx & (\rho\,\sigma_1 \ldots \sigma_n \,;\, \rho'\,\sigma_1' \ldots \sigma_n')\,(\sigma_{n+1} \,;\, \sigma_{n+1}) & \text{by } \approx\text{-App} \\ \approx & (\rho \,;\, \rho')\,(\sigma_1 \,;\, \sigma_1') \ldots (\sigma_n \,;\, \sigma_n')\,(\sigma_{n+1} \,;\, \sigma_{n+1}) & \text{by IH} \end{array}$$

◀

▶ **Lemma 72** (Generalized $\approx$-BetaRR rule). *Suppose that:*
- $\Gamma, x_1 : A_1, \ldots, x_n : A_n \vdash \rho : s \twoheadrightarrow s' : B$
- $\Gamma \vdash \sigma_i : t_i \twoheadrightarrow t_i' : B_i$ *for all* $1 \le i \le n$

Then:

$$(\lambda x_1 \ldots x_n.\rho)\,\sigma_1 \ldots \sigma_n \approx \rho\{x_1 \backslash\!\!\backslash \sigma_1\} \ldots \{x_n \backslash\!\!\backslash \sigma_n\}$$

Proof. By induction on n. The base case when $n = 0$ is immediate. In the inductive case:

$$\begin{array}{lll} & (\lambda x_1\,x_2 \ldots x_n.\rho)\,\sigma_1\,\sigma_2 \ldots \sigma_n & \\ \approx & ((\lambda x_2 \ldots x_n.\rho)\{x_1 \backslash\!\!\backslash \sigma_1\})\,\sigma_2 \ldots \sigma_n & \text{by } \approx\text{-BetaRR (Lem. 56)} \\ = & ((\lambda x_2 \ldots x_n.\rho)\{x_1 \backslash t_1\} \,;\, (\lambda x_2 \ldots x_n.s)\{x_1 \backslash \sigma_1\})\,\sigma_2 \ldots \sigma_n & \\ = & ((\lambda x_2 \ldots x_n.\rho\{x_1 \backslash t_1\}) \,;\, (\lambda x_2 \ldots x_n.s\{x_1 \backslash \sigma_1\}))\,\sigma_2 \ldots \sigma_n & \\ \approx & (\lambda x_2 \ldots x_n.(\rho\{x_1 \backslash t_1\} \,;\, s\{x_1 \backslash \sigma_1\}))\,\sigma_2 \ldots \sigma_n & \text{by } \approx\text{-Abs } (n-1 \text{ times}) \\ = & (\lambda x_2 \ldots x_n.\rho\{x_1 \backslash\!\!\backslash \sigma_1\})\,\sigma_2 \ldots \sigma_n & \text{by definition} \\ = & \rho\{x_1 \backslash\!\!\backslash \sigma_1\}\{x_2 \backslash\!\!\backslash \sigma_2\} \ldots \{x_n \backslash\!\!\backslash \sigma_n\} & \text{by IH} \end{array}$$

◀

▶ **Proposition 73** (Permutation). *Suppose that:*
- $\Gamma \vdash \rho : \lambda x_1 \ldots x_n.s \twoheadrightarrow \lambda x_1 \ldots x_n.s' : A_1 \to \ldots \to A_n \to B$
- $\Gamma \vdash \sigma_i : t_i \twoheadrightarrow t_i' : A_i$ *for each* $1 \le i \le n$.

Then:

1. $\rho\,\sigma_1 \ldots \sigma_n \approx \rho\,t_1 \ldots t_n \,;\, s'\{x_1 \backslash\!\!\backslash \sigma_1\} \ldots \{x_n \backslash\!\!\backslash \sigma_n\}$
2. $\rho\,\sigma_1 \ldots \sigma_n \approx s\{x_1 \backslash\!\!\backslash \sigma_1\} \ldots \{x_n \backslash\!\!\backslash \sigma_n\} \,;\, \rho\,t_1' \ldots t_n'$

Proof. For item 1.:

$$\begin{array}{llll} \rho\,\sigma_1 \ldots \sigma_n & \approx & (\rho \,;\, \lambda x_1 \ldots x_n.s')\,\sigma_1 \ldots \sigma_n & \text{by } \approx\text{-IdR} \\ & \approx & (\rho \,;\, \lambda x_1 \ldots x_n.s')\,(t_1 \,;\, \sigma_1) \ldots (t_n \,;\, \sigma_n) & \text{by } \approx\text{-IdL } (n \text{ times}) \\ & \approx & (\rho\,t_1 \ldots t_n) \,;\, ((\lambda x_1 \ldots x_n.s')\,\sigma_1 \ldots \sigma_n) & \text{by generalized } \approx\text{-App (Lem. 71)} \\ & \approx & (\rho\,t_1 \ldots t_n) \,;\, ((\lambda x_1 \ldots x_n.s')\,\sigma_1 \ldots \sigma_n) & \text{by } \approx\text{-1Abs } (n \text{ times}) \\ & \approx & (\rho\,t_1 \ldots t_n) \,;\, (s'\{x_1 \backslash\!\!\backslash \sigma_1\} \ldots \{x_n \backslash\!\!\backslash \sigma_n\}) & \text{by generalized } \approx\text{-BetaRR (Lem. 72)} \end{array}$$

Item 2. is similar:

$$
\begin{array}{rcll}
\rho\,\sigma_1 \dots \sigma_n & \approx & ((\lambda x_1 \dots x_n.s)\,;\,\rho)\,\sigma_1 \dots \sigma_n & \text{by } \approx\text{-IdL} \\
& \approx & ((\lambda x_1 \dots x_n.s)\,;\,\rho)\,(\sigma_1\,;\,t'_1) \dots (\sigma_n\,;\,t'_n) & \text{by } \approx\text{-IdR } (n \text{ times}) \\
& \approx & ((\lambda x_1 \dots x_n.s)\,\sigma_1 \dots \sigma_n)\,;\,(\rho\,t'_1 \dots t'_n) & \text{by generalized } \approx\text{-App (Lem. 71)} \\
& \approx & ((\lambda x_1 \dots x_n.s)\,\sigma_1 \dots \sigma_n)\,;\,(\rho\,t'_1 \dots t'_n) & \text{by } \approx\text{-1Abs } (n \text{ times}) \\
& \approx & (s\{x \backslash\!\!\backslash \sigma_1\} \dots \{x \backslash\!\!\backslash \sigma_n\})\,;\,(\rho\,t'_1 \dots t'_n) & \text{by generalized } \approx\text{-BetaRR (Lem. 72)}
\end{array}
$$

◀

B.5 Summary of properties of the substitution operators and ≈

$$
\begin{array}{ll}
s\{x\backslash\backslash\rho\}\{y\backslash t\} = s\{y\backslash t\}\{x\backslash\backslash\rho\{y\backslash t\}\} & (\textit{Lem.}\ 45) \\
\underline{s}\{x\backslash t\} = \underline{s\{x\backslash t\}} & (\text{trivial}) \\
\underline{s\{x\backslash t\}} = s\{x\backslash\backslash\underline{t}\} & (\textit{Lem.}\ 47) \\
s =_{\beta\eta} s' \text{ implies } \underline{s} \approx \underline{s'} & (\textit{Lem.}\ 50) \\
\rho \approx \sigma \text{ implies } \rho^{\mathsf{src}} =_{\beta\eta} \sigma^{\mathsf{src}} \text{ and } \rho^{\mathsf{tgt}} =_{\beta\eta} \sigma^{\mathsf{tgt}} & (\textit{Lem.}\ 52) \\
\mathtt{C}\langle\rho\,;\sigma\rangle \approx \mathtt{C}\langle\rho\rangle\,;\mathtt{C}\langle\sigma\rangle & (\textit{Lem.}\ 53) \\
s\{x\backslash\backslash\rho\}\,;s\{x\backslash\backslash\sigma\} \approx s\{x\backslash\backslash\rho\,;\sigma\} & (\textit{Lem.}\ 54) \\
(\lambda x.\rho)\,\sigma \approx \rho\{x\backslash\backslash\backslash\sigma\} & (\textit{Lem.}\ 56) \\
\rho\{x\backslash t\} \approx \rho\{x\backslash\backslash\backslash\underline{t}\} & (\textit{Rem.}\ 58) \\
s\{x\backslash\backslash\rho\} \approx \underline{s}\{x\backslash\backslash\backslash\rho\} & (\textit{Rem.}\ 59) \\
\rho\{x\backslash\backslash\backslash\sigma\} \approx \rho \quad \text{ if } x \notin \mathsf{fv}(\rho) & (\textit{Lem.}\ 60) \\
(\lambda y.\rho)\{x\backslash\backslash\backslash\sigma\} \approx \lambda y.\rho\{y\backslash\backslash\backslash\sigma\} & (\textit{Lem.}\ 61) \\
(\rho_1\,\rho_2)\{x\backslash\backslash\backslash\sigma\} \approx \rho_1\{x\backslash\backslash\backslash\sigma\}\,\rho_2\{x\backslash\backslash\backslash\sigma\} & (\textit{Lem.}\ 61) \\
s\{x\backslash q\}\{y\backslash\backslash\tau\} \approx s\{y\backslash\backslash\tau\}\{x\backslash\backslash\backslash q\{y\backslash\backslash\tau\}\} & (\textit{Lem.}\ 62) \\
s =_{\beta\eta} s' \text{ implies } \rho\{x\backslash s\} \approx \rho\{x\backslash s'\} & (\textit{Lem.}\ 63) \\
s =_{\beta\eta} s' \text{ implies } s\{x\backslash\backslash\rho\} \approx s'\{x\backslash\backslash\rho\} & (\textit{Lem.}\ 64) \\
\rho \approx \rho' \text{ implies } \rho\{x\backslash s\} \approx \rho'\{x\backslash s\} & (\textit{Lem.}\ 65) \\
\rho \approx \rho' \text{ implies } s\{x\backslash\backslash\rho\} \approx s\{x\backslash\backslash\rho'\} & (\textit{Lem.}\ 66) \\
\rho \approx \rho' \text{ and } \sigma \approx \sigma' \text{ imply } \rho\{x\backslash\backslash\backslash\sigma\} \approx \rho'\{x\backslash\backslash\backslash\sigma'\} & (\textit{Prop.}\ 67) \\
\rho\{x\backslash q_0\}\,;p_1\{x\backslash\backslash\sigma\} \approx p_0\{x\backslash\backslash\sigma\}\,;\rho\{x\backslash q_1\} & (\textit{Lem.}\ 68) \\
(\rho_1\,;\rho_2)\{x\backslash\backslash\backslash\sigma_1\,;\sigma_2\} \approx \rho_1\{x\backslash\backslash\backslash\sigma_1\}\,;\rho_2\{x\backslash\backslash\backslash\sigma_2\} & (\textit{Prop.}\ 69) \\
\rho\{x\backslash\backslash\backslash\sigma\}\{y\backslash\backslash\backslash\tau\} \approx \rho\{y\backslash\backslash\backslash\tau\}\{x\backslash\backslash\backslash\sigma\{y\backslash\backslash\backslash\tau\}\} & (\textit{Prop.}\ 70) \\
(\rho\,\sigma_1\ldots\sigma_n)\,;(\rho'\,\sigma'_1\ldots\sigma'_n) \approx (\rho\,;\rho')\,(\sigma_1\,;\sigma'_1)\ldots(\sigma_n\,;\sigma'_n) & (\textit{Lem.}\ 71) \\
(\lambda x_1\ldots x_n.\rho)\,\sigma_1\ldots\sigma_n \approx \rho\{x_1\backslash\backslash\backslash\sigma_1\}\ldots\{x_n\backslash\backslash\backslash\sigma_n\} & (\textit{Lem.}\ 72) \\
\rho\,\sigma_1\ldots\sigma_n \approx \rho\,t_1\ldots t_n\,;s'\{x_1\backslash\backslash\backslash\sigma_1\}\ldots\{x_n\backslash\backslash\backslash\sigma_n\} & (\textit{Prop.}\ 73) \\
\rho\,\sigma_1\ldots\sigma_n \approx s\{x_1\backslash\backslash\backslash\sigma_1\}\ldots\{x_n\backslash\backslash\backslash\sigma_n\}\,;\rho\,t'_1\ldots t'_n & (\textit{Prop.}\ 73)
\end{array}
$$

C Restricted η-expansion

Recall from Def. 32 the notions of rewrite context ($\mathtt{R}$, $\mathtt{R}'$, $\ldots$), applicative rewrite context and strongly applicative term context.

▶ **Definition 74.** *The relation of* restricted η-expansion *written* $\to_{\overline{\eta}}$*, is defined as follows. Let* $\mathtt{R}\langle\rho\rangle$ *be a rewrite such that* ρ *is of function type (*i.e. $A \to B$*),* $\mathtt{R}$ *is not applicative and* ρ *is not a* λ*-abstraction nor a composition ("*;*"). Then given a variable* $x \notin \mathsf{fv}(\rho)$*:*

$$\mathtt{R}\langle\rho\rangle \to_{\overline{\eta}} \mathtt{R}\langle\lambda x.\rho\, x\rangle$$

Observe that this notion is **not** *closed by arbitrary contexts.*

▶ Remark 75. If $\rho \to_{\overline{\eta}} \sigma$ then $\rho \approx \sigma$ using the $\approx$-Eta rule.

▶ **Proposition 76.** *Restricted* η*-expansion is SN and CR.*

Proof. **Strong normalization.** An $\overline{\eta}$*-redex occurrence* of a rewrite ρ is a pair $(\mathtt{R}, \rho')$ such that $\mathtt{R}\langle\rho'\rangle = \rho$ where ρ is of function type, $\mathtt{R}$ is not applicative and ρ' is not a λ-abstraction nor a composition. The *degree* of a redex occurrence is the size of the type $A \to B$. The *measure* of a rewrite ρ is the multiset of all the degrees of redex occurrences of ρ. To prove strong normalization, observe that $\overline{\eta}$-expansion decreases the measure of the rewrite. In fact, suppose that there is a step:

$$\mathtt{R}\langle\rho\rangle \to_{\overline{\eta}} \mathtt{R}\langle\lambda x.\rho\, x\rangle$$

where the type of ρ is $A \to B$. Consider a redex occurrence $(\mathtt{R}', \rho')$ of the right-hand side, *i.e.* $\mathtt{R}\langle\lambda x.\rho\, x\rangle = \mathtt{R}'\langle\rho'\rangle$ where ρ' is of function type, $\mathtt{R}'$ is not applicative, and ρ' is not a λ-abstraction nor a composition. We claim that either the degree of $(\mathtt{R}', \rho')$ is strictly less than the degree of $(\mathtt{R}, \rho)$ or, otherwise, that $(\mathtt{R}', \rho')$ can be mapped injectively to a redex occurrence $(\mathtt{R}'', \rho'')$ on the left-hand side. Note that $\mathtt{R} \neq \mathtt{R}'$ because $\lambda x.\rho\, x$ is a λ-abstraction so $(\mathtt{R}, \lambda x.\rho\, x)$ is not a redex occurrence. We consider three cases, depending on whether the contexts $\mathtt{R}$, $\mathtt{R}'$ are disjoint, $\mathtt{R}$ is a prefix of $\mathtt{R}'$, or $\mathtt{R}'$ is a prefix of $\mathtt{R}$:

1. If $\mathtt{R}$ and $\mathtt{R}'$ are disjoint, then there is a two-hole context $\hat{\mathtt{R}}$ such that $\mathtt{R} = \hat{\mathtt{R}}\langle\Box, \rho'\rangle$ and $\mathtt{R}' = \hat{\mathtt{R}}\langle(\lambda x.\rho\, x), \Box\rangle$. Then $(\mathtt{R}', \rho')$ can be mapped to the redex occurrence $(\hat{\mathtt{R}}\langle\rho, \Box\rangle, \rho')$ on the left-hand side.
2. If $\mathtt{R}$ is a prefix of $\mathtt{R}'$, then $\mathtt{R}' = \mathtt{R}\langle\mathtt{R}''\rangle$ and $\lambda x.\rho\, x = \mathtt{R}''\langle\rho'\rangle$. Note that $\mathtt{R}''$ is not empty because, as we have already argued, $\mathtt{R} \neq \mathtt{R}'$. We proceed by case analysis on the shape of $\mathtt{R}''$;
   - **2.1** If $\mathtt{R}'' = \lambda x.\Box$, then $(\mathtt{R}\langle\lambda x.\Box\rangle, \rho\, x)$ is a redex occurrence with degree strictly less than the degree of $(\mathtt{R}, \rho)$, given that the type of $\rho\, x$ is B.
   - **2.2** If $\mathtt{R}'' = \lambda x.\mathtt{R}'''\, x$, then $(\mathtt{R}\langle\lambda x.\mathtt{R}'''\, x\rangle, \rho')$ can be mapped to the redex occurrence $(\mathtt{R}\langle\mathtt{R}'''\rangle, \rho')$ on the left-hand side. Note that if $\mathtt{R}\langle\lambda x.\mathtt{R}'''\, x\rangle$ is not applicative, then $\mathtt{R}\langle\mathtt{R}'''\rangle$ is also not applicative.
   - **2.3** If $\mathtt{R}'' = \lambda x.\rho\,\Box$, then $(\mathtt{R}\langle\lambda x.\rho\,\Box\rangle, x)$ is a redex occurrence with degree strictly less than the degree of $(\mathtt{R}, \rho)$, given that the type of x is A.
3. If $\mathtt{R}'$ is a prefix of $\mathtt{R}$, then $\mathtt{R} = \mathtt{R}'\langle\mathtt{R}''\rangle$ and $\rho' = \mathtt{R}''\langle\lambda x.\rho\, x\rangle$. Then $(\mathtt{R}', \mathtt{R}''\langle\lambda x.\rho\, x\rangle)$ can be mapped to the redex occurrence $(\mathtt{R}', \mathtt{R}''\langle\rho\rangle)$ on the left-hand side. Note that if $\mathtt{R}''\langle\lambda x.\rho\, x\rangle$ is not a λ-abstraction nor a composition then $\mathtt{R}''$ is non-empty, and the outermost constructor is an application, hence $\mathtt{R}''\langle\rho\rangle$ is also not a λ-abstraction nor a composition.

The mapping thus defined is injective.

Confluence. By Newman's lemma, it suffices to show that restricted η-expansion is WCR. Indeed, suppose that $\rho \to_{\overline{\eta}} \rho_1$ and $\rho \to_{\overline{\eta}} \rho_2$, and let us show that there is a rewrite ρ_3 such that $\rho_1 \to_{\overline{\eta}}{}^* \rho_3$ and $\rho_2 \to_{\overline{\eta}}{}^* \rho_3$. More precisely, let:

$$\begin{array}{lll} \rho = \mathtt{R}_1\langle\rho'_1\rangle & \to_{\overline{\eta}} \quad \mathtt{R}_1\langle\lambda x.\rho'_1\,x\rangle = \rho_1 & \text{for } x \notin \mathsf{fv}(\rho'_1) \\ \rho = \mathtt{R}_2\langle\rho'_2\rangle & \to_{\overline{\eta}} \quad \mathtt{R}_2\langle\lambda y.\rho'_2\,y\rangle = \rho_2 & \text{for } y \notin \mathsf{fv}(\rho'_2) \end{array}$$

where the contexts $\mathtt{R}_1, \mathtt{R}_2$ are not applicative, and the rewrites ρ_1, ρ_2 are of function type and not abstractions nor compositions. If $\mathtt{R}_1 = \mathtt{R}_2$ then it is trivial to conclude in zero rewriting steps. Otherwise, there are three subcases, depending on whether the contexts $\mathtt{R}_1, \mathtt{R}_2$ are disjoint, or $\mathtt{R}_1$ is a prefix of $\mathtt{R}_2$, or $\mathtt{R}_2$ is a prefix of $\mathtt{R}_1$. The last two cases are symmetric so we only consider the first one:

1. If $\mathtt{R}_1$ and $\mathtt{R}_2$ are disjoint, then there is a two-hole context $\hat{\mathtt{R}}$ such that $\mathtt{R}_1 = \hat{\mathtt{R}}\langle\Box, \rho'_2\rangle$ and $\mathtt{R}_2 = \hat{\mathtt{R}}\langle\rho'_1, \Box\rangle$. Then the situation is:

$$\begin{array}{ccc} \hat{\mathtt{R}}\langle\rho'_1, \rho'_2\rangle & \longrightarrow & \hat{\mathtt{R}}\langle(\lambda x.\rho'_1\,x), \rho'_2\rangle \\ \downarrow & & \downarrow \\ \hat{\mathtt{R}}\langle\rho'_1, (\lambda y.\rho'_2\,y)\rangle & \longrightarrow & \hat{\mathtt{R}}\langle(\lambda x.\rho'_1\,x), (\lambda y.\rho'_2\,y)\rangle \end{array}$$

   To be able to close the diagram, note that $\hat{\mathtt{R}}\langle\Box, \rho'_2\rangle$ is applicative if and only if $\hat{\mathtt{R}}\langle\Box, (\lambda y.\rho'_2)\rangle$ is applicative. Similarly, $\hat{\mathtt{R}}\langle\rho'_1, \Box\rangle$ is applicative if and only if $\hat{\mathtt{R}}\langle(\lambda x.\rho'_1\,x), \Box\rangle$ is applicative.

2. If $\mathtt{R}_1$ is a prefix of $\mathtt{R}_2$, then $\mathtt{R}_2 = \mathtt{R}_1\langle\mathtt{R}'\rangle$ and $\rho'_1 = \mathtt{R}'\langle\rho'_2\rangle$. Then the situation is:

$$\begin{array}{ccc} \mathtt{R}_1\langle\mathtt{R}'\langle\rho'_2\rangle\rangle & \longrightarrow & \mathtt{R}_1\langle\lambda x.\mathtt{R}'\langle\rho'_2\rangle\,x\rangle \\ \downarrow & & \downarrow \\ \mathtt{R}_1\langle\mathtt{R}'\langle\lambda y.\rho'_2\,y\rangle\rangle & \longrightarrow & \mathtt{R}_1\langle\lambda x.\mathtt{R}'\langle\lambda y.\rho'_2\,y\rangle\,x\rangle \end{array}$$

   To justify the step on the right-hand side of the diagram, note that $\mathtt{R}'$ is not empty because we already know that $\mathtt{R}_1 \neq \mathtt{R}_2$. Moreover, the outermost constructor of $\mathtt{R}'$ cannot be a λ-abstraction nor a composition, because $\rho_1 = \mathtt{R}'\langle\rho'_2\rangle$ is not a λ-abstraction nor a composition. This means that $\mathtt{R}'$ must be either of the form $\mathtt{R}''\,\sigma$ or of the form $\sigma\,\mathtt{R}''$ and it is not applicative. Hence, in any of these two cases, the context $\mathtt{R}_1\langle\lambda x.\mathtt{R}'\,x\rangle$ is not applicative.

   To justify the step on the bottom of the diagram, note, again, that $\mathtt{R}'$ is non-empty and its outermost constructor is an application, hence $\mathtt{R}'\langle\lambda y.\rho'_2\,y\rangle$ is not a λ-abstraction nor a composition.

◀

C.1 $\overline{\eta}$-normal forms

▶ **Definition 77** ($\overline{\eta}$-normal form)**.** *We recall the standard notion of $\overline{\eta}$-normal form for terms: a typable term $\Gamma \vdash s : A$. is in* $\overline{\eta}$-normal form *if whenever s can be written as of the form $s = \mathtt{C}\langle s'\rangle$ such that s' is of function type (*i.e. *$A \to B$) then either $\mathtt{C}$ is strongly applicative or s' is a λ-abstraction.*

The notion is extended for rewrites as follows. A typable rewrite $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ is in $\overline{\eta}$-normal form *if whenever ρ can be written as of the form $\rho = \mathtt{R}\langle\rho'\rangle$ such that ρ' is of function type (*i.e. *$A \to B$) then either $\mathtt{R}$ is applicative or ρ' is a λ-abstraction or a composition.*

▶ Remark 78. A rewrite is in $\overline{\eta}$-normal form precisely if it is a normal form for $\to_{\overline{\eta}}$.

▶ Remark 79. A term s is in $\overline{\eta}$-normal form in the standard sense if and only if the corresponding rewrite $\underline{s}$ is in $\overline{\eta}$-normal form. This is a consequence of the two following observations:

1. A term context $\mathtt{C}$ is applicative if and only if it is strongly applicative, given that it has no compositions.
2. A term s has no compositions.

▶ **Definition 80** ($\overline{\eta}$-condition). *The set of rewriting rule symbols $\mathcal{R}$ is said to verify the* $\overline{\eta}$-condition *if for every $\varrho : s \twoheadrightarrow t : A \in \mathcal{R}$. the terms s and t are in $\overline{\eta}$-normal form.*

▶ **Lemma 81** (Endpoints of rewrites in $\overline{\eta}$-normal form). *Assume that the set of rewriting rule symbols $\mathcal{R}$ verifies the $\overline{\eta}$-condition. Let $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ be a rewrite in $\overline{\eta}$-normal form. Then $\underline{\rho^{\mathsf{src}}}$ and $\underline{\rho^{\mathsf{tgt}}}$ are in $\overline{\eta}$-normal form.*

Proof. Let us prove that the source $\underline{\rho^{\mathsf{src}}}$ is in $\overline{\eta}$-normal form; the proof for the target $\underline{\rho^{\mathsf{tgt}}}$ is similar. By contradiction, suppose that $\underline{\rho^{\mathsf{src}}}$ is not in $\overline{\eta}$-normal form. Note that ρ^{src} is of the form $\mathtt{C}\langle s\rangle$ where $\mathtt{C}$ is not applicative and s is not a λ-abstraction nor a composition. Since ρ^{src} is a term, without compositions ("**;**"), this means that s is not a λ-abstraction and that $\mathtt{C}$ is not strongly applicative. By Lem. 33 there are two possibilities:

(A) In this case, $\rho = \mathtt{R}\langle\alpha\rangle$ where $\mathtt{R}^{\mathsf{src}} = \mathtt{C}$ and $\alpha^{\mathsf{src}} = s$. Suppose, without loss of generality, that α is the smallest possible term that satisfies these equations. In particular, note that α cannot be a composition $(\alpha_1 \,;\, \alpha_2)$, because this would allow us to write $\rho = \mathtt{R}'\langle\alpha_1\rangle$ with $\mathtt{R}' := \mathtt{R}\langle\Box \,;\, \alpha_2\rangle$ and this also verifies the equations.
Then $\mathtt{R}$ is not applicative, as this would imply that $\mathtt{C}$ is applicative. Note that α is not a λ-abstraction, as this would imply that s is a λ-abstraction. Finally, α is not a composition, as we have already noted. This contradicts the fact that ρ is in $\overline{\eta}$-normal form.

(B) In this case, $\rho = \mathtt{R}\langle\varrho\rangle$ where $\mathtt{R}^{\mathsf{src}} = \mathtt{C}_1$ and $\varrho^{\mathsf{src}} = \mathtt{C}_2\langle s\rangle$ and $\mathtt{C} = \mathtt{C}_1\langle\mathtt{C}_2\rangle$. Note that $\mathtt{C}_2$ is not strongly applicative, as this would imply that $\mathtt{C}$ is strongly applicative. Moreover, as already noted before, s is not a λ-abstraction. This means that ϱ^{src} is not in $\overline{\eta}$-normal form, contradicting the hypothesis that the set of rewriting rule symbols $\mathcal{R}$ verifies the $\overline{\eta}$-condition.

◀

D Flattening

▶ **Lemma 82** (Typing rule for multistep substitution). *If $\Gamma, x : A \vdash \mu : s_0 \twoheadrightarrow s_1 : B$ and $\Gamma \vdash \nu : t_0 \twoheadrightarrow t_1 : A$ then $\Gamma \vdash \mu\{x\backslash\nu\} : s_0\{x\backslash t_0\} \twoheadrightarrow s_1\{x\backslash t_1\} : B$.*

Proof. Straightforward by induction on the derivation of the judgment. ◀

▶ **Lemma 83** (Substitution lemma for multisteps). *The following substitution property holds:*

$$\mu\{x\backslash\nu\}\{y\backslash\nu'\} = \mu\{y\backslash\nu'\}\{x\backslash\nu\{y\backslash\nu'\}\}$$

Note, in particular, that $\mu\{x\backslash\nu\}\{y\backslash s\} = \mu\{y\backslash s\}\{x\backslash\nu\{y\backslash s\}\}$.

Proof. Routine by induction on μ. ◀

D.1 Termination of flattening

▶ Remark 84. If $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$ and $\mathtt{x} \notin \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$ then $\rho^{\mathsf{src}} = \sigma^{\mathsf{src}}$ and $\rho^{\mathsf{tgt}} = \sigma^{\mathsf{tgt}}$.

▶ Remark 85. Let $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$ where $\mathtt{x} \neq \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$. Then neither ρ nor σ are multisteps. Indeed, note that, in all the rules other than $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM, there must be at least one composition ("**;**") on the left-hand side, and at least one composition on the right-hand side.

▶ **Lemma 86.** *Let μ, ν be arbitrary multisteps. Then:*

1. $\mu\{x\backslash\nu\}^{\mathsf{src}} = \mu^{\mathsf{src}}\{x\backslash\nu^{\mathsf{src}}\}$
2. $\mu\{x\backslash\nu\}^{\mathsf{tgt}} = \mu^{\mathsf{tgt}}\{x\backslash\nu^{\mathsf{tgt}}\}$

Proof. Straightforward by induction on μ. ◀

▶ **Lemma 87.** *If $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$ where $\mathtt{x} \in \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$, then $\underline{\rho^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^{=}_{\mathtt{x}} \underline{\sigma^{\mathsf{src}}}$ and $\underline{\rho^{\mathsf{tgt}}} \overset{\flat}{\mapsto}{}^{=}_{\mathtt{x}} \underline{\sigma^{\mathsf{tgt}}}$. Here $\overset{\flat}{\mapsto}{}^{=}_{\mathtt{x}}$ denotes the* reflexive *closure of $\overset{\flat}{\mapsto}_{\mathtt{x}}$.*

Proof. By induction on ρ:

1. **Variable $(\rho = x)$, constant $(\rho = \mathbf{c})$, or rule symbol $(\rho = \varrho)$.** There cannot be a step $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$ using the $\mathcal{F}$-BetaM or $\mathcal{F}$-EtaM rules, so this case trivially holds.
2. **Abstraction $(\rho = \lambda x.\rho')$.** There are two subcases, depending on whether the step takes place at the root or under the abstraction:
   2.1 Reduction at the root: then ρ' is a multistep of the form $\mu\, x$ where $x \notin \mathsf{fv}(\mu)$, and the step is of the form $\lambda x.\mu\, x \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}} \mu$. Then for the source we have that $\lambda x.\mu^{\mathsf{src}}\, x \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}} \mu^{\mathsf{src}}$, observing that x cannot occur free in μ^{src}, and similarly for the target.
   2.2 Under the abstraction: then the step is of the form $\lambda x.\rho' \overset{\flat}{\mapsto}_{\mathtt{x}} \lambda x.\sigma'$ with $\rho' \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma'$. By IH we have that $\underline{\rho'^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^{=}_{\mathtt{x}} \underline{\sigma'^{\mathsf{src}}}$ so also $\underline{\lambda x.\rho'^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^{=}_{\mathtt{x}} \underline{\lambda x.\sigma'^{\mathsf{src}}}$, and similarly for the target.
3. **Application $(\rho = \rho_1\, \rho_2)$.** There are three subcases, depending on whether the step takes place at the root, to the left, or to the right of the application:
   3.1 Reduction at the root: then ρ_1 and ρ_2 must be multisteps of the forms $\rho_1 = \lambda x.\mu$ and $\rho_2 = \nu$, and the step is of the form $(\lambda x.\mu)\, \nu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM}} \mu\{x\backslash\nu\}$. Then for the source we have that $(\lambda x.\mu^{\mathsf{src}})\, \nu^{\mathsf{src}} \overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM}} \mu^{\mathsf{src}}\{x\backslash\nu^{\mathsf{src}}\} = \mu\{x\backslash\nu\}^{\mathsf{src}}$ by Lem. 86, and similarly for the target.
   3.2 Left of the application: then the step is of the form $\rho_1\, \rho_2 \overset{\flat}{\mapsto}_{\mathtt{x}} \rho_1'\, \rho_2$ with $\rho_1 \overset{\flat}{\mapsto}_{\mathtt{x}} \rho_1'$. By IH we have that $\underline{\rho_1^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^{=}_{\mathtt{x}} \underline{\rho_1'^{\mathsf{src}}}$ so also $\underline{\rho_1^{\mathsf{src}}}\, \underline{\rho_2^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^{=}_{\mathtt{x}} \underline{\rho_1'^{\mathsf{src}}}\, \underline{\rho_2^{\mathsf{src}}}$, and similarly for the target.
   3.3 Right of the application: symmetric to the previous case.
4. **Composition, $(\rho = \rho_1 \,;\, \rho_2)$.** There are two subcases, depending on whether the step takes place to the left or to the right of the composition:
   4.1 Left of the composition: then the step is of the form $\rho_1 \,;\, \rho_2 \overset{\flat}{\mapsto}_{\mathtt{x}} \rho_1' \,;\, \rho_2$ with $\rho_1 \overset{\flat}{\mapsto}_{\mathtt{x}} \rho_1'$. For the source, note that by IH we have that $\underline{\rho_1^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^{=}_{\mathtt{x}} \underline{\rho_1'^{\mathsf{src}}}$ so indeed $\underline{(\rho_1 \,;\, \rho_2)^{\mathsf{src}}} = \underline{\rho_1^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^{=}_{\mathtt{x}} \underline{\rho_1'^{\mathsf{src}}} = \underline{(\rho_1' \,;\, \rho_2)^{\mathsf{src}}}$. For the target, simply note that $\underline{(\rho_1 \,;\, \rho_2)^{\mathsf{tgt}}} = \underline{\rho_2^{\mathsf{tgt}}} = \underline{(\rho_1' \,;\, \rho_2)^{\mathsf{tgt}}}$, so we conclude in zero reduction steps.
   4.2 Right of the composition: symmetric to the previous case.

◀

▶ **Lemma 88** ($\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM postponement). *Let* $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma \overset{\flat}{\mapsto}_{\mathtt{y}} \tau$ *where* $\mathtt{x} \in \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$ *and* $\mathtt{y} \notin \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$*. Then there is a rewrite* υ *such that* $\rho \overset{\flat}{\mapsto}_{\mathtt{y}} \upsilon \overset{\flat}{\mapsto}{}^{+}_{\mathtt{x}} \tau$*.*

Proof. By induction on the context under which the step $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$ takes place. We analyze the cases for reduction at the root as well as for congruence closure:

1. $\mathcal{F}$-**BetaM reduction at the root.** Let $(\lambda x.\mu)\,\nu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM}} \mu\{x\backslash\nu\} \overset{\flat}{\mapsto}_{\mathtt{y}} \tau$. It suffices to note that the left-hand side of $\mathtt{y}$ contains at least one composition (Rem. 85), given that $\mathtt{y} \notin \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$. Hence the step $\mu\{x\backslash\nu\} \overset{\flat}{\mapsto}_{\mathtt{y}} \tau$ is impossible.
2. $\mathcal{F}$-**EtaM reduction at the root.** Similar to the previous case.
3. **Under a λ-abstraction.** Let $\lambda x.\rho' \overset{\flat}{\mapsto}_{\mathtt{x}} \lambda x.\sigma' \overset{\flat}{\mapsto}_{\mathtt{y}} \tau$ where $\rho' \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma'$.
If the second step is internal to σ', then the conclusion follows easily by IH. More precisely, suppose that $\tau = \lambda x.\tau'$ with $\sigma' \overset{\flat}{\mapsto}_{\mathtt{y}} \tau'$. By IH there exists a rewrite υ' such that $\rho' \overset{\flat}{\mapsto}_{\mathtt{y}} \upsilon' \overset{\flat}{\mapsto}{}^{+}_{\mathtt{x}} \tau'$. Taking $\upsilon := \lambda x.\upsilon'$ we have that $\lambda x.\rho' \overset{\flat}{\mapsto}_{\mathtt{y}} \lambda x.\upsilon' \overset{\flat}{\mapsto}{}^{+}_{\mathtt{x}} \lambda x.\tau'$ as required.
If the second step is at the root, then the only rule that can apply is $\mathcal{F}$-Abs, so $\sigma' = (\sigma_1 \,;\, \sigma_2)$ and $\tau = (\lambda x.\sigma_1) \,;\, (\lambda x.\sigma_2)$, that is, the situation is:
$$\lambda x.\rho' \overset{\flat}{\mapsto}_{\mathtt{x}} \lambda x.(\sigma_1 \,;\, \sigma_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} (\lambda x.\sigma_1) \,;\, (\lambda x.\sigma_2)$$
Note that the right-hand side of the first step is a multistep which does not contain any composition (";"), because it is either a $\mathcal{F}$-BetaM or a $\mathcal{F}$-EtaM step. Hence ρ' must be of the form $(\rho_1 \,;\, \rho_2)$ where the first step is either internal to ρ_1 or internal to ρ_2. So there are two subcases:
   - **3.1** First step internal to ρ_1: then $\lambda x.(\rho_1 \,;\, \sigma_2) \overset{\flat}{\mapsto}_{\mathtt{x}} \lambda x.(\sigma_1 \,;\, \sigma_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} (\lambda x.\sigma_1) \,;\, (\lambda x.\sigma_2)$ where $\rho_1 \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma_1$. Taking $\upsilon := \lambda x.(\rho_1 \,;\, \sigma_2)$ we have that $\lambda x.(\rho_1 \,;\, \sigma_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} (\lambda x.\rho_1) \,;\, (\lambda x.\sigma_2) \overset{\flat}{\mapsto}_{\mathtt{x}} (\lambda x.\sigma_1) \,;\, (\lambda x.\sigma_2)$ as required.
   - **3.2** First step internal to ρ_2: symmetric to the previous case.
4. **Left of an application.** Let $\rho'\,\alpha \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma'\,\alpha \overset{\flat}{\mapsto}_{\mathtt{y}} \tau$ where $\rho' \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma'$.
If the second step is internal to σ', then the conclusion follows easily by IH. More precisely, suppose that $\tau = \tau'\,\alpha$ with $\sigma' \overset{\flat}{\mapsto}_{\mathtt{y}} \tau'$. Then by IH there exists a rewrite υ' such that $\rho' \overset{\flat}{\mapsto}_{\mathtt{y}} \upsilon' \overset{\flat}{\mapsto}{}^{+}_{\mathtt{x}} \tau'$. Taking $\upsilon := \upsilon'\,\alpha$ we have that $\rho'\,\alpha \overset{\flat}{\mapsto}_{\mathtt{y}} \upsilon'\,\alpha \overset{\flat}{\mapsto}{}^{+}_{\mathtt{x}} \tau'\,\alpha$ as required.
If the second step is internal to α, then the conclusion follows easily given that the steps are disjoint. More precisely, suppose that $\tau = \sigma'\,\beta$ with $\alpha \overset{\flat}{\mapsto}_{\mathtt{y}} \beta$. Taking $\upsilon := \rho'\,\beta$ we have that $\rho'\,\alpha \overset{\flat}{\mapsto}_{\mathtt{y}} \rho'\,\beta \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma'\,\beta$ as required.
If the second step is at the root, there are only three rules that can apply ($\mathcal{F}$-App1, $\mathcal{F}$-App2, and $\mathcal{F}$-App3), so we consider three subcases:
   - **4.1** $\mathcal{F}$-App1: then $\sigma' = (\sigma_1 \,;\, \sigma_2)$, and $\alpha = \mu$ is a multistep, and moreover $\tau = (\sigma_1\,\underline{\mu^{\mathsf{src}}}) \,;\, (\sigma_2\,\mu)$. The situation is:
$$\rho'\,\mu \overset{\flat}{\mapsto}_{\mathtt{x}} (\sigma_1 \,;\, \sigma_2)\,\mu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} (\sigma_1\,\underline{\mu^{\mathsf{src}}}) \,;\, (\sigma_2\,\mu)$$
where $\rho' \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma_1 \,;\, \sigma_2$. Note that the right-hand side of the first step must be a multistep which does not contain any composition (";"), given that $\mathtt{x} \in \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$. Hence ρ' must be of the form $(\rho_1 \,;\, \rho_2)$ where the first step is either internal to ρ_1 or internal to ρ_2. So there are two subcases:

4.1.1 First step internal to ρ_1: then $(\rho_1, \sigma_2)\,\mu \overset{\flat}{\mapsto}_{\mathtt{x}} (\sigma_1 \,;\, \sigma_2)\,\mu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} (\sigma_1\,\underline{\mu^{\mathsf{src}}}) \,;\, (\sigma_2\,\mu)$ where $\rho_1 \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma_1$. Taking $\upsilon := (\rho_1\,\underline{\mu^{\mathsf{src}}}) \,;\, (\sigma_2\,\mu)$ we have that: $(\rho_1, \sigma_2)\,\mu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} (\rho_1\,\underline{\mu^{\mathsf{src}}}) \,;\, (\sigma_2\,\mu) \overset{\flat}{\mapsto}_{\mathtt{x}} (\sigma_1\,\underline{\mu^{\mathsf{src}}}) \,;\, (\sigma_2\,\mu)$ as required.

4.1.2 First step internal to ρ_2: symmetric to the previous case.

4.2 $\mathcal{F}$-App2: then $\sigma' = \mu$ is a multistep, and $\alpha = (\alpha_1 \,;\, \alpha_2)$, and moreover $\tau = (\mu\,\alpha_1) \,;\, (\underline{\mu^{\mathsf{tgt}}}\,\alpha_2)$. The situation is:

$$\rho'\,(\alpha_1 \,;\, \alpha_2) \overset{\flat}{\mapsto}_{\mathtt{x}} \mu\,(\alpha_1 \,;\, \alpha_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App2}} (\mu\,\alpha_1) \,;\, (\underline{\mu^{\mathsf{tgt}}}\,\alpha_2)$$

where $\rho' \overset{\flat}{\mapsto}_{\mathtt{x}} \mu$. Note that the first step step is either a $\mathcal{F}$-BetaM or an $\mathcal{F}$-EtaM step, so it cannot erase compositions ("$;$"). Therefore ρ' does not contain any composition, *i.e.* $\rho' = \mu_0$ is also a multistep. Hence taking $\upsilon := (\mu_0\,\alpha_1) \,;\, (\underline{\mu_0^{\mathsf{tgt}}}\,\alpha_2)$ we have that

$$\begin{array}{lll} \mu_0\,(\alpha_1 \,;\, \alpha_2) & \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App2}} & (\mu_0\,\alpha_1) \,;\, (\underline{\mu_0^{\mathsf{tgt}}}\,\alpha_2) \\ & \overset{\flat}{\mapsto}_{\mathtt{x}} & (\mu\,\alpha_1) \,;\, (\underline{\mu_0^{\mathsf{tgt}}}\,\alpha_2) \\ & \overset{\flat}{\mapsto}^{=}_{\mathtt{x}} & (\mu\,\alpha_1) \,;\, (\underline{\mu^{\mathsf{tgt}}}\,\alpha_2) \quad \text{by Lem. 87} \end{array}$$

4.3 $\mathcal{F}$-App3: then $\sigma' = (\sigma_1 \,;\, \sigma_2)$ and $\alpha = (\alpha_1 \,;\, \alpha_2)$, and moreover $\tau = ((\sigma_1 \,;\, \sigma_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2))$. The situation is:

$$\rho'\,(\alpha_1 \,;\, \alpha_2) \overset{\flat}{\mapsto}_{\mathtt{x}} (\sigma_1 \,;\, \sigma_2)\,(\alpha_1 \,;\, \alpha_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App3}} ((\sigma_1 \,;\, \sigma_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2))$$

Note that the right-hand side of the first step is a multistep which does not contain any composition ("$;$"), given that $\mathtt{x} \in \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$. Hence ρ' must be of the form $(\rho_1 \,;\, \rho_2)$ where the first step is either internal to ρ_1 or internal to ρ_2. So there are two subcases:

4.3.1 First step internal to ρ_1: then $(\rho_1 \,;\, \sigma_2)\,(\alpha_1 \,;\, \alpha_2) \overset{\flat}{\mapsto}_{\mathtt{x}} (\sigma_1 \,;\, \sigma_2)\,(\alpha_1 \,;\, \alpha_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App3}} ((\sigma_1 \,;\, \sigma_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2))$ where $\rho_1 \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma_1$. Taking $\upsilon := ((\rho_1 \,;\, \sigma_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2))$ we have that:

$$\begin{array}{lll} (\rho_1 \,;\, \sigma_2)\,(\alpha_1 \,;\, \alpha_2) & \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App3}} & ((\rho_1 \,;\, \sigma_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2)) \\ & \overset{\flat}{\mapsto}_{\mathtt{x}} & ((\sigma_1 \,;\, \sigma_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2)) \end{array}$$

4.3.2 First step internal to ρ_2: then $(\sigma_1 \,;\, \rho_2)\,(\alpha_1 \,;\, \alpha_2) \overset{\flat}{\mapsto}_{\mathtt{x}} (\sigma_1 \,;\, \sigma_2)\,(\alpha_1 \,;\, \alpha_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App3}} ((\sigma_1 \,;\, \sigma_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2))$ where $\rho_2 \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma_2$. Taking $\upsilon := ((\sigma_1 \,;\, \rho_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\rho_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2))$ we have that:

$$\begin{array}{lll} (\sigma_1 \,;\, \rho_2)\,(\alpha_1 \,;\, \alpha_2) & \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App3}} & ((\sigma_1 \,;\, \rho_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\rho_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2)) \\ & \overset{\flat}{\mapsto}_{\mathtt{x}} & ((\sigma_1 \,;\, \sigma_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\rho_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2)) \\ & \overset{\flat}{\mapsto}^{*}_{\mathtt{x}} & ((\sigma_1 \,;\, \sigma_2)\,\underline{\alpha_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\alpha_1 \,;\, \alpha_2)) \quad \text{by Lem. 87} \end{array}$$

5. Right of an application. Symmetric to the previous case.

6. Left of a composition. Let $\rho' \,;\, \alpha \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma' \,;\, \alpha \overset{\flat}{\mapsto}_{\mathtt{y}} \tau$ where $\rho' \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma'$. Note that there is no rewriting rule whose left-hand side is a composition ("$;$"). Hence the second step must necessarily be internal to σ' or to α.

If the second step is internal to σ', *i.e.* $\tau = \tau' \,;\, \alpha$ with $\sigma' \overset{\flat}{\mapsto}_{\mathtt{y}} \tau'$, then the conclusion follows easily by IH (similarly as in the *Left of an application* case).

If the second step is internal to α, *i.e.* $\tau = \sigma' \,;\, \beta$ with $\alpha \overset{\flat}{\mapsto}_{\mathtt{y}} \beta$, then the conclusion follows easily given that the steps are disjoint (similarly as in the *Left of an application* case).

7. **Right of a composition.** Symmetric to the previous case. ◀

▶ **Definition 89** (Heavy applications). *An application $\rho\,\sigma$ is* heavy *if ρ and σ are not multisteps,* i.e. *if both ρ and σ contain compositions ("$;$"). We write $\#_{\mathsf{h}}(\rho)$ to stand for the number of heavy applications in ρ. More precisely:*

$$\begin{aligned}
\#_{\mathsf{h}}(x) = \#_{\mathsf{h}}(\mathbf{c}) = \#_{\mathsf{h}}(\varrho) &\overset{\text{def}}{=} 0\\
\#_{\mathsf{h}}(\lambda x.\rho) &\overset{\text{def}}{=} \#_{\mathsf{h}}(\rho)\\
\#_{\mathsf{h}}(\rho\,\sigma) &\overset{\text{def}}{=} \#_{\mathsf{h}}(\rho) + \#_{\mathsf{h}}(\sigma) + \begin{cases} 1 & \text{if } \rho\,\sigma \text{ is heavy}\\ 0 & \text{otherwise}\end{cases}\\
\#_{\mathsf{h}}(\rho \,;\, \sigma) &\overset{\text{def}}{=} \#_{\mathsf{h}}(\rho) + \#_{\mathsf{h}}(\sigma)
\end{aligned}$$

▶ Remark 90. Multisteps and (lifted) terms have no heavy applications, *i.e.* $\#_{\mathsf{h}}(\mu) = 0$ and $\#_{\mathsf{h}}(\underline{s}) = 0$.

▶ **Lemma 91** (Decrease of heavy applications). *Let $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$ where $\mathtt{x} \notin \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$. Then $\#_{\mathsf{h}}(\rho) \geq \#_{\mathsf{h}}(\sigma)$. Furthermore if $\mathtt{x} = \mathcal{F}\text{-App3}$ then $\#_{\mathsf{h}}(\rho) > \#_{\mathsf{h}}(\sigma)$.*

Proof. By induction on the context under which the step $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$ takes place. We consider all the cases for reduction at the root as well as for congruence closure:

1. **Root $\mathcal{F}$-Abs step.** Let $\lambda x.(\rho \,;\, \sigma) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} (\lambda x.\rho) \,;\, (\lambda x.\sigma)$. Then $\#_{\mathsf{h}}(\lambda x.(\rho \,;\, \sigma)) = \#_{\mathsf{h}}(\rho) + \#_{\mathsf{h}}(\sigma) = \#_{\mathsf{h}}((\lambda x.\rho) \,;\, (\lambda x.\sigma))$.
2. **Root $\mathcal{F}$-App1 step.** Let $(\rho \,;\, \sigma)\,\mu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} (\rho\,\underline{\mu^{\mathsf{src}}}) \,;\, (\sigma\,\mu)$. Note that all the explicitly written applications, *i.e.* $(\rho \,;\, \sigma)\,\mu$, and $\rho\,\underline{\mu^{\mathsf{src}}}$, and $\sigma\,\mu$ are not heavy. Hence, using the fact that multisteps and terms have no heavy applications (Rem. 90) we have that $\#_{\mathsf{h}}((\rho \,;\, \sigma)\,\mu) = \#_{\mathsf{h}}(\rho) + \#_{\mathsf{h}}(\sigma) = \#_{\mathsf{h}}((\rho\,\underline{\mu^{\mathsf{src}}}) \,;\, (\sigma\,\mu))$.
3. **Root $\mathcal{F}$-App2 step.** Symmetric to the previous case.
4. **Root $\mathcal{F}$-App3 step.** Let $(\rho_1 \,;\, \rho_2)\,(\sigma_1 \,;\, \sigma_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App3}} ((\rho_1 \,;\, \rho_2)\,\underline{\sigma_1^{\mathsf{src}}}) \,;\, (\underline{\rho_2^{\mathsf{tgt}}}\,(\sigma_1 \,;\, \sigma_2))$. Note that the explicitly written application on the left-hand side, *i.e.* $(\rho_1 \,;\, \rho_2)\,(\sigma_1 \,;\, \sigma_2)$ is heavy, whereas the explicitly written applications on the right-hand side, *i.e.* $(\rho_1 \,;\, \rho_2)\,\underline{\sigma_1^{\mathsf{src}}}$ and $\underline{\rho_2^{\mathsf{tgt}}}\,(\sigma_1 \,;\, \sigma_2)$, are not heavy. Hence $\#_{\mathsf{h}}((\rho_1 \,;\, \rho_2)\,(\sigma_1 \,;\, \sigma_2)) = 1 + \#_{\mathsf{h}}(\rho_1) + \#_{\mathsf{h}}(\rho_2) + \#_{\mathsf{h}}(\sigma_1) + \#_{\mathsf{h}}(\sigma_2) > \#_{\mathsf{h}}(\rho_1) + \#_{\mathsf{h}}(\rho_2) + \#_{\mathsf{h}}(\sigma_1) + \#_{\mathsf{h}}(\sigma_2) = \#_{\mathsf{h}}(((\rho_1 \,;\, \rho_2)\,\underline{\sigma_1^{\mathsf{src}}}) \,;\, (\underline{\rho_2^{\mathsf{tgt}}}\,(\sigma_1 \,;\, \sigma_2)))$.
5. **Congruence, under an abstraction.** let $\lambda x.\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \lambda x.\sigma$ with $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$. Note that $\#_{\mathsf{h}}(\lambda x.\rho) = \#_{\mathsf{h}}(\rho)$ and $\#_{\mathsf{h}}(\lambda x.\sigma) = \#_{\mathsf{h}}(\sigma)$, so it is immediate to conclude by resorting to the IH.
6. **Congruence, left of an application.** let $\rho\,\alpha \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma\,\alpha$ with $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$. Recall that in a reduction step (other than $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM) the left and the right-hand sides are not multisteps (Rem. 85). This implies that ρ and σ are not multisteps. This means that the application $\rho\,\alpha$ is heavy if and only if the application $\sigma\,\alpha$ is heavy. Let $k := 1$ if $\rho\,\alpha$ is heavy, and $k := 0$ otherwise. We have that $\#_{\mathsf{h}}(\rho\,\alpha) = k + \#_{\mathsf{h}}(\rho) + \#_{\mathsf{h}}(\alpha)$ and $\#_{\mathsf{h}}(\sigma\,\alpha) = k + \#_{\mathsf{h}}(\sigma) + \#_{\mathsf{h}}(\alpha)$. Hence it is immediate to conclude by resorting to the IH.
7. **Congruence, right of an application.** Symmetric to the previous case.

8. **Congruence, left of a composition.** Let $\rho\,;\alpha \overset{\flat}{\mapsto}_{\mathsf{x}} \sigma\,;\alpha$ with $\rho \overset{\flat}{\mapsto}_{\mathsf{x}} \sigma$. Then $\#_{\mathsf{h}}(\rho\,;\alpha) = \#_{\mathsf{h}}(\rho)$ and $\#_{\mathsf{h}}(\sigma\,;\alpha) = \#_{\mathsf{h}}(\sigma)$, so it is immediate to conclude by resorting to the IH.
9. **Congruence, right of a composition.** Symmetric to the previous case.

◀

▶ **Definition 92.** *The* weight *of a rewrite ρ is a non-negative integer $\#_{\mathsf{w}}(\rho)$ defined inductively as follows:*

$$\begin{array}{rcl} \#_{\mathsf{w}}(x) = \#_{\mathsf{w}}(\mathbf{c}) = \#_{\mathsf{w}}(\varrho) & \overset{\text{def}}{=} & 0 \\ \#_{\mathsf{w}}(\lambda x.\rho) & \overset{\text{def}}{=} & 2\,\#_{\mathsf{w}}(\rho) \\ \#_{\mathsf{w}}(\rho\,\sigma) & \overset{\text{def}}{=} & 2\,\#_{\mathsf{w}}(\rho) + 2\,\#_{\mathsf{w}}(\sigma) \\ \#_{\mathsf{w}}(\rho\,;\sigma) & \overset{\text{def}}{=} & 1 + \#_{\mathsf{w}}(\rho) + \#_{\mathsf{w}}(\sigma) \end{array}$$

▶ Remark 93. Multisteps and (lifted) terms have zero weight, *i.e.* $\#_{\mathsf{w}}(\mu) = 0$ and $\#_{\mathsf{w}}(\underline{s}) = 0$.

▶ **Lemma 94** (Decrease of weight)**.** *Let $\rho \overset{\flat}{\mapsto}_{\mathsf{x}} \sigma$ where $\mathsf{x} \in \{\mathcal{F}\text{-Abs}, \mathcal{F}\text{-App1}, \mathcal{F}\text{-App2}\}$. Then $\#_{\mathsf{w}}(\rho) > \#_{\mathsf{w}}(\sigma)$.*

Proof. By induction on the context under which the step $\rho \overset{\flat}{\mapsto}_{\mathsf{x}} \sigma$ takes place. We consider all the cases for reduction at the root as well as for congruence closure:

1. **Root $\mathcal{F}$-Abs step.** Let $\lambda x.(\rho\,;\sigma) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} (\lambda x.\rho)\,;(\lambda x.\sigma)$. Then:

$$\begin{array}{rcl} \#_{\mathsf{w}}(\lambda x.(\rho\,;\sigma)) & = & 2\,(1 + \#_{\mathsf{w}}(\rho) + \#_{\mathsf{w}}(\sigma)) \\ & > & 1 + 2\,\#_{\mathsf{w}}(\rho) + 2\,\#_{\mathsf{w}}(\sigma) \\ & = & \#_{\mathsf{w}}((\lambda x.\rho)\,;(\lambda x.\sigma)) \end{array}$$

2. **Root $\mathcal{F}$-App1 step.** Let $(\rho\,;\sigma)\,\mu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} (\rho\,\underline{\mu^{\mathsf{src}}})\,;(\sigma\,\mu)$. Then:

$$\begin{array}{rcll} \#_{\mathsf{w}}((\rho\,;\sigma)\,\mu) & = & 2\,(1 + \#_{\mathsf{w}}(\rho) + \#_{\mathsf{w}}(\sigma)) + 2\,\#_{\mathsf{w}}(\mu) & \\ & > & 1 + 2\,\#_{\mathsf{w}}(\rho) + 2\,\#_{\mathsf{w}}(\sigma) + 2\,\#_{\mathsf{w}}(\mu) & \\ & = & 1 + 2\,\#_{\mathsf{w}}(\rho) + 2\,\#_{\mathsf{w}}(\underline{\mu^{\mathsf{src}}}) + 2\,\#_{\mathsf{w}}(\sigma) + 2\,\#_{\mathsf{w}}(\mu) & \text{by Rem. 93} \\ & = & \#_{\mathsf{w}}((\rho\,\underline{\mu^{\mathsf{src}}})\,;(\sigma\,\mu)) & \end{array}$$

3. **Root $\mathcal{F}$-App2 step.** Symmetric to the previous case.
4. **Congruence closure.** Congruence under abstraction, application and composition are straightforward given that the functions $\#_{\mathsf{w}}(\lambda x.-)$, $\#_{\mathsf{w}}(--)$, and $\#_{\mathsf{w}}(-\,;-)$ are monotonic.

◀

▶ **Proposition 95.** *The flattening system $\mathcal{F}$ is strongly normalizing.*

Proof. Recall that $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM steps can be postponed after steps of other kinds (Lem. 88). Hence, by standard rewriting techniques, SN of $\mathcal{F}$ can be reduced to SN of $\overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM}} \cup \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}}$ on one hand, plus SN of $\overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} \cup \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} \cup \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App2}} \cup \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App3}}$ on the other one.

It is immediate to show that the union of $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM is SN, given that (typable) multisteps can be understood as simply typed λ-terms, by regarding constants $(\mathbf{c}, \mathbf{d}, \ldots)$ and rule symbols $(\varrho, \vartheta, \ldots)$ as free variables of their corresponding types. Hence termination of $\overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM}} \cup \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}}$ is reduced to termination of $\beta\eta$-reduction in the simply-typed λ-calculus.

To show that the system without $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM is SN, consider the measure on rewrites given by $\#(\rho) \stackrel{\text{def}}{=} (\#_{\mathsf{h}}(\rho), \#_{\mathsf{w}}(\rho))$ with the lexicographic order. It is then easy to show that if $\rho \overset{\flat}{\mapsto}_{\mathtt{x}} \sigma$ with $\mathtt{x} \notin \{\mathcal{F}\text{-BetaM}, \mathcal{F}\text{-EtaM}\}$ then $\#(\rho) > \#(\sigma)$. Indeed, by Lem. 91 we know that $\mathcal{F}$-App3 steps strictly decrease the first component and other kinds of steps do not increase it. Moreover, by Lem. 94, we know that $\mathcal{F}$-Abs, $\mathcal{F}$-App1, and $\mathcal{F}$-App2 steps strictly decrease the second component. ◀

D.2 Confluence of flattening

▶ **Proposition 96.** *The flattening system $\mathcal{F}$ is confluent.*

Proof. By Newman's lemma, given that $\mathcal{F}$ is SN (Prop. 95), it suffices to show that it is WCR. Indeed, let $\rho \overset{\flat}{\mapsto} \rho_1$ and $\rho \overset{\flat}{\mapsto} \rho_2$ and let us show that there exists a rewrite ρ_3 such that $\rho \overset{\flat}{\mapsto}^* \rho_3$ and $\rho_1 \overset{\flat}{\mapsto}^* \rho_3$. We proceed by induction on ρ.

1. **Variable ($\rho = x$), constant ($\rho = \mathbf{c}$), or rule symbol ($\rho = \varrho$).** A rewrite of any of these shapes does not reduce, so the statements holds vacuously.
2. **Abstraction ($\rho = \lambda x.\rho'$).** If both steps are internal to ρ', it is immediate to conclude by resorting to the IH. If both steps are at the root, note that the rules $\mathcal{F}$-Abs and $\mathcal{F}$-EtaM are mutually exclusive, given that $\mathcal{F}$-Abs requires that the left-hand side has a composition, whereas $\mathcal{F}$-EtaM requires that it be a multistep. So if both steps are at the root, they must be instances of the same rule, and actually the same instance, so this case is trivial.

   The remaining case is when one of the steps is at the root and the other one is internal. We proceed by case analysis, depending on the kind of step that is performed at the root:

   2.1 $\mathcal{F}$-Abs: Then the step at the root is of the form $\lambda x.(\rho_1 \mathbin{;} \rho_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} (\lambda x.\rho_1) \mathbin{;} (\lambda x.\rho_2)$. Note that the internal step cannot be at the root of $\rho_1 \mathbin{;} \rho_2$ given that there are no rewriting rules whose left-hand side is a composition (";"). Hence the internal step must be either internal to ρ_1 or internal to ρ_2, so there are two subcases:

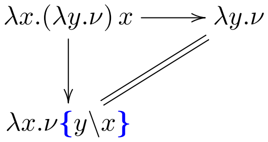
   2.1.1 If the internal step is of the form $\lambda x.(\rho_1 \mathbin{;} \rho_2) \overset{\flat}{\mapsto} \lambda x.(\rho_1' \mathbin{;} \rho_2)$ with $\rho_1 \overset{\flat}{\mapsto} \rho_1'$, the situation is:

$$\begin{array}{ccc} \lambda x.(\rho_1 \mathbin{;} \rho_2) & \longrightarrow & (\lambda x.\rho_1) \mathbin{;} (\lambda x.\rho_2) \\ \downarrow & & \downarrow \\ \lambda x.(\rho_1' \mathbin{;} \rho_2) & \longrightarrow & (\lambda x.\rho_1') \mathbin{;} (\lambda x.\rho_2) \end{array}$$

   2.1.2 If the internal step is of the form $\lambda x.(\rho_1 \mathbin{;} \rho_2) \overset{\flat}{\mapsto} \lambda x.(\rho_1 \mathbin{;} \rho_2')$ with $\rho_2 \overset{\flat}{\mapsto} \rho_2'$, the proof is similar as for the previous case.

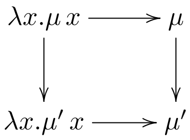
   2.2 $\mathcal{F}$-EtaM: Then the step at the root is of the form $\lambda x.\mu\, x \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}} \mu$ with $x \notin \mathsf{fv}(\mu)$. There are two subcases, depending on whether the internal step is at the root of $\mu\, x$ or internal to μ:

   2.2.1 If the internal step is at the root of $\mu\, x$ then it can only be an instance of the $\mathcal{F}$-BetaM rule, given that the remaining rewriting rules whose left-hand side is an application require that it contains at least one composition (";"), while $\mu\, x$ is a multistep. Hence $\mu = \lambda y.\nu$ and the situation is:



Note that the two rewrites are α-equivalent because $x \notin \mathsf{fv}(\nu)$.

2.2.2 If the internal step is internal to μ, then the situation is:

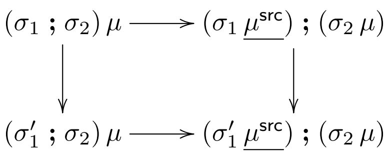


3. **Application ($\rho = \rho_1\,\rho_2$).** If both steps are internal to ρ_1, it is immediate to conclude by resorting to the IH. Similarly, if both steps are internal to ρ_2, it is immediate to conclude by IH. If one step is internal to ρ_1 and the other one is internal to ρ_2, it is also immediate to conclude given that the steps are disjoint. If both steps are at the root, note that the rules $\mathcal{F}$-App1, $\mathcal{F}$-App2, $\mathcal{F}$-App3, $\mathcal{F}$-BetaM are all mutually exclusive; for example there cannot simultaneously be a $\mathcal{F}$-App1 step and a $\mathcal{F}$-BetaM step at the root, given that $\mathcal{F}$-App1 requires that the rewrite on the left is not a multistep whereas $\mathcal{F}$-BetaM requires that it be a multistep. So if both steps are at the root they must be instances of the same rule, and actually the same instance, so this case is trivial.

The remaining case is when one of the steps it at the root and the other one is internal. We proceed by case analysis, depending on the kind of step that is performed at the root:

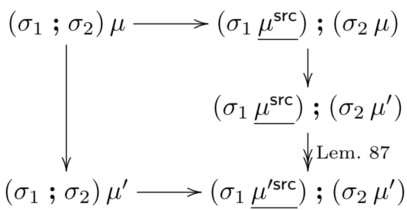
3.1 $\mathcal{F}$-App1: Then $\rho_1 = \sigma_1 \mathbin{;} \sigma_2$ and $\rho_2 = \mu$ is a multistep, and the step at the root is of the form $(\sigma_1 \mathbin{;} \sigma_2)\,\mu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} (\sigma_1\,\underline{\mu^{\mathsf{src}}}) \mathbin{;} (\sigma_2\,\mu)$. Note that the internal step must cannot be at the root of $\sigma_1 \mathbin{;} \sigma_2$ given that there are no rewriting rules whose left-hand side is a composition ("$;$"). Hence there are three subcases, depending on whether the internal step is internal to σ_1, internal to σ_2, or internal to μ:

3.1.1 If the internal step is of the form $(\sigma_1 \mathbin{;} \sigma_2)\,\mu \overset{\flat}{\mapsto} (\sigma_1' \mathbin{;} \sigma_2)\,\mu$ with $\sigma_1 \overset{\flat}{\mapsto} \sigma_1'$, the situation is:



3.1.2 If the internal step is of the form $(\sigma_1 \mathbin{;} \sigma_2)\,\mu \overset{\flat}{\mapsto} (\sigma_1 \mathbin{;} \sigma_2')\,\mu$ with $\sigma_2 \overset{\flat}{\mapsto} \sigma_2'$, the proof is similar as for the previous case.

3.1.3 If the internal step is of the form $(\sigma_1 \mathbin{;} \sigma_2)\,\mu \overset{\flat}{\mapsto} (\sigma_1 \mathbin{;} \sigma_2)\,\mu'$ with $\mu \overset{\flat}{\mapsto} \mu'$, then note that it must be either a $\mathcal{F}$-BetaM or a $\mathcal{F}$-EtaM step given that the other rules cannot reduce a multistep (Rem. 85). Then the situation is:



3.2 $\mathcal{F}$-$\mathsf{App2}$: Symmetric to the previous case.

3.3 $\mathcal{F}$-$\mathsf{App3}$: Then $\rho_1 = \sigma_1 \,;\, \sigma_2$ and $\rho_2 = \tau_1 \,;\, \tau_2$, and the step at the root is of the form $(\sigma_1 \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2) \overset{\flat}{\mapsto} ((\sigma_1 \,;\, \sigma_2)\,\underline{\tau_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\tau_1 \,;\, \tau_2))$. Note that the internal step must cannot be at the root of $\sigma_1 \,;\, \sigma_2$ nor at the root of $\tau_1 \,;\, \tau_2$ given that there are no rewriting rules whose left-hand side is a composition (" ; "). Hence there are four subcases, depending on whether the internal step is internal to σ_1, internal to σ_2, internal to τ_1, or internal to τ_2.

3.3.1 If the internal step is of the form $(\sigma_1 \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2) \overset{\flat}{\mapsto} (\sigma_1' \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2)$ with $\sigma_1 \overset{\flat}{\mapsto} \sigma_1'$, then the situation is:

$$\begin{array}{ccc}
(\sigma_1 \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2) & \longrightarrow & ((\sigma_1 \,;\, \sigma_2)\,\underline{\tau_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\tau_1 \,;\, \tau_2)) \\
\downarrow & & \downarrow \\
(\sigma_1' \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2) & \longrightarrow & ((\sigma_1' \,;\, \sigma_2)\,\underline{\tau_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\tau_1 \,;\, \tau_2))
\end{array}$$

3.3.2 If the internal step is of the form $(\sigma_1 \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2) \overset{\flat}{\mapsto} (\sigma_1 \,;\, \sigma_2')\,(\tau_1 \,;\, \tau_2)$ with $\sigma_2 \overset{\flat}{\mapsto} \sigma_2'$, then we claim that $\underline{\sigma_2^{\mathsf{tgt}}} \overset{\flat}{\mapsto}{}^{*} \underline{\sigma_2'^{\mathsf{tgt}}}$. Indeed, if the step $\sigma_2 \overset{\flat}{\mapsto} \sigma_2'$ is a $\mathcal{F}$-BetaM or an $\mathcal{F}$-EtaM step this is a consequence of Lem. 87, and if the step is not an instance of the $\mathcal{F}$-BetaM rule, we have that $\underline{\sigma_2^{\mathsf{tgt}}} = \underline{\sigma_2'^{\mathsf{tgt}}}$ as has already been observed (Rem. 84). In any case, the situation is:

$$\begin{array}{ccc}
(\sigma_1 \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2) & \longrightarrow & ((\sigma_1 \,;\, \sigma_2)\,\underline{\tau_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\tau_1 \,;\, \tau_2)) \\
\downarrow & & \downarrow \\
 & & ((\sigma_1 \,;\, \sigma_2')\,\underline{\tau_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\tau_1 \,;\, \tau_2)) \\
\downarrow & & \downarrow \text{by the claim} \\
(\sigma_1 \,;\, \sigma_2')\,(\tau_1 \,;\, \tau_2) & \longrightarrow & ((\sigma_1 \,;\, \sigma_2')\,\underline{\tau_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2'^{\mathsf{tgt}}}\,(\tau_1 \,;\, \tau_2))
\end{array}$$

3.3.3 If the internal step is of the form $(\sigma_1 \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2) \overset{\flat}{\mapsto} (\sigma_1 \,;\, \sigma_2)\,(\tau_1' \,;\, \tau_2)$ with $\tau_1 \overset{\flat}{\mapsto} \tau_1'$, the proof is similar as for the previous case.

3.3.4 If the internal step is of the form $(\sigma_1 \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2) \overset{\flat}{\mapsto} (\sigma_1 \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2')$ with $\tau_2 \overset{\flat}{\mapsto} \tau_2'$, the proof is similar as for when the internal step is internal to σ_1 (subcase 3.3.1).

3.4 $\mathcal{F}$-BetaM: Then $\rho_1 = \lambda x.\mu$ and $\rho_2 = \nu$ are both multisteps. The internal step may be an $\mathcal{F}$-EtaM step at the root of $\lambda x.\mu$, or a step internal to μ or a step internal to ν. Note by Rem. 85 that all of these steps must necessarily be $\mathcal{F}$-BetaM or $\mathcal{F}$-EtaM steps. Therefore these cases correspond to typical critical pairs for $\beta\eta$-reduction, namely:

$$\begin{array}{ccc}
(\lambda x.\mu\, x)\,\nu & \longrightarrow & \mu\,\nu \\
\downarrow & \mathrel{/\!\!/} & \\
\mu\,\nu & &
\end{array}
\qquad \text{where } x \notin \mathsf{fv}(\mu)$$

$$\begin{array}{ccc}
(\lambda x.\mu)\,\nu & \longrightarrow & \mu\{x\backslash\nu\} \\
\downarrow & & \downarrow \\
(\lambda x.\mu')\,\nu & \longrightarrow & \mu'\{x\backslash\nu\}
\end{array}
\qquad
\begin{array}{ccc}
(\lambda x.\mu)\,\nu & \longrightarrow & \mu\{x\backslash\nu\} \\
\downarrow & & \downarrow \\
(\lambda x.\mu)\,\nu' & \longrightarrow & \mu\{x\backslash\nu'\}
\end{array}$$

The diagrams on the bottom rely, respectively, on the following properties. If $\mathtt{x} \in \{\mathcal{F}\text{-}\mathsf{BetaM}, \mathcal{F}\text{-}\mathsf{EtaM}\}$ then:

- $\mu \overset{\flat}{\mapsto}_{\mathtt{x}} \mu'$ implies $\mu\{x\backslash\nu\} \overset{\flat}{\mapsto}_{\mathtt{x}} \mu'\{x\backslash\nu\}$
- $\nu \overset{\flat}{\mapsto}_{\mathtt{x}} \nu'$ implies $\mu\{x\backslash\nu\} \overset{\flat}{\mapsto}^{*}_{\mathtt{x}} \mu\{x\backslash\nu'\}$.

These are straightforward to prove, resorting to Lem. 83 when appropriate.

4. **Composition ($\rho = \rho_1 \,;\, \rho_2$).** Note that the steps cannot be at the root, given that there are no rewriting rules whose left-hand side is a composition ("**;**"). If both steps are internal to ρ_1, it is immediate to conclude by resorting to the IH. Similarly, if both steps are internal to ρ_2, it is immediate to conclude by IH. Finally, if one step is internal to ρ_1 and the other one is internal to ρ_2, it is also immediate to conclude given that the steps are disjoint.

◀

▶ **Definition 97.** *The reduction relation $\overset{\circ}{\mapsto}$ is defined as $\overset{\flat}{\mapsto}$ but excluding the $\mathcal{F}$-EtaM rule.*

▶ Remark 98. The reduction relation $\overset{\circ}{\mapsto}$ is also SN and CR. Strong normalization is immediate by Prop. 95, since $\overset{\circ}{\mapsto} \subseteq \overset{\flat}{\mapsto}$. The proof of confluence is the same as in Prop. 96, ignoring all the cases involving the $\mathcal{F}$-EtaM rule, and observing that peaks not involving the $\mathcal{F}$-EtaM rule may be closed without using the $\mathcal{F}$-EtaM rule.

D.3 Soundness with respect to permutation equivalence

▶ **Lemma 99.** *Let $\Gamma, x : A \vdash \mu : p_0 \twoheadrightarrow p_1 : B$ and $\Gamma \vdash \nu : q_0 \twoheadrightarrow q_1 : A$. Then $\mu\{x\backslash\nu\} \approx \mu\{x\backslash\!\!\backslash\nu\}$.*

Proof. By induction on the derivation of $\Gamma, x : A \vdash \mu : p_0 \twoheadrightarrow p_1 : B$:

1. RVar: Let $\Gamma, x : A \vdash y : y \twoheadrightarrow y : B$ with $(y : B) \in \Gamma$. There are two subcases, depending on whether $x = y$ or not.
   - **1.1** If $x = y$ then: $x\{x\backslash\nu\} = \nu \approx (\underline{t_0} \,;\, \nu) = x\{x\backslash\!\!\backslash\nu\}$ by $\approx$-IdL.
   - **1.2** If $x \neq y$ then $y\{x\backslash\nu\} = y \approx (y \,;\, y) = y\{x\backslash\!\!\backslash\nu\}$ by $\approx$-IdL.
2. RCon: Let $\Gamma, x : A \vdash \mathbf{c} : \mathbf{c} \twoheadrightarrow \mathbf{c} : B$ with $(\mathbf{c} : B) \in \mathcal{C}$. Then $\mathbf{c}\{x\backslash\nu\} = \mathbf{c} \approx (\mathbf{c} \,;\, \mathbf{c}) = \mathbf{c}\{x\backslash\!\!\backslash\nu\}$ by $\approx$-IdL.
3. RRule: Let $\Gamma, x : A \vdash \varrho : s_0 \twoheadrightarrow s_1 : B$ be derived from $\cdot \vdash s_0 : B$ and $\cdot \vdash s_1 : B$ with $(\varrho : s_0 \twoheadrightarrow s_1 : B) \in \mathcal{R}$. Then $\varrho\{x\backslash\nu\} = \varrho \approx (\varrho \,;\, \underline{s_1}) = \varrho\{x\backslash\!\!\backslash\nu\}$ by $\approx$-IdR. Note that s_1 is a closed term by Lem. 38, so $s_1\{x\backslash\!\!\backslash\nu\} = \underline{s_1}$.
4. RAbs: Let $\Gamma, x : A \vdash \lambda y.\mu : \lambda y.s_0 \twoheadrightarrow \lambda y.s_1 : B \to C$ be derived from $\Gamma, x : A, y : B \vdash \mu : s_0 \twoheadrightarrow s_1 : C$. Then:

$$\begin{array}{rcll} (\lambda y.\mu)\{x\backslash\nu\} & = & \lambda y.\mu\{x\backslash\nu\} & \\ & \approx & \lambda y.\mu\{x\backslash\!\!\backslash\nu\} & \text{by IH} \\ & \approx & (\lambda y.\mu)\{x\backslash\!\!\backslash\nu\} & \text{by Lem. 61} \end{array}$$

5. RApp: Let $\Gamma, x : A \vdash \mu_1\,\mu_2 : s_0\,t_0 \twoheadrightarrow s_1\,t_1 : C$ be derived from $\Gamma, x : A \vdash \mu_1 : s_0 \twoheadrightarrow s_1 : B \to C$ and $\Gamma, x : A \vdash \mu_2 : t_0 \twoheadrightarrow t_1 : B$. Then:

$$\begin{array}{rcll} (\mu_1\,\mu_2)\{x\backslash\nu\} & = & \mu_1\{x\backslash\nu\}\,\mu_2\{x\backslash\nu\} & \\ & \approx & \mu_1\{x\backslash\!\!\backslash\nu\}\,\mu_2\{x\backslash\!\!\backslash\nu\} & \text{by IH} \\ & \approx & (\mu_1\,\mu_2)\{x\backslash\!\!\backslash\nu\} & \text{by Lem. 61} \end{array}$$

6. RTrans: Impossible, as μ is a multistep without compositions ("**;**").

7. REq: Let $\Gamma, x : A \vdash \mu : s_0 \multimap s_1 : B$ be derived from $\Gamma, x : A \vdash \mu : s'_0 \multimap s'_1 : B$ with $\Gamma, x : A \vdash s_0 =_{\beta\eta} s'_0 : B$ and $\Gamma, x : A \vdash s'_1 =_{\beta\eta} s_1 : B$. Then:

$$\begin{array}{rcll} \mu\{x\backslash\nu\} & \approx & \mu\{x\backslash\underline{q_0}\} \,;\, s'_1\{x\backslash\!\backslash\nu\} & \text{by IH} \\ & \approx & \mu\{x\backslash\underline{q_0}\} \,;\, s_1\{x\backslash\!\backslash\nu\} & \text{by Lem. 64} \\ & = & \mu\{x\backslash\!\backslash\nu\} & \end{array}$$

◀

▶ **Lemma 100** (Soundness). *If $\rho \overset{\flat}{\mapsto} \sigma$ then $\rho \approx \sigma$.*

Proof. It suffices to show that all the axioms of the flattening system $\mathcal{F}$ relate permutation equivalent rewrites:

1. $\mathcal{F}$-Abs:

$$\lambda x.(\rho \,;\, \sigma) \approx (\lambda x.\rho) \,;\, (\lambda x.\sigma) \quad \text{by } \approx\text{-Abs}$$

2. $\mathcal{F}$-App1:

$$\begin{array}{rcll} (\rho \,;\, \sigma)\, \mu & \approx & (\rho \,;\, \sigma)\, (\underline{\mu^{\mathsf{src}}} \,;\, \mu) & \text{by } \approx\text{-IdL} \\ & \approx & (\rho\, \underline{\mu^{\mathsf{src}}}) \,;\, (\sigma\, \mu) & \text{by } \approx\text{-App} \end{array}$$

3. $\mathcal{F}$-App2:

$$\begin{array}{rcll} \mu\, (\rho \,;\, \sigma) & \approx & (\mu \,;\, \underline{\mu^{\mathsf{tgt}}})\, (\rho \,;\, \sigma) & \text{by } \approx\text{-IdR} \\ & \approx & (\mu\, \rho) \,;\, (\underline{\mu^{\mathsf{tgt}}}\, \sigma) & \text{by } \approx\text{-App} \end{array}$$

4. $\mathcal{F}$-App3:

$$\begin{array}{rcll} (\rho_1 \,;\, \rho_2)(\sigma_1 \,;\, \sigma_2) & \approx & ((\rho_1 \,;\, \rho_2) \,;\, \underline{\rho_2^{\mathsf{tgt}}})(\underline{\sigma_1^{\mathsf{src}}} \,;\, (\sigma_1 \,;\, \sigma_2)) & \text{by } \approx\text{-IdL and } \approx\text{-IdR} \\ & \approx & ((\rho_1 \,;\, \rho_2)\, \underline{\sigma_1^{\mathsf{src}}}) \,;\, (\underline{\rho_2^{\mathsf{tgt}}}\, (\sigma_1 \,;\, \sigma_2)) & \text{by } \approx\text{-App} \end{array}$$

5. $\mathcal{F}$-BetaM:

$$\begin{array}{rcll} (\lambda x.\mu)\, \nu & \approx & \mu\{x\backslash\!\backslash\nu\} & \text{by } \approx\text{-BetaRR (Lem. 56)} \\ & \approx & \mu\{x\backslash\nu\} & \text{by Lem. 99} \end{array}$$

6. $\mathcal{F}$-EtaM:

$$(\lambda x.\mu)\, x \approx \mu \quad \text{by } \approx\text{-Eta, if } x \notin \mathsf{fv}(\mu)$$

◀

D.4 Characterization of normal forms

▶ **Lemma 101** (Characterization of normal multisteps).

1. *The set of multisteps in $\overset{\circ}{\mapsto}$-normal form is exactly the set of (typable) flat multisteps.*
2. *The set of multisteps in $\overset{\flat}{\mapsto}$-normal form is a subset of the set of (typable) flat multisteps.*

Proof. Recall that multisteps are only subject to $\overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM},\mathcal{F}\text{-EtaM}}$ reduction (Rem. 85), so this result is immediate if typable multisteps are understood as simply typed λ-terms where constants $(\mathbf{c}, \mathbf{d}, \ldots)$ and rule symbols $(\varrho, \vartheta, \ldots)$ are regarded as free variables of their corresponding types. ◀

▶ **Proposition 102** (Characterization of normal rewrites).

1. *The set of rewrites in* $\overset{\circ}{\mapsto}$*-normal form is exactly the set of (typable) flat rewrites.*
2. *The set of rewrites in* $\overset{\flat}{\mapsto}$*-normal form is a subset of the set of (typable) flat rewrites.*

Proof. Item 2. is an easy consequence of item 1. For item 1., we prove the two implications:

($\Rightarrow$) Let ρ be a rewrite in $\overset{\circ}{\mapsto}$-normal form, and let us show that it is a flat rewrite. We proceed by induction on ρ:

1. **Variable ($\rho = x$), constant ($\rho = \mathbf{c}$), or rule symbol ($\rho = \varrho$).** Immediate.
2. **Abstraction ($\rho = \lambda x.\sigma$).** By IH σ is a flat rewrite. If $\sigma = \hat{\mu}$ is a flat multistep, it is easy to check that $\lambda x.\hat{\mu}$ is also a flat multistep. If $\sigma = (\hat{\sigma}_1 \,;\, \hat{\sigma}_2)$ is a composition then $\rho = \lambda x.(\hat{\sigma}_1 \,;\, \hat{\sigma}_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} (\lambda x.\hat{\sigma}_1) \,;\, (\lambda x.\hat{\sigma}_2)$ contradicting the fact that it is a $\overset{\circ}{\mapsto}$-normal form.
3. **Application ($\rho = \sigma\,\tau$).** By IH σ and τ are flat rewrites. Note that σ cannot be a composition of the form $\sigma = (\hat{\sigma}_1 \,;\, \hat{\sigma}_2)$ because then $\rho = (\hat{\sigma}_1 \,;\, \hat{\sigma}_2)\,\tau$ would reduce, either applying $\mathcal{F}$-App1 at the root (if τ is a multistep) or $\mathcal{F}$-App3 at the root (if τ is a composition), and this would contradict the fact that ρ is $\overset{\circ}{\mapsto}$-normal. Hence $\sigma = \hat{\mu}$ is a flat multistep. Moreover, τ cannot be a composition of the form $\tau = (\hat{\tau}_1 \,;\, \hat{\tau}_2)$ because then $\rho = \hat{\mu}\,(\hat{\tau}_1 \,;\, \hat{\tau}_2)$ would reduce applying $\mathcal{F}$-App2 at the root, and this would contradict the fact that ρ is $\overset{\circ}{\mapsto}$-normal. Hence $\tau = \hat{\nu}$ is also a flat multistep. Finally, $\rho = \hat{\mu}\,\hat{\nu}$ is a multistep in $\overset{\circ}{\mapsto}$-normal form, so by Lem. 101 it is a flat multistep.
4. **Composition ($\rho = \sigma \,;\, \tau$).** By IH, $\sigma = \hat{\sigma}$ and $\tau = \hat{\tau}$ are flat rewrites, so $\hat{\sigma} \,;\, \hat{\tau}$ is also a flat rewrite.

($\Leftarrow$) Let $\hat{\rho}$ be a flat rewrite. Let us prove that it is $\overset{\circ}{\mapsto}$-normal by induction on the derivation that it is a flat rewrite.

1. **Flat multistep, $\hat{\rho} = \hat{\mu}$.** Then $\hat{\mu}$ is in $\overset{\circ}{\mapsto}$-normal form by Lem. 101.
2. **Composition, $\hat{\rho} = \hat{\rho}_1 \,;\, \hat{\rho}_2$.** Then by IH $\hat{\rho}_1$ and $\hat{\rho}_2$ are $\overset{\circ}{\mapsto}$-normal. Moreover, there cannot be a reduction step at the root, given there are no rewriting rules in the flattening system $\mathcal{F}$ whose left-hand side is a composition ("**;**").

◀

D.5 $\overline{\eta}$-normal forms are closed by flattening

▶ **Lemma 103** (Flattening preserves $\overline{\eta}$-normal forms)**.** *Assume that the set of rewriting rule symbols $\mathcal{R}$ verifies the $\overline{\eta}$-condition. Let $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ be a $\overline{\eta}$-normal form, and suppose that $\rho \overset{\circ}{\mapsto} \sigma$ is a step other than an $\mathcal{F}$-EtaM step. Then σ is also a $\overline{\eta}$-normal form.*

Proof. The rewriting step must be of the form $\mathtt{R}\langle\rho_1\rangle \overset{\circ}{\mapsto} \mathtt{R}\langle\rho_2\rangle$ where $\rho_1 \overset{\flat}{\mapsto} \rho_2$ is an instance of one of the axioms of the flattening system $\mathcal{F}$ other than the $\mathcal{F}$-EtaM rule. By hypothesis, $\mathtt{R}\langle\rho_1\rangle$ is in $\overline{\eta}$-normal form, and we are to show that $\mathtt{R}\langle\rho_2\rangle$ is also in $\overline{\eta}$-normal form. By contradiction, suppose that the right-hand side can be written as of the form $\mathtt{R}\langle\rho_2\rangle = \mathtt{R}'\langle\alpha\rangle$ such that $\mathtt{R}'$ is not applicative and α is not a λ-abstraction nor a composition.

The proof proceeds by case analysis on the relative positions of the holes of $\mathtt{R}$ and $\mathtt{R}'$. We consider three cases, depending on whether the holes of $\mathtt{R}$ and $\mathtt{R}'$ lie at disjoint positions, or $\mathtt{R}'$ is a prefix of $\mathtt{R}$ (with $\mathtt{R} \neq \mathtt{R}'$), or $\mathtt{R}$ is a prefix of $\mathtt{R}'$ (including the case $\mathtt{R} = \mathtt{R}'$).

1. **$\mathtt{R}$ and $\mathtt{R}'$ are disjoint.** That is, there is a context $\hat{\mathtt{R}}$ with two holes such that $\mathtt{R} = \hat{\mathtt{R}}\langle\Box, \alpha\rangle$ and $\mathtt{R}' = \hat{\mathtt{R}}\langle\rho_2, \Box\rangle$. Then the left-hand side of the step is of the form $\mathtt{R}\langle\rho_1\rangle = \hat{\mathtt{R}}\langle\rho_1, \alpha\rangle$. Take $\mathtt{R}'' := \hat{\mathtt{R}}\langle\rho_1, \Box\rangle$. Note that $\mathtt{R}''$ cannot be applicative, since this would imply that $\mathtt{R}'$ is applicative. Hence the left-hand side of the step can be written as of the form $\mathtt{R}\langle\rho_1\rangle = \mathtt{R}''\langle\alpha\rangle$ where $\mathtt{R}''$ is not applicative and α is not a λ-abstraction nor a composition. This contradicts the fact that the left-hand side of the step is in $\overline{\eta}$-normal form.

2. **$\mathtt{R}'$ is a strict prefix of $\mathtt{R}$.** That is, $\mathtt{R} = \mathtt{R}'\langle\mathtt{R}''\rangle$ with $\mathtt{R}'' \neq \Box$ and $\alpha = \mathtt{R}''\langle\rho_2\rangle$. Then the left-hand side of the step is of the form $\mathtt{R}\langle\rho_1\rangle = \mathtt{R}'\langle\mathtt{R}''\langle\rho_1\rangle\rangle$. But $\mathtt{R}'$ is not applicative and $\mathtt{R}''\langle\rho_1\rangle$ is not a λ-abstraction nor a composition, because we know that $\mathtt{R}''$ is non-empty and that α is not a λ-abstraction nor a composition. This contradicts the fact that the left-hand side of the step is in $\overline{\eta}$-normal form.
3. **$\mathtt{R}$ is a non-strict prefix of $\mathtt{R}'$.** That is, $\mathtt{R}' = \mathtt{R}\langle\mathtt{R}''\rangle$ with $\rho_2 = \mathtt{R}''\langle\alpha\rangle$. We analyze all the possible cases, depending on the axiom used to derive the step $\rho_1 \overset{\flat}{\mapsto} \rho_2$. Recall that, by hypothesis, the step is not an $\mathcal{F}$-EtaM step:

3.1 $\mathcal{F}$-Abs: Let $\rho_1 = \lambda x.(\sigma \mathbin{;} \tau) \overset{\flat}{\mapsto} (\lambda x.\sigma) \mathbin{;} (\lambda x.\tau) = \rho_2$. Note that α cannot be at the root of ρ_2 because it is assumed that α is not a composition. Similarly, α it cannot be immediately to the left or to the right of the composition, because it is assumed that α is not a λ-abstraction. Hence there are two subcases, depending on the position of α on the right-hand side, *i.e.* on the shape of $\mathtt{R}''$:

3.1.1 *If α is internal to σ,* i.e. $\mathtt{R}'' = (\lambda x.\mathtt{R}''') \mathbin{;} (\lambda x.\tau)$. Note that $\mathtt{R}'''$ is not applicative. Then the left-hand side of the step is of the form $\mathtt{R}\langle\lambda x.(\mathtt{R}'''\langle\alpha\rangle \mathbin{;} \tau)\rangle$ where the context $\mathtt{R}\langle\lambda x.(\mathtt{R}'''\langle\Box\rangle \mathbin{;} \tau)\rangle$ is still not applicative. This contradicts the fact that the left-hand side is in $\overline{\eta}$-normal form.

3.1.2 *If α is internal to τ,* i.e. $\mathtt{R}'' = (\lambda x.\sigma) \mathbin{;} (\lambda x.\mathtt{R}''')$. Then the proof is similar as for the previous case.

3.2 $\mathcal{F}$-App1: Let $\rho_1 = (\sigma \mathbin{;} \tau)\,\mu \overset{\flat}{\mapsto} (\sigma\,\underline{\mu}^{\mathsf{src}}) \mathbin{;} (\tau\,\mu) = \rho_2$. Note that α cannot be at the root of ρ_2, *i.e.* it cannot be the case that $\mathtt{R}'' = \Box$, because α is assumed not to be a composition. We consider six subcases, depending on the position of α on the right-hand side, *i.e.* on the shape of $\mathtt{R}''$:

3.2.1 *If α is immediately to the left,* i.e. $\mathtt{R}'' = \Box \mathbin{;} (\tau\,\mu)$. Then the expression $\alpha = \sigma\,\underline{\mu}^{\mathsf{src}}$ is of function type, and $\mathtt{R}' = \mathtt{R}\langle\Box \mathbin{;} (\tau\,\mu)\rangle$ is not applicative. Hence the expression $\rho_1 = (\sigma \mathbin{;} \tau)\,\mu$ on the left-hand side is also of function type. Moreover, it is not a λ-abstraction nor a composition, and it lies below the context $\mathtt{R}$, which is not applicative (because this would imply that $\mathtt{R}'$ is applicative). This contradicts the fact that the left-hand side of the step is in $\overline{\eta}$-normal form.

3.2.2 *If α is immediately to the right,* i.e. $\mathtt{R}'' = (\sigma\,\underline{\mu}^{\mathsf{src}}) \mathbin{;} \Box$. Similar to the previous case.

3.2.3 *If α is internal to σ,* i.e. $\mathtt{R}'' = (\mathtt{R}'''\,\underline{\mu}^{\mathsf{src}}) \mathbin{;} (\tau\,\mu)$. Then $\sigma = \mathtt{R}'''\langle\alpha\rangle$ and $\mathtt{R}''$ is not applicative. So the left-hand side is of the form $\mathtt{R}\langle\mathtt{R}'''\langle\alpha\rangle\,\mu\rangle$. Note that the context $\mathtt{R}\langle\mathtt{R}'''\,\mu\rangle$ cannot be applicative, for this would imply that $\mathtt{R}''$ is applicative. This contradicts the fact that the left-hand side of the step is in $\overline{\eta}$-normal form.

3.2.4 *If α is internal to $\underline{\mu}^{\mathsf{src}}$,* i.e. $\mathtt{R}'' = (\sigma\,\mathtt{R}''') \mathbin{;} (\tau\,\mu)$. Then $\underline{\mu}^{\mathsf{src}} = \mathtt{R}'''\langle\alpha\rangle$ and $\mathtt{R}''$ is not applicative. In particular, $\mathtt{R}'''$ is not applicative. This means that $\underline{\mu}^{\mathsf{src}}$ is not in $\overline{\eta}$-normal form. We claim that this is impossible. To justify the claim, it suffices to show that μ is in $\overline{\eta}$-normal form, since by Lem. 81 this implies that $\underline{\mu}^{\mathsf{src}}$ is in $\overline{\eta}$-normal form. Indeed, suppose that $\mu = \mathtt{R}^*\langle\beta\rangle$ where $\mathtt{R}^*$ is not applicative and β is not a λ-abstraction nor a composition. Then the left-hand side of the step is of the form $\mathtt{R}\langle(\sigma \mathbin{;} \tau)\,\mathtt{R}^*\langle\beta\rangle\rangle$. Note that $\mathtt{R}\langle(\sigma \mathbin{;} \tau)\,\mathtt{R}^*\rangle$ is not applicative given that $\mathtt{R}^*$ is not applicative. This contradicts the fact that the left-hand side of the step is in $\overline{\eta}$-normal form.

3.2.5 *If α is internal to τ,* i.e. $\mathtt{R}'' = (\sigma\,\underline{\mu}^{\mathsf{src}}) \mathbin{;} (\mathtt{R}'''\,\mu)$. Similar to the case in which α is internal to σ (subcase 3.2.3).

3.2.6 *If α is internal to μ,* i.e. $\mathtt{R}'' = (\sigma\,\underline{\mu}^{\mathsf{src}}) \mathbin{;} (\tau\,\mathtt{R}''')$. Similar to the case in which α is internal to σ (subcase 3.2.3).

3.3 $\mathcal{F}$-App2: Let $\rho_1 = \mu\,(\sigma \,;\, \tau) \overset{\flat}{\mapsto} (\mu\,\sigma) \,;\, (\underline{\mu^{\mathsf{tgt}}}\,\tau) = \rho_2$. The proof is similar as for the previous case.

3.4 $\mathcal{F}$-App3: Let $\rho_1 = (\sigma_1 \,;\, \sigma_2)\,(\tau_1 \,;\, \tau_2) \overset{\flat}{\mapsto} ((\sigma_1 \,;\, \sigma_2)\,\underline{\tau_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\,(\tau_1 \,;\, \tau_2)) = \rho_2$. The proof is similar as for the previous case.

3.5 $\mathcal{F}$-BetaM: Let $\rho_1 = (\lambda x.\mu)\,\nu \overset{\flat}{\mapsto} \mu\{x\backslash\nu\} = \rho_2$. Note that ρ_2 has no compositions ("**;**"), so the proof of this case is a straightforward adaptation of the proof that β-reduction preserves $\overline{\eta}$-normal forms in the simply typed λ-calculus. ◀

D.6 More properties of flattening

The following properties are used to prove completeness of flat permutation equivalence.

▶ **Definition 104** (Flattening to normal form). *If ρ is a rewrite, we write $\rho^\flat$ to denote the $\overset{\flat}{\mapsto}$-normal form of ρ. Note that the $\mathcal{F}$-EtaM reduction rule is included. The expressions $\rho^\blacktriangleleft$ and $\rho^\blacktriangleright$ denote the $\overset{\flat}{\mapsto}$-normal forms of the source and target, respectively, that is, $(\underline{\rho^{\mathsf{src}}})^\flat$ and $(\underline{\rho^{\mathsf{tgt}}})^\flat$.*

▶ **Lemma 105** (Coherence of the flat source and target).

1. $\rho^\blacktriangleleft \approx \underline{(\rho^\flat)^{\mathsf{src}}}$ *and, even more strongly,* $\underline{(\rho^\flat)^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^* \rho^\blacktriangleleft$.
2. $\rho^\blacktriangleright \approx \underline{(\rho^\flat)^{\mathsf{tgt}}}$ *and, even more strongly,* $\underline{(\rho^\flat)^{\mathsf{tgt}}} \overset{\flat}{\mapsto}{}^* \rho^\blacktriangleright$.

Proof. We prove item 1, the proof for item 2. is similar. By definition, $\rho^\blacktriangleleft = (\underline{\rho^{\mathsf{src}}})^\flat$. Note that $\rho \overset{\flat}{\mapsto}{}^* \rho^\flat$. Recall by Rem. 84 that steps other than $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM preserve the endpoints, while by Lem. 87 we know that $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM reduction steps commute with taking the endpoints. Hence we have that $\underline{\rho^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^* \underline{(\rho^\flat)^{\mathsf{src}}}$. By confluence of flattening Prop. 96, $\underline{(\rho^\flat)^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^* (\underline{\rho^{\mathsf{src}}})^\flat = \rho^\blacktriangleleft$. Moreover, by soundness of flattening (Lem. 100) we have that $\underline{(\rho^\flat)^{\mathsf{src}}} \approx \rho^\blacktriangleleft$ as required. ◀

▶ **Lemma 106** (Generalized flattening for composition trees).

1. **Generalized $\mathcal{F}$-Abs.** $\lambda x.\mathtt{K}\langle\rho_1,\ldots,\rho_n\rangle \overset{\flat}{\mapsto}{}^* \mathtt{K}\langle\lambda x.\rho_1,\ldots,\lambda x.\rho_n\rangle$.
2. **Generalized $\mathcal{F}$-App1.** $\mathtt{K}\langle\rho_1,\ldots,\rho_{n-1},\rho_n\rangle\,\mu \overset{\flat}{\mapsto}{}^* \mathtt{K}\langle(\rho_1\,\underline{\mu^{\mathsf{src}}}),\ldots,(\rho_{n-1}\,\underline{\mu^{\mathsf{src}}}),(\rho_n\,\mu)\rangle$.
3. **Generalized $\mathcal{F}$-App2.** $\mu\,\mathtt{K}\langle\rho_1,\rho_2,\ldots,\rho_n\rangle \overset{\flat}{\mapsto}{}^* \mathtt{K}\langle(\mu\,\rho_1),(\underline{\mu^{\mathsf{tgt}}}\,\rho_2),\ldots,(\underline{\mu^{\mathsf{tgt}}}\,\rho_n)\rangle$.
4. **Generalized $\mathcal{F}$-App3.** *If $n, m > 1$ then:*

$$\mathtt{K}\langle\rho_1,\ldots,\rho_n\rangle\,\mathtt{K}'\langle\sigma_1,\ldots,\sigma_m\rangle \overset{\flat}{\mapsto}{}^* \mathtt{K}\langle(\rho_1\,\underline{\sigma_1^{\mathsf{src}}}),\ldots,(\rho_n\,\underline{\sigma_1^{\mathsf{src}}})\rangle \,;\, \mathtt{K}'\langle(\underline{\rho_n^{\mathsf{tgt}}}\,\sigma_1),\ldots,(\underline{\rho_n^{\mathsf{tgt}}}\,\sigma_m)\rangle$$

Proof. We prove each item:

1. **Generalized $\mathcal{F}$-Abs.** By induction on $\mathtt{K}$. If $\mathtt{K} = \Box$, then $n = 1$ and we have that $\lambda x.\rho_1 \overset{\flat}{\mapsto}{}^* \lambda x.\rho_1$ in zero reduction steps. If $\mathtt{K} = \mathtt{K}_1\,\mathtt{K}_2$ then $\mathtt{K}_1$ and $\mathtt{K}_2$ have at least one hole each, so there is an index $1 \leq i \leq n-1$ such that $\mathtt{K}\langle\rho_1,\ldots,\rho_n\rangle = \mathtt{K}_1\langle\rho_1,\ldots,\rho_i\rangle \,;\, \mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle$. Then:

$$\begin{array}{lll} & \lambda x.\mathtt{K}\langle\rho_1,\ldots,\rho_n\rangle & \\ = & \lambda x.(\mathtt{K}_1\langle\rho_1,\ldots,\rho_i\rangle \,;\, \mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle) & \\ \overset{\flat}{\mapsto} & (\lambda x.\mathtt{K}_1\langle\rho_1,\ldots,\rho_i\rangle) \,;\, (\lambda x.\mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle) & \text{by } \mathcal{F}\text{-Abs} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle\lambda x.\rho_1,\ldots,\lambda x.\rho_i\rangle \,;\, (\lambda x.\mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle) & \text{by IH} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle\lambda x.\rho_1,\ldots,\lambda x.\rho_i\rangle \,;\, \mathtt{K}_2\langle\lambda x.\rho_{i+1},\ldots,\lambda x.\rho_n\rangle & \text{by IH} \\ = & \mathtt{K}\langle\lambda x.\rho_1,\ldots,\lambda x.\rho_n\rangle & \end{array}$$

2. **Generalized $\mathcal{F}$-App1.** By induction on $\mathtt{K}$. If $\mathtt{K} = \Box$, then $n = 1$ and we have that $\rho_1\,\mu \overset{\flat}{\mapsto}{}^* \rho_1\,\mu$ in zero reduction steps. If $\mathtt{K} = \mathtt{K}_1\,\mathtt{K}_2$ then $\mathtt{K}_1$ and $\mathtt{K}_2$ have at least one hole each, so there is an index $1 \leq i \leq n-1$ such that $\mathtt{K}\langle\rho_1,\ldots,\rho_{n-1},\rho_n\rangle = \mathtt{K}_1\langle\rho_1,\ldots,\rho_i\rangle \,;\, \mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_{n-1},\rho_n\rangle$. Then:

$$\begin{array}{lll} & \mathtt{K}\langle\rho_1,\ldots,\rho_{n-1},\rho_n\rangle\,\mu & \\ = & (\mathtt{K}_1\langle\rho_1,\ldots,\rho_i\rangle \,;\, \mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_{n-1},\rho_n\rangle)\,\mu & \\ \overset{\flat}{\mapsto} & (\mathtt{K}_1\langle\rho_1,\ldots,\rho_i\rangle\,\underline{\mu^{\mathsf{src}}}) \,;\, (\mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_{n-1},\rho_n\rangle\,\mu) & \text{by } \mathcal{F}\text{-App1} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle(\rho_1\,\underline{\mu^{\mathsf{src}}}),\ldots,(\rho_i\,\underline{\mu^{\mathsf{src}}})\rangle \,;\, (\mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_{n-1},\rho_n\rangle\,\mu) & \text{by IH} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle(\rho_1\,\underline{\mu^{\mathsf{src}}}),\ldots,(\rho_i\,\underline{\mu^{\mathsf{src}}})\rangle \,;\, \mathtt{K}_2\langle(\rho_{i+1},\underline{\mu^{\mathsf{src}}}),\ldots,(\rho_{n-1}\,\underline{\mu^{\mathsf{src}}}),(\rho_n\,\mu)\rangle & \text{by IH} \\ = & \mathtt{K}\langle(\rho_1\,\underline{\mu^{\mathsf{src}}}),\ldots,(\rho_{n-1}\,\underline{\mu^{\mathsf{src}}}),(\rho_n\,\mu)\rangle & \end{array}$$

3. **Generalized $\mathcal{F}$-App2.** By induction on $\mathtt{K}$. If $\mathtt{K} = \Box$, then $n = 1$ and we have that $\mu\,\rho_1 \overset{\flat}{\mapsto}{}^* \mu\,\rho_1$ in zero reduction steps. If $\mathtt{K} = \mathtt{K}_1\,\mathtt{K}_2$ then $\mathtt{K}_1$ and $\mathtt{K}_2$ have at least one hole each, so there is an index $1 \leq i \leq n-1$ such that $\mathtt{K}\langle\rho_1,\rho_2,\ldots,\rho_n\rangle = \mathtt{K}_1\langle\rho_1,\rho_2,\ldots,\rho_i\rangle \,;\, \mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle$. Then:

$$\begin{array}{lll} & \mu\,\mathtt{K}\langle\rho_1,\rho_2,\ldots,\rho_n\rangle & \\ = & \mu\,(\mathtt{K}_1\langle\rho_1,\rho_2,\ldots,\rho_i\rangle \,;\, \mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle) & \\ \overset{\flat}{\mapsto} & (\mu\,\mathtt{K}_1\langle\rho_1,\rho_2,\ldots,\rho_i\rangle) \,;\, (\underline{\mu^{\mathsf{tgt}}}\,\mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle) & \text{by } \mathcal{F}\text{-App2} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle(\mu\,\rho_1),(\underline{\mu^{\mathsf{tgt}}}\,\rho_2),\ldots,(\underline{\mu^{\mathsf{tgt}}}\,\rho_i)\rangle \,;\, (\underline{\mu^{\mathsf{tgt}}}\,\mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle) & \text{by IH} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle(\mu\,\rho_1),(\underline{\mu^{\mathsf{tgt}}}\,\rho_2),\ldots,(\underline{\mu^{\mathsf{tgt}}}\,\rho_i)\rangle \,;\, \mathtt{K}_2\langle(\underline{\mu^{\mathsf{tgt}}}\,\rho_{i+1}),\ldots,(\underline{\mu^{\mathsf{tgt}}}\,\rho_n)\rangle & \text{by IH} \\ = & \mathtt{K}\langle(\mu\,\rho_1),(\underline{\mu^{\mathsf{tgt}}}\,\rho_2),\ldots,(\underline{\mu^{\mathsf{tgt}}}\,\rho_n)\rangle & \end{array}$$

4. **Generalized $\mathcal{F}$-App3.** Let $n, m > 1$. Since $n > 1$, the composition tree $\mathtt{K}$ has at least two holes, so it must be of the form $\mathtt{K} = \mathtt{K}_1 \,;\, \mathtt{K}_2$. Similarly, since $m > 1$, the composition tree $\mathtt{K}'$ has at least two holes, so it must be of the form $\mathtt{K}' = \mathtt{K}'_1 \,;\, \mathtt{K}'_2$. Moreover, since each of $\mathtt{K}_1$, $\mathtt{K}_2$ $\mathtt{K}'_1$, and $\mathtt{K}'_2$ have at least one hole, there must be indices $1 \leq i \leq n-1$ and $1 \leq j \leq n-1$ such that:

$$\begin{array}{rcl} \mathtt{K}\langle\rho_1,\ldots,\rho_n\rangle & = & \mathtt{K}_1\langle\rho_1,\ldots,\rho_i\rangle \,;\, \mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle \\ \mathtt{K}'\langle\sigma_1,\ldots,\sigma_n\rangle & = & \mathtt{K}'_1\langle\sigma_1,\ldots,\sigma_j\rangle \,;\, \mathtt{K}'_2\langle\sigma_{j+1},\ldots,\sigma_m\rangle \end{array}$$

Hence we have that:

$$\begin{array}{lll} & \mathtt{K}\langle\rho_1,\ldots,\rho_n\rangle\,\mathtt{K}'\langle\sigma_1,\ldots,\sigma_n\rangle & \\ = & (\mathtt{K}_1\langle\rho_1,\ldots,\rho_i\rangle \,;\, \mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle)\,(\mathtt{K}'_1\langle\sigma_1,\ldots,\sigma_j\rangle \,;\, \mathtt{K}'_2\langle\sigma_{j+1},\ldots,\sigma_m\rangle) & \\ \overset{\flat}{\mapsto} & ((\mathtt{K}_1\langle\rho_1,\ldots,\rho_i\rangle \,;\, \mathtt{K}_2\langle\rho_{i+1},\ldots,\rho_n\rangle)\,\underline{\sigma_1^{\mathsf{src}}}) \,;\, (\underline{\rho_n^{\mathsf{tgt}}}\,(\mathtt{K}'_1\langle\sigma_1,\ldots,\sigma_j\rangle \,;\, \mathtt{K}'_2\langle\sigma_{j+1},\ldots,\sigma_m\rangle)) & \\ & \quad\text{by } \mathcal{F}\text{-App3} & \\ = & (\mathtt{K}\langle\rho_1,\ldots,\rho_n\rangle\,\underline{\sigma_1^{\mathsf{src}}}) \,;\, (\underline{\rho_n^{\mathsf{tgt}}}\,\mathtt{K}'\langle\sigma_1,\ldots,\sigma_m\rangle) & \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}\langle(\rho_1\,\underline{\sigma_1^{\mathsf{src}}}),\ldots,(\rho_n\,\underline{\sigma_1^{\mathsf{src}}})\rangle \,;\, (\underline{\rho_n^{\mathsf{tgt}}}\,\mathtt{K}'\langle\sigma_1,\ldots,\sigma_m\rangle) & \text{by generalized } \mathcal{F}\text{-App1} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}\langle(\rho_1\,\underline{\sigma_1^{\mathsf{src}}}),\ldots,(\rho_n\,\underline{\sigma_1^{\mathsf{src}}})\rangle \,;\, \mathtt{K}'\langle(\underline{\rho_n^{\mathsf{tgt}}}\,\sigma_1),\ldots,(\underline{\rho_n^{\mathsf{tgt}}}\,\sigma_m)\rangle & \text{by generalized } \mathcal{F}\text{-App2} \end{array}$$

We implicitly use the fact that, in general, $\mathtt{K}\langle\rho_1,\ldots,\rho_n\rangle^{\mathsf{src}} = \rho_1^{\mathsf{src}}$ and $\mathtt{K}\langle\rho_1,\ldots,\rho_n\rangle^{\mathsf{tgt}} = \rho_n^{\mathsf{tgt}}$, which is easy to check by induction on $\mathtt{K}$.

◀

▶ **Lemma 107** (Flattening term/rewrite substitution of a composition). *Let s be a term with n free occurrences of x, that is $s = s\langle x, x, \ldots, x\rangle$ where, by abuse of notation, we write s for the* *term itself and also for a context with n holes that do not bind x. Moreover, let $\rho_1, \ldots, \rho_m$ be fixed rewrites with $m > 0$. If i is an index $1 \leq i \leq n$, and τ is a rewrite, we write $s\langle\tau\rangle_i$ for the rewrite that results from replacing the i-th free occurrence of x in s by τ, the free occurrences of x at positions $j < i$ by $\underline{\rho_m^{\mathsf{tgt}}}$, and the free occurrences of x at positions $j > i$ by $\underline{\rho_1^{\mathsf{src}}}$. That is:*

$$s\langle\tau\rangle_i^x := s\langle\underbrace{\underline{\rho_m^{\mathsf{tgt}}}, \ldots, \underline{\rho_m^{\mathsf{tgt}}}}_{i-1}, \tau, \underbrace{\underline{\rho_1^{\mathsf{src}}}, \ldots, \underline{\rho_1^{\mathsf{src}}}}_{n-i}\rangle$$

Then, if $n > 0$, for any m-hole composition tree $\mathtt{K}$ there exists an $(n \cdot m)$-hole composition tree $\mathtt{K}'$ such that:

$$s\{x\backslash\!\backslash\mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle\} \overset{\flat}{\mapsto}{}^* \mathtt{K}'\langle\underbrace{s\langle\rho_1\rangle_1^x, \ldots s\langle\rho_m\rangle_1^x}_{(m \text{ rewrites})}, \underbrace{s\langle\rho_1\rangle_2^x, \ldots s\langle\rho_m\rangle_2^x}_{(m \text{ rewrites})}, \ldots \underbrace{s\langle\rho_1\rangle_n^x, \ldots s\langle\rho_m\rangle_n^x}_{(m \text{ rewrites})}\rangle$$

Informally, this expresses that the flattening of $s\{x\backslash\!\backslash\mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle\}$ is the composition of substituting first the first occurrence of x by $\rho_1, \ldots, \rho_m$, leaving the remaining ocurrences fixed, then substituting the second occurrence of x by $\rho_1, \ldots, \rho_m$, and so on.

Proof. We proceed by induction on s. If s is a variable other than x, a constant, or a rule symbol, then $n = 0$ and the implication holds vacuously. The remaining cases are:

1. **Substituted variable, $s = x$.** Taking $\mathtt{K}' := \mathtt{K}$ we have that

$$x\{x\backslash\!\backslash\mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle\} = \mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle = \mathtt{K}\langle x\langle\rho_1\rangle_1^x, \ldots, x\langle\rho_m\rangle_1^x\rangle$$

2. **Abstraction, $\lambda y.s$.** Note that there are n free occurrences of x in s. Then:

$$\begin{array}{lll} \lambda y.s\{x\backslash\!\backslash\mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle\} & \overset{\flat}{\mapsto}{}^* & \lambda y.\mathtt{K}'\langle s\langle\rho_1\rangle_1^x, \ldots s\langle\rho_m\rangle_1^x, \ldots s\langle\rho_1\rangle_n^x, s\langle\rho_m\rangle_n^x\rangle \\ & & \quad \text{by IH} \\ & \overset{\flat}{\mapsto}{}^* & \mathtt{K}'\langle\lambda y.s\langle\rho_1\rangle_1^x, \ldots \lambda y.s\langle\rho_m\rangle_1^x, \ldots \lambda y.s\langle\rho_1\rangle_n^x, \ldots \lambda y.s\langle\rho_m\rangle_n^x\rangle \\ & & \quad \text{by generalized } \mathcal{F}\text{-Abs Lem. 106} \end{array}$$

3. **Application, $s\,t$.** Then $n = i + j$ where i is the number of free occurrences of x in s, and j is the number of free occurrences of x in t. By hypothesis, $n > 0$. We consider three subcases, depending on whether $i = 0$, or $j = 0$, or both i and j are strictly positive:

3.1 If $i = 0$ and $j = n$, then:

$$\begin{array}{lll} & s\{x\backslash\!\backslash\mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle\}\, t\{x\backslash\!\backslash\mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle\} & \\ = & s\,t\{x\backslash\!\backslash\mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle\} & \\ \overset{\flat}{\mapsto}{}^* & s\,\mathtt{K}'\langle t\langle\rho_1\rangle_1^x, \ldots, t\langle\rho_m\rangle_1^x, \ldots t\langle\rho_1\rangle_n^x, \ldots, t\langle\rho_m\rangle_n^x\rangle & \text{by IH} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}'\langle(s\,t\langle\rho_1\rangle_1^x), \ldots, (s\,t\langle\rho_m\rangle_1^x), \ldots (s\,t\langle\rho_1\rangle_n^x), \ldots, (s\,t\langle\rho_m\rangle_n^x)\rangle & \\ & \quad \text{by generalized } \mathcal{F}\text{-App2 Lem. 106} & \end{array}$$

3.2 If $j = 0$ and $i = n$, the proof is symmetric to the previous case.

3.3 If $i > 0$ and $j > 0$, note that $(t\langle\rho_1\rangle_1^x)^{\mathsf{src}} = t\{x\backslash\!\backslash\rho_1^{\mathsf{src}}\}$ and that $(s\langle\rho_m\rangle_i^x)^{\mathsf{tgt}} = s\{x\backslash\!\backslash\rho_m^{\mathsf{tgt}}\}$. Then:

$$\begin{array}{ll} & s\{x\backslash\!\backslash\mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle\}\, t\{x\backslash\!\backslash\mathtt{K}\langle\rho_1, \ldots, \rho_m\rangle\} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle s\langle\rho_1\rangle_1^x, \ldots, s\langle\rho_m\rangle_1^x, \ldots, s\langle\rho_1\rangle_i^x, \ldots, s\langle\rho_m\rangle_i^x\rangle \\ & \mathtt{K}_2\langle t\langle\rho_1\rangle_1^x, \ldots, t\langle\rho_m\rangle_1^x, \ldots, t\langle\rho_1\rangle_j^x, \ldots, t\langle\rho_m\rangle_j^x\rangle \\ & \quad \text{by IH} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle s\langle\rho_1\rangle_1^x\, t\{x\backslash\!\backslash\underline{\rho_1^{\mathsf{src}}}\}, \ldots, s\langle\rho_m\rangle_1^x\, t\{x\backslash\!\backslash\underline{\rho_1^{\mathsf{src}}}\}, \ldots, s\langle\rho_1\rangle_i^x\, t\{x\backslash\!\backslash\underline{\rho_1^{\mathsf{src}}}\}, \ldots, s\langle\rho_m\rangle_i^x\, t\{x\backslash\!\backslash\underline{\rho_1^{\mathsf{src}}}\}\rangle \,; \\ & \mathtt{K}_2\langle s\{x\backslash\!\backslash\underline{\rho_m^{\mathsf{tgt}}}\}\, t\langle\rho_1\rangle_1^x, \ldots, s\{x\backslash\!\backslash\underline{\rho_m^{\mathsf{tgt}}}\}\, t\langle\rho_m\rangle_1^x, \ldots, s\{x\backslash\!\backslash\underline{\rho_m^{\mathsf{tgt}}}\}\, t\langle\rho_1\rangle_j^x, \ldots, s\{x\backslash\!\backslash\underline{\rho_m^{\mathsf{tgt}}}\}\, t\langle\rho_m\rangle_j^x\rangle \\ & \quad \text{by generalized } \mathcal{F}\text{-App3 Lem. 106} \\ = & \mathtt{K}_1\langle(s\,t)\langle\rho_1\rangle_1^x, \ldots, (s\,t)\langle\rho_m\rangle_1^x, \ldots, (s\,t)\langle\rho_1\rangle_i^x, \ldots, (s\,t)\langle\rho_m\rangle_i^x\rangle \,; \\ & \mathtt{K}_2\langle(s\,t)\langle\rho_1\rangle_{i+1}^x, \ldots, (s\,t)\langle\rho_m\rangle_{i+1}^x, \ldots, (s\,t)\langle\rho_1\rangle_n^x, \ldots, (s\,t)\langle\rho_m\rangle_n^x\rangle \end{array}$$

Taking $\mathtt{K}' := \mathtt{K}_1 \,;\, \mathtt{K}_2$ we conclude. ◀

▶ **Lemma 108** (Flattening below term/rewrite substitution). *If $\rho \overset{\flat}{\mapsto} \rho'$ then $s\{x \backslash\!\backslash \rho\} \overset{\flat}{\mapsto}{}^* s\{x \backslash\!\backslash \rho'\}$.*

Proof. Straightforward by induction on s. ◀

▶ **Lemma 109** (Flattening below rewrite/term substitution). *If $\rho \overset{\flat}{\mapsto} \rho'$ then $\rho\{x \backslash s\} \overset{\flat}{\mapsto} \rho'\{x \backslash s\}$.*

Proof. By induction on the context under which the step $\rho \overset{\flat}{\mapsto} \rho'$ takes place. Congruence closure is straightforward by resorting to the induction hypothesis, for example if $\rho_1\, \rho_2 \overset{\flat}{\mapsto} \rho_1'\, \rho_2$ with $\rho_1 \overset{\flat}{\mapsto} \rho_1'$ then by IH $\rho_1\{x \backslash s\} \overset{\flat}{\mapsto} \rho_1'\{x \backslash s\}$, so $\rho_1\{x \backslash s\}\, \rho_2\{x \backslash s\} \overset{\flat}{\mapsto} \rho_1'\{x \backslash s\}\, \rho_2\{x \backslash s\}$. The interesting cases is when a rewriting rule is applied at the root:

1. $\mathcal{F}$-Abs: Let $\lambda y.(\rho \,;\, \sigma) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} (\lambda y.\rho) \,;\, (\lambda x.\sigma)$. Then $\lambda y.(\rho\{x \backslash s\} \,;\, \sigma\{x \backslash s\}) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Abs}} (\lambda y.\rho\{x \backslash s\}) \,;\, (\lambda x.\sigma\{x \backslash s\})$.
2. $\mathcal{F}$-App1: Let $(\rho \,;\, \sigma)\, \mu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} (\rho\, \underline{\mu^{\mathsf{src}}}) \,;\, (\sigma\, \mu)$. Then note that $\mu\{x \backslash s\}$ is a multistep and that $\mu^{\mathsf{src}}\{x \backslash s\} = \mu\{x \backslash s\}^{\mathsf{src}}$ by Lem. 48. Hence:
$$(\rho\{x \backslash s\} \,;\, \sigma\{x \backslash s\})\, \mu\{x \backslash s\} \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App1}} (\rho\{x \backslash s\}\, \underline{\mu\{x \backslash s\}^{\mathsf{src}}}) \,;\, (\sigma\{x \backslash s\}\, \mu\{x \backslash s\})$$
3. $\mathcal{F}$-App2: Symmetric to the previous case.
4. $\mathcal{F}$-App3: Let $(\rho_1 \,;\, \rho_2)\, (\sigma_1 \,;\, \sigma_2) \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App3}} ((\rho_1 \,;\, \rho_2)\, \underline{\sigma_1^{\mathsf{src}}}) \,;\, (\underline{\sigma_2^{\mathsf{tgt}}}\, (\sigma_1 \,;\, \sigma_2))$. Note that $\sigma_1^{\mathsf{src}}\{x \backslash s\} = \sigma_1\{x \backslash s\}^{\mathsf{src}}$ and $\sigma_2^{\mathsf{tgt}}\{x \backslash s\} = \sigma_2\{x \backslash s\}^{\mathsf{tgt}}$ by Lem. 48. Hence:
$$\begin{array}{rl} & (\rho_1\{x \backslash s\} \,;\, \rho_2\{x \backslash s\})\, (\sigma_1\{x \backslash s\} \,;\, \sigma_2\{x \backslash s\}) \\ \overset{\flat}{\mapsto}_{\mathcal{F}\text{-App3}} & ((\rho_1\{x \backslash s\} \,;\, \rho_2\{x \backslash s\})\, \underline{\sigma_1\{x \backslash s\}^{\mathsf{src}}}) \,;\, (\underline{\sigma_2\{x \backslash s\}^{\mathsf{tgt}}}\, (\sigma_1\{x \backslash s\} \,;\, \sigma_2\{x \backslash s\})) \end{array}$$
5. $\mathcal{F}$-BetaM: Let $(\lambda y.\mu)\, \nu \overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM}} \mu\{y \backslash \nu\}$. Then:
$$(\lambda y.\mu\{x \backslash s\})\, \nu\{x \backslash s\} \overset{\flat}{\mapsto}_{\mathcal{F}\text{-BetaM}} \mu\{x \backslash s\}\{y \backslash \nu\{x \backslash s\}\} = \mu\{y \backslash \nu\}\{x \backslash s\}$$
The last equality is justified using Lem. 83.
6. $\mathcal{F}$-EtaM: Let $\lambda y.\mu\, y \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}} \mu$, where $y \notin \mathsf{fv}(\mu)$. We may assume that $y \notin \mathsf{fv}(s)$, renaming y if needed, so in particular $y \notin \mathsf{fv}(\mu\{x \backslash s\})$. Then we have that $\lambda y.\mu\{x \backslash s\}\, y \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}} \mu\{x \backslash s\}$.

◀

▶ **Lemma 110** (Flattening of η-expanded multisteps). *Let μ, ν be multisteps in $\overline{\eta}$-normal form such that $\mu^\flat = \nu^\flat$. Then $\mu^\circ = \nu^\circ$.*

Proof. Consider the reduction sequences $\mu \overset{\flat}{\mapsto}{}^* \mu^\flat$ and $\nu \overset{\flat}{\mapsto}{}^* \nu^\flat$. Since $\mathcal{F}$-EtaM-redexes may be postponed after $\mathcal{F}$-BetaM-redexes (a standard result, regarding multisteps as terms of the simply-typed λ-calculus), these reductions factorize as $\mu \overset{\flat}{\mapsto}{}^*_{\mathcal{F}\text{-BetaM}} \mu^\circ \overset{\flat}{\mapsto}{}^*_{\mathcal{F}\text{-EtaM}} \mu^\flat$ and $\nu \overset{\flat}{\mapsto}{}^*_{\mathcal{F}\text{-BetaM}} \nu^\circ \overset{\flat}{\mapsto}{}^*_{\mathcal{F}\text{-EtaM}} \nu^\flat$. Moreover, recall that flattening preserves $\overline{\eta}$-normal forms (Lem. 103), so μ° and ν° are $\overline{\eta}$-normal forms. As an auxiliary claim, observe that if ξ is a multistep in $\mathcal{F}$-BetaM-normal form then any $\mathcal{F}$-EtaM reduction step $\xi \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}} \xi'$ corresponds to a backwards expansion step $\xi' \to_{\overline{\eta}} \xi$; this can be easily checked by induction on ξ following the characterization of flat multisteps (Lem. 101). Hence $\mu^\flat \to_{\overline{\eta}} \mu^\circ$ and $\mu^\flat = \nu^\flat \to_{\overline{\eta}} \nu^\circ$. Finally, since μ° and ν° are $\overline{\eta}$-normal forms and the expansion relation $\to_{\overline{\eta}}$ is confluent (Prop. 76), we obtain that $\mu^\circ = \nu^\circ$, as required. ◀

D.7 Flat permutation equivalence

▶ Remark 111. Every time that flattening $-^\flat$ is used in the rules defining $\sim$, it operates over a multistep. So the only rules that are needed are the $\mathcal{F}$-BetaM and $\mathcal{F}$-EtaM rules.

▶ Remark 112. Recall that, by definition, flat rewrites are given by the grammar $\hat{\rho} ::= \hat{\mu} \mid \hat{\rho}\,;\,\hat{\rho}$. This corresponds to the set of all and only the rewrites of the form $\mathtt{K}\langle\hat{\mu}_1,\dots,\hat{\mu}_n\rangle$.

▶ **Lemma 113** (Soundness of splitting with respect to permutation equivalence). *Let $\Gamma \vdash \mu : s \twoheadrightarrow t : A$ and $\Gamma \vdash \mu_1 : s' \twoheadrightarrow r_1 : A$ and $\Gamma \vdash \mu_2 : r_2 \twoheadrightarrow t' : A$ be such that $\mu \Leftrightarrow \mu_1\,;\,\mu_2$. Then $\mu \approx \mu_1\,;\,\mu_2$*

Proof. By induction on the derivation of $\mu \Leftrightarrow \mu_1\,;\,\mu_2$:
1. SVar: Let $x \Leftrightarrow x\,;\,x$. Then $x \approx x\,;\,x$ by $\approx$-IdL.
2. SCon: Let $\mathbf{c} \Leftrightarrow \mathbf{c}\,;\,\mathbf{c}$. Then $\mathbf{c} \approx \mathbf{c}\,;\,\mathbf{c}$ by $\approx$-IdL.
3. SRuleL: Let $\varrho \Leftrightarrow \varrho\,;\,\underline{\varrho^{\mathsf{tgt}}}$. Then $\varrho \approx \varrho\,;\,\underline{\varrho^{\mathsf{tgt}}}$ by $\approx$-IdR.
4. SRuleR: Let $\varrho \Leftrightarrow \underline{\varrho^{\mathsf{src}}}\,;\,\varrho$. Then $\varrho \approx \underline{\varrho^{\mathsf{src}}}\,;\,\varrho$ by $\approx$-IdL.
5. SAbs: Let $\lambda x.\mu \Leftrightarrow \lambda x.\mu_1\,;\,\lambda x.\mu_2$ be derived from $\mu \Leftrightarrow \mu_1\,;\,\mu_2$. Then:

$$\begin{array}{lcll} \lambda x.\mu & \approx & \lambda x.(\mu_1\,;\,\mu_2) & \text{by IH} \\ & \approx & (\lambda x.\mu_1)\,;\,(\lambda x.\mu_2) & \text{by } \approx\text{-Abs} \end{array}$$

6. SApp: Let $\mu\,\nu \Leftrightarrow \mu_1\,\nu_1\,;\,\mu_2\,\nu_2$ be derived from $\mu \Leftrightarrow \mu_1\,;\,\mu_2$ and $\nu \Leftrightarrow \nu_1\,;\,\nu_2$. Then:

$$\begin{array}{lcll} \mu\,\nu & \approx & (\mu_1\,;\,\mu_2)\,\nu & \text{by IH} \\ & \approx & (\mu_1\,;\,\mu_2)\,(\nu_1\,;\,\nu_2) & \text{by IH} \\ & \approx & (\mu_1\,\nu_1)\,;\,(\mu_2\,\nu_2) & \text{by } \approx\text{-App} \end{array}$$

◀

▶ **Lemma 114** (Soundness of flat permutation equivalence with respect to permutation equivalence). *Let $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ and $\Gamma \vdash \sigma : s' \twoheadrightarrow t' : A$ be such that $\rho \sim \sigma$. Then $\rho \approx \sigma$.*

Proof. By induction on the derivation of $\rho \sim \sigma$. Reflexivity, transitivity, symmetry, and closure under composition contexts is immediate. The interesting case is when an axiom is applied at the root:
1. $\sim$-Assoc: Let $(\rho\,;\,\sigma)\,;\,\tau \sim \rho\,;\,(\sigma\,;\,\tau)$. Then by $\approx$-Assoc also $(\rho\,;\,\sigma)\,;\,\tau \approx \rho\,;\,(\sigma\,;\,\tau)$.
2. $\sim$-Perm: Let $\mu \sim \mu_1\,;\,\mu_2$ where $\mu \Leftrightarrow \mu_1\,;\,\mu_2$. Then by Lem. 113 we have that $\mu \approx \mu_1\,;\,\mu_2$.

◀

D.8 Completeness of flat permutation equivalence with respect to permutation equivalence

Before proving completeness, we need a few auxiliary results.

▶ **Lemma 115** (Generalized $\sim$-Assoc rule). *Let $\mu, \nu_1, \dots, \nu_n$ be multisteps where $n \geq 1$, and let $\mathtt{K}$ be a composition tree. Then:*
1. $\mu\,;\,\mathtt{K}\langle\nu_1,\nu_2,\dots,\nu_n\rangle \sim \mathtt{K}\langle(\mu\,;\,\nu_1),\nu_2,\dots,\nu_n\rangle$
2. $\mathtt{K}\langle\nu_1,\nu_2,\dots,\nu_n\rangle\,;\,\mu \sim \mathtt{K}\langle\nu_1,\nu_2,\dots,(\nu_n\,;\,\mu)\rangle$

Proof. We only prove item 1. (item 2. is similar). We proceed by induction on $\mathtt{K}$:
1. **Empty,** $\mathtt{K} = \Box$**.** Then $\mu\,;\,\nu_1 \sim \mu\,;\,\nu_1$ by reflexivity.

2. **Composition,** $\mathtt{K} = \mathtt{K}_1 \mathbin{;} \mathtt{K}_2$**.** Then $n > 1$ and there is an index $1 \le i \le n$ such that $\mathtt{K}\langle \nu_1, \ldots, \nu_n\rangle = \mathtt{K}_1\langle \nu_1, \nu_2, \ldots, \nu_i\rangle \mathbin{;} \mathtt{K}_2\langle \nu_{i+1}, \ldots, \nu_n\rangle$. Hence:

$$\begin{array}{rcll} \mu \mathbin{;} \mathtt{K}\langle \nu_1, \ldots, \nu_n\rangle & = & \mu \mathbin{;} (\mathtt{K}_1\langle \nu_1, \nu_2, \ldots, \nu_i\rangle \mathbin{;} \mathtt{K}_2\langle \nu_{i+1}, \ldots, \nu_n\rangle) & \\ & \sim & (\mu \mathbin{;} \mathtt{K}_1\langle \nu_1, \nu_2, \ldots, \nu_i\rangle) \mathbin{;} \mathtt{K}_2\langle \nu_{i+1}, \ldots, \nu_n\rangle & \text{by } \sim\text{-Assoc} \\ & \sim & \mathtt{K}_1\langle (\mu \mathbin{;} \nu_1), \nu_2, \ldots, \nu_i\rangle \mathbin{;} \mathtt{K}_2\langle \nu_{i+1}, \ldots, \nu_n\rangle & \text{by IH} \\ & = & \mathtt{K}\langle (\mu \mathbin{;} \nu_1), \nu_2, \ldots, \nu_n\rangle & \end{array}$$

◀

▶ **Lemma 116** (Left/right splitting)**.** *Let μ be a multistep. Then:*
1. $\mu \Leftrightarrow \mu \mathbin{;} \underline{\mu^{\mathsf{tgt}}}$
2. $\mu \Leftrightarrow \underline{\mu^{\mathsf{src}}} \mathbin{;} \mu$

Proof. We only prove item 1. (item 2. is similar). We proceed by induction on μ:
1. **Variable,** $\mu = x$**.** By SVar, $x \Leftrightarrow x \mathbin{;} x$.
2. **Constant,** $\mu = \mathbf{c}$**.** By SCon, $\mathbf{c} \Leftrightarrow \mathbf{c} \mathbin{;} \mathbf{c}$.
3. **Rule symbol,** $\mu = \varrho$**.** By SRuleL, $\varrho \Leftrightarrow \varrho \mathbin{;} \underline{\varrho^{\mathsf{tgt}}}$.
4. **Abstraction,** $\mu = \lambda x.\nu$**.** By IH $\nu \Leftrightarrow \nu \mathbin{;} \underline{\nu^{\mathsf{tgt}}}$ so by SAbs, $\lambda x.\nu \Leftrightarrow \lambda x.\nu \mathbin{;} \lambda x.\underline{\nu^{\mathsf{tgt}}}$.
5. **Application,** $\mu = \nu_1\,\nu_2$**.** By IH $\nu_1 \Leftrightarrow \nu_1 \mathbin{;} \underline{\nu_1^{\mathsf{tgt}}}$ and $\nu_2 \Leftrightarrow \nu_2 \mathbin{;} \underline{\nu_2^{\mathsf{tgt}}}$ so $\nu_1\,\nu_2 \Leftrightarrow \nu_1\,\nu_2 \mathbin{;} \underline{\nu_1^{\mathsf{tgt}}}\,\underline{\nu_2^{\mathsf{tgt}}}$.

◀

▶ **Lemma 117** (Free variables of splitting)**.** *If $\mu \Leftrightarrow \mu_1 \mathbin{;} \mu_2$ then $\mathsf{fv}(\mu) = \mathsf{fv}(\mu_1) \cup \mathsf{fv}(\mu_2)$.*

Proof. Straightforward by induction on the derivation of $\mu \Leftrightarrow \mu_1 \mathbin{;} \mu_2$. The interesting cases are the SRuleL and SRuleR rules. For example, for the SRuleL case, note that $\mathsf{fv}(\varrho) = \mathsf{fv}(\varrho) \cup \mathsf{fv}(\varrho^{\mathsf{tgt}})$ given that $\mathsf{fv}(\varrho^{\mathsf{tgt}}) = \varnothing$, as the source and the target of a given rule symbol are closed terms. ◀

▶ **Lemma 118** (Splitting commutes with substitution)**.** *If $\mu \Leftrightarrow \mu_1 \mathbin{;} \mu_2$ and $\nu \Leftrightarrow \nu_1 \mathbin{;} \nu_2$ then $\mu\{x\backslash\nu\} \Leftrightarrow \mu_1\{x\backslash\nu_1\} \mathbin{;} \mu_2\{x\backslash\nu_2\}$.*

Proof. By induction on the derivation of $\mu \Leftrightarrow \mu_1 \mathbin{;} \mu_2$:
1. SVar: Let $y \Leftrightarrow y \mathbin{;} y$. If $x \neq y$, it is immediate. If $x = y$, then indeed $\nu \Leftrightarrow \nu_1 \mathbin{;} \nu_2$.
2. SCon: Immediate.
3. SRuleL: Let $\varrho \Leftrightarrow \varrho \mathbin{;} \underline{\varrho^{\mathsf{tgt}}}$. Recall that the target of a rule symbol is always a closed term, so $\underline{\varrho^{\mathsf{tgt}}}\{x\backslash\xi\} = \underline{\varrho^{\mathsf{tgt}}}$. Then it is immediate, given that $\varrho \Leftrightarrow \varrho \mathbin{;} \underline{\varrho^{\mathsf{tgt}}}$.
4. SRuleR: Similar to the previous case.
5. SAbs: Let $\lambda y.\mu \Leftrightarrow \lambda y.\mu_1 \mathbin{;} \lambda y.\mu_2$ be derived from $\mu \Leftrightarrow \mu_1 \mathbin{;} \mu_2$. Then by IH we have that $\mu\{x\backslash\nu\} \Leftrightarrow \mu_1\{x\backslash\nu_1\} \mathbin{;} \mu_2\{x\backslash\nu_2\}$, so applying the SAbs rule $\lambda y.\mu\{x\backslash\nu\} \Leftrightarrow \lambda y.\mu_1\{x\backslash\nu_1\} \mathbin{;} \lambda y.\mu_2\{x\backslash\nu_2\}$.
6. SApp: Let $\mu_1\,\mu_2 \Leftrightarrow \mu_{11}\,\mu_{21} \mathbin{;} \mu_{12}\,\mu_{22}$ be derived from $\mu_1 \Leftrightarrow \mu_{11} \mathbin{;} \mu_{12}$ and $\mu_1 \Leftrightarrow \mu_{21} \mathbin{;} \mu_{22}$. Then by IH we have that $\mu_1\{x\backslash\nu\} \Leftrightarrow \mu_{11}\{x\backslash\nu_1\} \mathbin{;} \mu_{12}\{x\backslash\nu_2\}$ and $\mu_2\{x\backslash\nu\} \Leftrightarrow \mu_{21}\{x\backslash\nu_1\} \mathbin{;} \mu_{22}\{x\backslash\nu_2\}$, so applying the SApp rule $(\mu_1\,\mu_2)\{x\backslash\nu\} \Leftrightarrow (\mu_{11}\,\mu_{21})\{x\backslash\nu_1\} \mathbin{;} (\mu_{12}\,\mu_{22})\{x\backslash\nu_2\}$.

◀

▶ **Lemma 119** (Coherence of splitting and flattening)**.** *Let μ, μ_1, μ_2 be multisteps not necessarily in normal form, and suppose that $\mu \Leftrightarrow \mu_1 \mathbin{;} \mu_2$. Then $\mu^\flat \Leftrightarrow \mu_1' \mathbin{;} \mu_2'$ where μ_1' and μ_2' are such that $\mu_1 \overset{\flat}{\mapsto}{}^* \mu_1'$ and $\mu_2 \overset{\flat}{\mapsto}{}^* \mu_2'$.*

Proof. It suffices to show that if $\mu \overset{\flat}{\mapsto} \nu$ then there exist multisteps ν_1 and ν_2 such that $\mu_1 \overset{\flat}{\mapsto}^* \nu_1$ and $\mu_2 \overset{\flat}{\mapsto}^* \nu_2$ and $\nu \Leftrightarrow \nu_1 \mathbin{;} \nu_2$. With this property, the proof of the lemma is immediate by induction on the length of a reduction to normal form $\mu \overset{\flat}{\mapsto}^* \mu^\flat$.

We proceed by induction on μ. If μ is a variable, a constant, or a rule symbol, it is immediate as there cannot be a reduction step $\mu \overset{\flat}{\mapsto} \nu$. There are two remaining cases:

1. **Abstraction, $\mu = \lambda x.\xi$.** Then note that $\mu \Leftrightarrow \mu_1 \mathbin{;} \mu_2$ must be derived using the SAbs rule, so $\xi \Leftrightarrow \xi_1 \mathbin{;} \xi_2$ where $\mu_1 = \lambda x.\xi_1$ and $\mu_2 = \lambda x.\xi_2$. We consider two subcases, depending on whether the step $\mu = \lambda x.\xi \overset{\flat}{\mapsto} \nu$, is internal to ξ or an $\mathcal{F}$-EtaM step at the root:
   - **1.1** If the step is internal to ξ, *i.e.* $\xi \overset{\flat}{\mapsto} \psi$ and $\nu = \lambda x.\psi$, then by IH we have that $\psi \Leftrightarrow \psi_1 \mathbin{;} \psi_2$ such that $\xi_1 \overset{\flat}{\mapsto}^* \psi_1$ and $\xi_2 \overset{\flat}{\mapsto}^* \psi_2$. Therefore, by the SAbs rule, $\nu \Leftrightarrow \lambda x.\psi_1 \mathbin{;} \lambda x.\psi_2$ where $\mu_1 = \lambda x.\xi_1 \overset{\flat}{\mapsto}^* \lambda x.\psi_1$ and $\mu_2 = \lambda x.\xi_2 \overset{\flat}{\mapsto}^* \lambda x.\psi_2$, as required.
   - **1.2** If the step is an $\mathcal{F}$-EtaM step at the root, *i.e.* $\xi = \nu\, x$ with $x \notin \mathsf{fv}(\nu)$, then note that $\xi \Leftrightarrow \xi_1 \mathbin{;} \xi_2$ must be derived using the SApp rule, so $\nu \Leftrightarrow \nu_1 \mathbin{;} \nu_2$ where $\xi_1 = \nu_1\, x$ and $\xi_2 = \nu_2\, x$. To conclude, note that $\mu_1 = \lambda x.\nu_1\, x \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}} \nu_1$ and $\mu_2 = \lambda x.\nu_2\, x \overset{\flat}{\mapsto}_{\mathcal{F}\text{-EtaM}} \nu_2$ noting that $\mathsf{fv}(\nu_1), \mathsf{fv}(\nu_2) \subseteq \mathsf{fv}(\nu)$ by Lem. 117.
2. **Application, $\mu = \xi_1\, \xi_2$.** Then note that $\mu \Leftrightarrow \mu_1 \mathbin{;} \mu_2$ must be derived using the SApp rule, so $\xi_1 \Leftrightarrow \xi_{11} \mathbin{;} \xi_{12}$ and $\xi_2 \Leftrightarrow \xi_{21} \mathbin{;} \xi_{22}$ where $\mu_1 = \xi_{11}\, \xi_{21}$ and $\mu_2 = \xi_{12}\, \xi_{22}$. We consider three subcases, depending on whether the step $\mu = \xi_1\, \xi_2 \overset{\flat}{\mapsto} \nu$ is internal to ξ_1, internal to ξ_2, or a $\mathcal{F}$-Beta step at the root:
   - **2.1** If the step is internal to ξ_1, *i.e.* $\xi_1 \overset{\flat}{\mapsto} \psi_1$ and $\nu = \psi_1\, \xi_2$, then by IH we have that $\psi_1 \Leftrightarrow \psi_{11} \mathbin{;} \psi_{12}$ such that $\xi_{11} \overset{\flat}{\mapsto}^* \psi_{11}$ and $\xi_{12} \overset{\flat}{\mapsto}^* \psi_{12}$. Therefore, by the SApp rule, $\nu \Leftrightarrow \psi_{11}\, \xi_{21} \mathbin{;} \psi_{12}\, \xi_{22}$ where $\mu_1 = \xi_{11}\, \xi_{21} \overset{\flat}{\mapsto}^* \psi_{11}\, \xi_{21}$ and $\mu_2 = \xi_{12}\, \xi_{22} \overset{\flat}{\mapsto}^* \psi_{12}\, \xi_{22}$.
   - **2.2** If the step is internal to ξ_2, *i.e.* $\xi_2 \overset{\flat}{\mapsto} \psi_2$ then by IH we have that $\psi_2 \Leftrightarrow \psi_{21} \mathbin{;} \psi_{22}$ such that $\xi_{21} \overset{\flat}{\mapsto}^* \psi_{21}$ and $\xi_{22} \overset{\flat}{\mapsto}^* \psi_{22}$. Therefore, by the SApp rule, $\nu \Leftrightarrow \xi_{11}\, \psi_{21} \mathbin{;} \xi_{12}\, \psi_{22}$ where $\mu_1 = \xi_{11}\, \xi_{21} \overset{\flat}{\mapsto}^* \xi_{11}\, \psi_{21}$ and $\mu_2 = \xi_{12}\, \xi_{22} \overset{\flat}{\mapsto}^* \xi_{12}\, \psi_{22}$.
   - **2.3** If the step is a $\mathcal{F}$-Beta step at the root, *i.e.* the step is of the form $\mu = (\lambda x.\xi_1')\, \xi_2 \overset{\flat}{\mapsto} \xi_1'\{x\backslash\xi_2\} = \nu$ with $\xi_1 = \lambda x.\xi_1'$, then note that $\xi_1 \Leftrightarrow \xi_{11} \mathbin{;} \xi_{12}$ must be derived using the SAbs rule, so $\xi_1' \Leftrightarrow \xi_{11}' \mathbin{;} \xi_{12}'$ with $\xi_{11} = \lambda x.\xi_{11}'$ and $\xi_{12} = \lambda x.\xi_{12}'$. Then by Lem. 118 we have that $\nu \Leftrightarrow \xi_{11}'\{x\backslash\xi_{21}\} \mathbin{;} \xi_{12}'\{x\backslash\xi_{22}\}$ where, moreover, we have that $\mu_1 = (\lambda x.\xi_{11}')\, \xi_{21} \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Beta}} \xi_{11}'\{x\backslash\xi_{21}\}$ and $\mu_2 = (\lambda x.\xi_{12}')\, \xi_{22} \overset{\flat}{\mapsto}_{\mathcal{F}\text{-Beta}} \xi_{12}'\{x\backslash\xi_{22}\}$.

◀

▶ **Lemma 120** (Canonical $\mapsto^\circ, \overline{\eta}$-normal splitting). *If $\mu_1 \Leftrightarrow \mu_2 \mathbin{;} \mu_3$ then there exist μ_1', μ_2', μ_3' such that $\mu_1' \Leftrightarrow \mu_2' \mathbin{;} \mu_3'$ where $\mu_1^\flat = (\mu_1')^\flat$ and $\mu_2^\flat = (\mu_2')^\flat$ and $\mu_3^\flat = (\mu_3')^\flat$, and moreover μ' is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form.*

Proof. By Lem. 119, we know that $\mu_1'' \Leftrightarrow \mu_2'' \mathbin{;} \mu_3''$ where $\mu_1'' = \mu_1^\flat$ and $\mu_2 \overset{\flat}{\mapsto} \mu_2''$ and $\mu_3 \overset{\flat}{\mapsto} \mu_3''$. By induction on the shape of μ_1'', it suffices to show that there exist μ_1', μ_2', μ_3' such that $\mu_1' \Leftrightarrow \mu_2' \mathbin{;} \mu_3'$ where $(\mu_1')^\flat = (\mu_1'')^\flat$ and $(\mu_2')^\flat = (\mu_2'')^\flat$ and $(\mu_3')^\flat = (\mu_3'')^\flat$, and moreover μ_1' is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form:

1. **μ_1'' headed by a variable.** Then $\mu_1'' = \lambda x_1 \ldots x_n.y\, \mu_{11}'' \ldots \mu_{1m}''$ and $\mu_2'' = \lambda x_1 \ldots x_n.y\, \mu_{21}'' \ldots \mu_{2m}''$ and $\mu_3'' = \lambda x_1 \ldots x_n.y\, \mu_{31}'' \ldots \mu_{3m}''$ where $\mu_{1i}'' \Leftrightarrow \mu_{2i}'' \mathbin{;} \mu_{3i}''$ for each $1 \leq i \leq m$. By IH there are multisteps such that $\mu_{1i}' \Leftrightarrow \mu_{2i}' \mathbin{;} \mu_{3i}'$ where $(\mu_{1i}'')^\flat = (\mu_{1i}')^\flat$ and $(\mu_{2i}'')^\flat = (\mu_{2i}')^\flat$ and

$(\mu''_{3i})^\flat = (\mu'_{3i})^\flat$, and moreover μ'_{1i} is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form. Suppose that μ''_1 is of arity N, *i.e.* that its type is of the form $A_1 \to \ldots \to A_N \to \alpha$ with α a base type. Take:

$$\begin{array}{lll} \mu'_1 & := & \lambda x_1 \ldots x_n x_{n+1} \ldots x_N . y\, \mu'_{11} \ldots \mu'_{1m}\, x_{n+1} \ldots x_N \\ \mu'_2 & := & \lambda x_1 \ldots x_n x_{n+1} \ldots x_N . y\, \mu'_{21} \ldots \mu'_{2m}\, x_{n+1} \ldots x_N \\ \mu'_3 & := & \lambda x_1 \ldots x_n x_{n+1} \ldots x_N . y\, \mu'_{31} \ldots \mu'_{3m}\, x_{n+1} \ldots x_N \end{array}$$

Then it is straightforward to check that $\mu'_1 \Leftrightarrow \mu'_2 \,;\, \mu'_3$ and $(\mu'_1)^\flat = (\mu''_1)^\flat$ and $(\mu'_2)^\flat = (\mu''_2)^\flat$ and $(\mu'_3)^\flat = (\mu''_3)^\flat$, and moreover μ'_1 is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form.

2. μ''_1 **headed by a constant.** Similar to the previous case.
3. μ''_1 **headed by a rule symbol.** Similar to the previous case.

◀

▶ **Proposition 121** (Generalized $\sim$-Perm rule). *If $\mu \Leftrightarrow \mu_1 \,;\, \mu_2$ then $\mu^\flat \sim \mu_1^\flat \,;\, \mu_2^\flat$.*
Note that this generalizes the $\sim$-Perm *rule, which requires μ to be in $\overset{\flat}{\mapsto}$-normal form.*

Proof. By the previous coherence lemma (Lem. 119), we have that $\mu^\flat \Leftrightarrow \mu'_1 \,;\, \mu'_2$ such that $\mu_1 \overset{\flat}{\mapsto}^* \mu'_1$ and $\mu_2 \overset{\flat}{\mapsto}^* \mu'_2$. By the $\sim$-Perm rule, $\mu^\flat \sim (\mu'_1)^\flat \,;\, (\mu'_2)^\flat$. Moreover, by strong normalization (Prop. 95) and confluence (Prop. 96) of flattening, $\mu_1^\flat = (\mu'_1)^\flat$ and $\mu_2^\flat = (\mu'_2)^\flat$, which means that $\mu^\flat \sim \mu_1^\flat \,;\, \mu_2^\flat$. ◀

▶ **Lemma 122** (Swap).
1. $(\mu\,\nu)^\flat \sim (\underline{\mu^{\mathsf{src}}}\,\nu)^\flat \,;\, (\mu\,\underline{\nu^{\mathsf{tgt}}})^\flat$
2. $(\mu\,\nu)^\flat \sim (\mu\,\underline{\nu^{\mathsf{src}}})^\flat \,;\, (\underline{\mu^{\mathsf{tgt}}}\,\nu)^\flat$

In particular, combining items 1. and 2. one has:

$$(\underline{\mu^{\mathsf{src}}}\,\nu)^\flat \,;\, (\mu\,\underline{\nu^{\mathsf{tgt}}})^\flat \sim (\mu\,\underline{\nu^{\mathsf{src}}})^\flat \,;\, (\underline{\mu^{\mathsf{tgt}}}\,\nu)^\flat$$

Proof. For item 1. note that, by Prop. 121, it suffices to show that $\mu\,\nu \Leftrightarrow \underline{\mu^{\mathsf{src}}}\,\nu \,;\, \mu\,\underline{\nu^{\mathsf{tgt}}}$. Indeed, by Lem. 116 we have that $\mu \Leftrightarrow \underline{\mu^{\mathsf{src}}} \,;\, \mu$ and that $\nu \Leftrightarrow \nu \,;\, \underline{\nu^{\mathsf{tgt}}}$, so by SApp $\mu\,\nu \Leftrightarrow \underline{\mu^{\mathsf{src}}}\,\nu \,;\, \mu\,\underline{\nu^{\mathsf{tgt}}}$. The proof of item 2. is symmetric. ◀

▶ **Lemma 123** (Generalized swap). *The following equivalence holds for arbitrary composition trees $\mathtt{K}_1, \mathtt{K}_2$ and arbitrary multisteps $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m$:*

$$\begin{array}{ll} & \mathtt{K}_1\langle(\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat, \ldots, (\mu_n\,\underline{\nu_1^{\mathsf{src}}})^\flat\rangle \,;\, \mathtt{K}_2\langle(\underline{\mu_n^{\mathsf{tgt}}}\,\nu_1)^\flat, \ldots, (\underline{\mu_n^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \\ \sim & \mathtt{K}_2\langle(\underline{\mu_1^{\mathsf{src}}}\,\nu_1)^\flat, \ldots, (\underline{\mu_1^{\mathsf{src}}}\,\nu_m)^\flat\rangle \,;\, \mathtt{K}_1\langle(\mu_1\,\underline{\nu_m^{\mathsf{tgt}}})^\flat, \ldots, (\mu_n\,\underline{\nu_m^{\mathsf{tgt}}})^\flat\rangle \end{array}$$

Proof. We proceed by induction on $\mathtt{K}_1$. To alleviate the notation we use the associativity rule ($\sim$-Assoc) implicitly.

1. **Empty,** $\mathtt{K}_1 = \Box$**.** Then $n = 1$. We proceed by a nested induction on $\mathtt{K}_2$:

   1.1 **Empty,** $\mathtt{K}_2 = \Box$**.** Then $m = 1$ and the following equivalence holds by Lem. 122:

   $$(\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat \,;\, (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_1)^\flat \sim (\underline{\mu_1^{\mathsf{src}}}\,\nu_1)^\flat \,;\, (\mu_1\,\underline{\nu_1^{\mathsf{tgt}}})^\flat$$

   1.2 **Composition,** $\mathtt{K}_2 = \mathtt{K}_{21} \,;\, \mathtt{K}_{22}$**.** Then $m > 1$ and there is an index $1 \leq j \leq m$ such that $\mathtt{K}_{21}$ has j holes and $\mathtt{K}_{22}$ has $m-j$ holes. Then:

$$
\begin{array}{cl}
 & (\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat \,;\, \mathtt{K}_2\langle(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \\
= & (\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat \,;\, \mathtt{K}_{21}\langle(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_j)^\flat\rangle \,;\, \mathtt{K}_{22}\langle(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_{j+1})^\flat,\ldots,(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \\
\sim & \mathtt{K}_{21}\langle(\underline{\mu_1^{\mathsf{src}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_1^{\mathsf{src}}}\,\nu_j)^\flat\rangle \,;\, (\mu_1\,\underline{\nu_j^{\mathsf{tgt}}})^\flat \,;\, \mathtt{K}_{22}\langle(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_{j+1})^\flat,\ldots,(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \\
 & \quad\text{by IH} \\
= & \mathtt{K}_{21}\langle(\underline{\mu_1^{\mathsf{src}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_1^{\mathsf{src}}}\,\nu_j)^\flat\rangle \,;\, (\mu_1\,\underline{\nu_{j+1}^{\mathsf{src}}})^\flat \,;\, \mathtt{K}_{22}\langle(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_{j+1})^\flat,\ldots,(\underline{\mu_1^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \\
 & \quad\text{as } \nu_j^{\mathsf{tgt}} =_{\beta\eta} \nu_{j+1}^{\mathsf{src}} \\
\sim & \mathtt{K}_{21}\langle(\underline{\mu_1^{\mathsf{src}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_1^{\mathsf{src}}}\,\nu_j)^\flat\rangle \,;\, \mathtt{K}_{22}\langle(\underline{\mu_1^{\mathsf{src}}}\,\nu_{j+1})^\flat,\ldots,(\underline{\mu_1^{\mathsf{src}}}\,\nu_m)^\flat\rangle \,;\, (\mu_1\,\underline{\nu_m^{\mathsf{tgt}}})^\flat \\
 & \quad\text{by IH} \\
= & \mathtt{K}_2\langle(\underline{\mu_1^{\mathsf{src}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_1^{\mathsf{src}}}\,\nu_m)^\flat\rangle \,;\, (\mu_1\,\underline{\nu_m^{\mathsf{tgt}}})^\flat
\end{array}
$$

2. **Composition,** $\mathtt{K}_1 = \mathtt{K}_{11}\,;\,\mathtt{K}_{12}$**.** Then $n > 1$ and there is an index $1 \le i \le n$ such that $\mathtt{K}_{11}$ has i holes and $\mathtt{K}_{12}$ has $n-i$ holes. Then:

$$
\begin{array}{cl}
 & \mathtt{K}_1\langle(\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat,\ldots,(\mu_n\,\underline{\nu_1^{\mathsf{src}}})^\flat\rangle \,;\, \mathtt{K}_2\langle(\underline{\mu_n^{\mathsf{tgt}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_n^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \\
= & \mathtt{K}_{11}\langle(\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat,\ldots,(\mu_i\,\underline{\nu_1^{\mathsf{src}}})^\flat\rangle \,; \\
 & \mathtt{K}_{12}\langle(\mu_{i+1}\,\underline{\nu_1^{\mathsf{src}}})^\flat,\ldots,(\mu_n\,\underline{\nu_1^{\mathsf{src}}})^\flat\rangle \,;\, \mathtt{K}_2\langle(\underline{\mu_n^{\mathsf{tgt}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_n^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \\
\sim & \mathtt{K}_{11}\langle(\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat,\ldots,(\mu_i\,\underline{\nu_1^{\mathsf{src}}})^\flat\rangle \,; \\
 & \mathtt{K}_2\langle(\underline{\mu_{i+1}^{\mathsf{src}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_{i+1}^{\mathsf{src}}}\,\nu_m)^\flat\rangle \,;\, \mathtt{K}_{12}\langle(\mu_{i+1}\,\underline{\nu_m^{\mathsf{tgt}}})^\flat,\ldots,(\mu_n\,\underline{\nu_m^{\mathsf{tgt}}})^\flat\rangle \\
 & \quad\text{by IH} \\
= & \mathtt{K}_{11}\langle(\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat,\ldots,(\mu_i\,\underline{\nu_1^{\mathsf{src}}})^\flat\rangle \,; \\
 & \mathtt{K}_2\langle(\underline{\mu_i^{\mathsf{tgt}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_i^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \,;\, \mathtt{K}_{12}\langle(\mu_{i+1}\,\underline{\nu_m^{\mathsf{tgt}}})^\flat,\ldots,(\mu_n\,\underline{\nu_m^{\mathsf{tgt}}})^\flat\rangle \\
 & \quad\text{since } \mu_i^{\mathsf{tgt}} =_{\beta\eta} \mu_{i+1}^{\mathsf{src}} \\
\sim & \mathtt{K}_2\langle(\underline{\mu_1^{\mathsf{src}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_1^{\mathsf{src}}}\,\nu_m)^\flat\rangle \,; \\
 & \mathtt{K}_{11}\langle(\mu_1\,\underline{\nu_m^{\mathsf{tgt}}})^\flat,\ldots,(\mu_i\,\underline{\nu_m^{\mathsf{tgt}}})^\flat\rangle \,;\, \mathtt{K}_{12}\langle(\mu_{i+1}\,\underline{\nu_m^{\mathsf{tgt}}})^\flat,\ldots,(\mu_n\,\underline{\nu_m^{\mathsf{tgt}}})^\flat\rangle \\
 & \quad\text{by IH} \\
= & \mathtt{K}_2\langle(\underline{\mu_1^{\mathsf{src}}}\,\nu_1)^\flat,\ldots,(\underline{\mu_1^{\mathsf{src}}}\,\nu_m)^\flat\rangle \,;\, \mathtt{K}_1\langle(\mu_1\,\underline{\nu_m^{\mathsf{tgt}}})^\flat,\ldots,(\mu_n\,\underline{\nu_m^{\mathsf{tgt}}})^\flat\rangle
\end{array}
$$

◀

▶ **Lemma 124** (Flattening of an application, up to $\sim$)**.** *Let* $\rho = \mathtt{K}_1\langle\mu_1,\ldots,\mu_n\rangle$ *and* $\sigma = \mathtt{K}_2\langle\nu_1,\ldots,\nu_m\rangle$ *be flat rewrites. Then*

$$\rho\,\sigma \;\overset{\flat}{\mapsto}{}^*\!\sim\; \mathtt{K}_1\langle(\mu_1\,\nu_1^{\blacktriangleleft})^\flat,\ldots,(\mu_n\,\nu_1^{\blacktriangleleft})^\flat\rangle \,;\, \mathtt{K}_2\langle(\mu_n^{\blacktriangleright}\,\nu_1)^\flat,\ldots,(\mu_n^{\blacktriangleright}\,\nu_m)^\flat\rangle$$

Proof. We consider four subcases, depending on whether $n = 1$ or $n > 1$, and on whether $m = 1$ or $m > 1$:

1. If $n = 1$ and $m = 1$, then:

$$
\begin{array}{cll}
\rho\,\sigma & \overset{\flat}{\mapsto}{}^* & (\mu_1\,\nu_1)^\flat \\
 & \sim & (\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat \,;\, (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_1)^\flat \quad \text{by Lem. 122} \\
 & = & (\mu_1\,\nu_1^{\blacktriangleleft})^\flat \,;\, (\mu_1^{\blacktriangleright}\,\nu_1)^\flat \quad \text{by confluence of flattening (Prop. 96)}
\end{array}
$$

2. If $n = 1$ and $m > 1$, then:

$$\begin{array}{ll} & \rho\,\sigma \\ \overset{\flat}{\mapsto}^* & \mu_1\,\mathtt{K}_2\langle \nu_1, \nu_2, \ldots, \nu_m\rangle \\ \overset{\flat}{\mapsto}^* & \mathtt{K}_2\langle \mu_1\,\nu_1, \underline{\mu_1^{\mathsf{tgt}}}\,\nu_2, \ldots, \underline{\mu_1^{\mathsf{tgt}}}\,\nu_m\rangle \\ & \quad \text{by generalized } \mathcal{F}\text{-App2 (Lem. 106)} \\ \overset{\flat}{\mapsto}^* & \mathtt{K}_2\langle (\mu_1\,\nu_1)^\flat, (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_2)^\flat, \ldots, (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \\ \sim & \mathtt{K}_2\langle ((\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat \,\boldsymbol{;}\, (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_1)^\flat), (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_2)^\flat, \ldots, (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \quad \text{by Lem. 122} \\ \sim & (\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat \,\boldsymbol{;}\, \mathtt{K}_2\langle (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_1)^\flat, (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_2)^\flat, \ldots, (\underline{\mu_1^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \quad \text{by Lem. 115} \\ = & (\mu_1\,\nu_1^{\blacktriangleleft})^\flat \,\boldsymbol{;}\, \mathtt{K}_2\langle (\mu_1^{\blacktriangleright}\,\nu_1)^\flat, (\mu_1^{\blacktriangleright}\,\nu_2)^\flat, \ldots, (\mu_1^{\blacktriangleright}\,\nu_m)^\flat\rangle \quad \text{by confluence of flattening (Prop. 96)} \end{array}$$

3. If $n > 1$ and $m = 1$, the proof is symmetric to the previous case.

4. If $n > 1$ and $m > 1$, then:

$$\begin{array}{lll} \rho\,\sigma & \overset{\flat}{\mapsto}^* & \mathtt{K}_1\langle \mu_1, \ldots, \mu_n\rangle\,\mathtt{K}_2\langle \nu_1, \ldots, \nu_m\rangle \\ & \overset{\flat}{\mapsto}^* & \mathtt{K}_1\langle \mu_1\,\underline{\nu_1^{\mathsf{src}}}, \ldots, \mu_n\,\underline{\nu_1^{\mathsf{src}}}\rangle \,\boldsymbol{;}\, \mathtt{K}_2\langle \underline{\mu_n^{\mathsf{tgt}}}\,\nu_1, \ldots, \underline{\mu_n^{\mathsf{tgt}}}\,\nu_m\rangle \\ & & \quad \text{by generalized } \mathcal{F}\text{-App3 (Lem. 106)} \\ & \overset{\flat}{\mapsto}^* & \mathtt{K}_1\langle (\mu_1\,\underline{\nu_1^{\mathsf{src}}})^\flat, \ldots, (\mu_n\,\underline{\nu_1^{\mathsf{src}}})^\flat\rangle \,\boldsymbol{;}\, \mathtt{K}_2\langle (\underline{\mu_n^{\mathsf{tgt}}}\,\nu_1)^\flat, \ldots, (\underline{\mu_n^{\mathsf{tgt}}}\,\nu_m)^\flat\rangle \\ & = & \mathtt{K}_1\langle (\mu_1\,\nu_1^{\blacktriangleleft})^\flat, \ldots, (\mu_n\,\nu_1^{\blacktriangleleft})^\flat\rangle \,\boldsymbol{;}\, \mathtt{K}_2\langle (\mu_n^{\blacktriangleright}\,\nu_1)^\flat, \ldots, (\mu_n^{\blacktriangleright}\,\nu_m)^\flat\rangle \\ & & \quad \text{by confluence of flattening (Prop. 96)} \end{array}$$

◀

▶ **Lemma 125** (Composition of a term with itself). *If s is a term such that $\underline{s}$ is in $\overset{\flat}{\mapsto}$-normal form, then $\underline{s} \sim \mathtt{K}\langle \underline{s}, \ldots, \underline{s}\rangle$ for any composition tree $\mathtt{K}$.*

Proof. By induction on $\mathtt{K}$. The key observation is that $\underline{s} \sim \underline{s} \,\boldsymbol{;}\, \underline{s}$ given that $s \Leftrightarrow s \,\boldsymbol{;}\, s$, as can be checked easily by induction on s. ◀

▶ **Lemma 126** (Arbitrary association). $\mathtt{K}\langle \rho_1, \rho_2, \ldots, \rho_n\rangle \sim \rho_1 \,\boldsymbol{;}\, \rho_2 \ldots \boldsymbol{;}\, \rho_n$ *where, on the right hand side, we assume that* **;** *is right-associative.*

Proof. Straightforward by induction on $\mathtt{K}$ using the $\sim$-Assoc rule. ◀

▶ **Lemma 127** (Equivalence for term/rewrite substitution of a composition). *Let s be a term, let $\mu_1, \ldots, \mu_n$ arbitrary multisteps, and let $\mathtt{K}$ a composition tree. Then:*

$$s\{x \backslash\!\backslash \mathtt{K}\langle \mu_1, \ldots, \mu_n\rangle\} \overset{\flat}{\mapsto}^* \sim \mathtt{K}\langle s\{x \backslash\!\backslash \mu_1\}, \ldots, s\{x \backslash\!\backslash \mu_n\}\rangle^\flat$$

Proof. If s has no free occurrences of x, the result holds trivially by Lem. 125. The interesting case is when s has $m > 0$ free occurrences of x. Following the notation of Lem. 107, let $s = s\langle x, \ldots, x\rangle$ where, by abuse of notation, we write s for the term itself and also for an m-hole context that does not bind x.

Recall from the notation introduced in Lem. 107 that given indices $1 \leq i \leq m$ and $1 \leq j \leq n$ we write $s\langle \mu_j\rangle_i^x$ to stand for the rewrite that results from substituting the i-th free occurrence of x by μ_j, the free occurrences of x at positions $i' < i$ by $\underline{\mu_n^{\mathsf{tgt}}}$, and the free occurrences of x at positions $i' > i$ by $\underline{\mu_1^{\mathsf{src}}}$, *i.e.*

$$s\langle \mu_j\rangle_i^x = s\langle \underline{\mu_n^{\mathsf{tgt}}}, \ldots, \underline{\mu_n^{\mathsf{tgt}}}, \underbrace{\mu_j}_{(i\text{-th position})}, \underline{\mu_1^{\mathsf{src}}}, \ldots, \underline{\mu_1^{\mathsf{src}}}\rangle$$

In order to prove this lemma, we first prove two auxiliary results. Intuitively, the first result allows one to swap consecutive multisteps that perform work at the positions of two different free occurrences of x. The second result allows one to join consecutive multisteps performing the same work for every position of x. Before we need to introduce some auxiliary notation:

Notation. Given a sequence of m indices $(j_1, \dots, j_m)$ such that $0 \leq j_k \leq n$ for all $1 \leq k \leq m$ and an index $1 \leq i \leq m$ such that $j_i > 0$, we write $\mathsf{r}^i_{(j_1,\dots,j_m)}$ and $\mathsf{R}^i_{(j_1,\dots,j_m)}$ for the following multisteps:

$$\begin{array}{lcl} \mathsf{r}^i_{(j_1,\dots,j_m)} & \stackrel{\text{def}}{=} & s\langle \underline{\mu^{\mathsf{tgt}}_{j_1}}, \dots, \underline{\mu^{\mathsf{tgt}}_{j_{i-1}}}, \underbrace{\mu_{j_i}}_{(i\text{-th position})}, \underline{\mu^{\mathsf{tgt}}_{j_{i+1}}}, \dots, \underline{\mu^{\mathsf{tgt}}_{j_m}} \rangle \\ \mathsf{R}^i_{(j_1,\dots,j_m)} & \stackrel{\text{def}}{=} & (\mathsf{r}^i_{(j_1,\dots,j_m)})^\flat \end{array}$$

where by convention $\mu^{\mathsf{tgt}}_0 := \mu^{\mathsf{src}}_1$. Note that $(s\langle\mu_j\rangle^x_i)^\flat = \mathsf{R}^i_{(n,\dots,n,j,0,\dots,0)}$ with j in the i-th position. Intuitively, $\mathsf{r}^i_{(j_1,\dots,j_m)}$ represents the transition from a state in which, for each $1 \leq k \leq m$, the k-th free ocurrence of x has been replaced by the sequence $(\mu_1 \,;\, \dots \,;\, \mu_{j_k})$, and to a state in which the sequence in the i-th free occurrence of x has been extended with the multistep μ_{j_i+1}.

Swapping consecutive steps. Consecutive multisteps affecting different positions can be swapped. More precisely, we claim that if $1 \leq i < k \leq m$ then:

$$\mathsf{R}^i_{(j_1,\dots,j_m)} \,;\, \mathsf{R}^k_{(j_1,\dots,j_{(i-1)},j_i+1,j_{(i+1)},\dots,j_m)} \sim \mathsf{R}^k_{(j_1,\dots,j_m)} \,;\, \mathsf{R}^i_{(j_1,\dots,j_{(k-1)},j_k+1,j_{(k+1)},\dots,j_m)} \quad (4)$$

Indeed, if we let $\mathsf{r}^{(i,k)}_{(j_1,\dots,j_m)}$ denote the step which, intuitively, combines the computational works of $\mathsf{r}^i_{(j_1,\dots,j_m)}$ and $\mathsf{r}^k_{(j_1,\dots,j_m)}$:

$$\mathsf{r}^{(i,k)}_{(j_1,\dots,j_m)} \stackrel{\text{def}}{=} s\langle \underline{\mu^{\mathsf{tgt}}_{j_1}}, \dots, \underline{\mu^{\mathsf{tgt}}_{j_{i-1}}}, \mu_{j_i}, \underline{\mu^{\mathsf{tgt}}_{j_{i+1}}}, \dots, \underline{\mu^{\mathsf{tgt}}_{j_{k-1}}}, \mu_{j_k}, \underline{\mu^{\mathsf{tgt}}_{j_{k+1}}}, \underline{\mu^{\mathsf{tgt}}_{j_m}} \rangle$$

then we may justify equation (4) by applying the generalized $\sim$-Perm rule (Prop. 121) and observing that the two following splittings hold:

$$\mathsf{r}^{(i,k)}_{(j_1,\dots,j_m)} \Leftrightarrow \mathsf{r}^i_{(j_1,\dots,j_m)} \,;\, \mathsf{r}^k_{(j_1,\dots,j_{(i-1)},j_i+1,j_{(i+1)},\dots,j_m)}$$

$$\mathsf{r}^{(i,k)}_{(j_1,\dots,j_m)} \Leftrightarrow \mathsf{r}^k_{(j_1,\dots,j_m)} \,;\, \mathsf{r}^i_{(j_1,\dots,j_{(k-1)},j_k+1,j_{(k+1)},\dots,j_m)}$$

Joining consecutive steps. Consecutive multisteps performing the same computational work in the positions of all the free occurrences of x can be joined. More precisely, we claim that if $1 \leq i < n$ then:

$$\mathsf{R}^1_{(i+1,i,\dots,i)} \,;\, \mathsf{R}^2_{(i+1,i+1,i,\dots,i)} \,;\, \dots \,;\, \mathsf{R}^m_{(i+1,i+1,\dots,i+1)} \sim s\{x\backslash\mu_{i+1}\}^\flat \quad (5)$$

Indeed, if for each $0 \leq i < n$ and each $0 \leq j \leq m$ we let $\mathsf{r}^{0..j}_i$ denote the step which, intuitively, combines the computational works of the first j steps above:

$$\mathsf{r}^{0..j}_i \stackrel{\text{def}}{=} s\langle \underbrace{\mu_{(i+1)}, \dots, \mu_{(i+1)}}_{(j)}, \underbrace{\underline{\mu^{\mathsf{tgt}}_i}, \dots, \underline{\mu^{\mathsf{tgt}}_i}}_{(m-j)} \rangle$$

Then we may observe that the following splitting holds for every $0 \leq j < m$:

$$\mathsf{r}^{0..(j+1)}_i \Leftrightarrow \mathsf{r}^{0..j}_i \,;\, \mathsf{r}^{j+1}_{(\underbrace{i+1,\dots,i+1}_{(j+1)}, \underbrace{i,\dots,i}_{(m-j-1)})}$$

Thus applying the generalized $\sim$-Perm rule (Prop. 121), and using the $\sim$-Assoc rule implicitly, we have that:

$$
\begin{array}{lll}
 & \mathsf{R}^1_{(i+1,i,\dots,i)} \,;\, \mathsf{R}^2_{(i+1,i+1,i,\dots,i)} \,;\, \dots \,;\, \mathsf{R}^m_{(i+1,\dots,i+1,i+1)} & \\
= & (\mathsf{r}_i^{0..1})^\flat \,;\, \mathsf{R}^2_{(i+1,i+1,i,\dots,i)} \,;\, \dots \,;\, \mathsf{R}^m_{(i+1,\dots,i+1,i+1)} & \\
\sim & (\mathsf{r}_i^{0..2})^\flat \,;\, \mathsf{R}^3_{(i+1,i+1,i+1,i,\dots,i)} \,;\, \dots \,;\, \mathsf{R}^m_{(i+1,\dots,i+1,i+1)} & \text{by Prop. 121} \\
\dots & & \\
\sim & (\mathsf{r}_i^{0..j})^\flat \,;\, \mathsf{R}^{j+1}_{(\underbrace{i+1,\dots,i+1}_{(j+1)},\underbrace{i,\dots,i}_{(m-j-1)})} \,;\, \dots \,;\, \mathsf{R}^m_{(i+1,i+1,\dots,i+1,i)} & \text{by Prop. 121} \\
\dots & & \\
\sim & (\mathsf{r}_i^{0..m})^\flat & \text{by Prop. 121} \\
= & s\{x\backslash\!\!\backslash\mu_{i+1}\}^\flat &
\end{array}
$$

To conclude the proof of this lemma, using the $\sim$-Assoc rule implicitly, note that:

$$
\begin{array}{ll}
 & s\{x\backslash\!\!\backslash \mathtt{K}\langle\mu_1,\dots,\mu_n\rangle\} \\
\overset{\flat}{\mapsto}{}^* & \mathtt{K}'\langle s\langle\mu_1\rangle^x_1,\dots,s\langle\mu_n\rangle^x_1,\dots,s\langle\mu_1\rangle^x_m,\dots,s\langle\mu_n\rangle^x_m\rangle \\
 & \quad \text{by Lem. 107} \\
\overset{\flat}{\mapsto}{}^* & \mathtt{K}'\langle (s\langle\mu_1\rangle^x_1)^\flat,\dots,(s\langle\mu_n\rangle^x_1)^\flat,\dots,(s\langle\mu_1\rangle^x_m)^\flat,\dots,(s\langle\mu_n\rangle^x_m)^\flat\rangle \\
\sim & (s\langle\mu_1\rangle^x_1)^\flat \,;\, \dots \,;\, (s\langle\mu_n\rangle^x_1)^\flat \,;\, \dots \,;\, (s\langle\mu_1\rangle^x_m)^\flat \,;\, \dots \,;\, (s\langle\mu_n\rangle^x_m)^\flat \\
 & \quad \text{by Lem. 126} \\
= & \mathsf{R}^1_{(1,0,\dots,0)} \,;\, \mathsf{R}^1_{(2,0,\dots,0)} \,;\, \dots \,;\, \mathsf{R}^1_{(n,0,\dots,0)} \,;\, \dots \,;\, \mathsf{R}^m_{(n,\dots,n,1)} \,;\, \mathsf{R}^m_{(n,\dots,n,2)} \,;\, \dots \,;\, \mathsf{R}^m_{(n,\dots,n,n)} \\
\sim & \mathsf{R}^1_{(1,0,\dots,0)} \,;\, \mathsf{R}^2_{(1,1,0,\dots,0)} \,;\, \dots \,;\, \mathsf{R}^m_{(1,1,\dots,1)} \,;\, \dots \,;\, \mathsf{R}^1_{(n,n-1,\dots,n-1)} \,;\, \mathsf{R}^2_{(n,n,n-1,\dots,n-1)} \,;\, \dots \,;\, \mathsf{R}^m_{(n,n,\dots,n)} \\
 & \quad \text{reordering the steps with equation (4)} \\
\sim & s\{x\backslash\!\!\backslash\mu_1\}^\flat \,;\, \dots \,;\, s\{x\backslash\!\!\backslash\mu_n\}^\flat \\
 & \quad \text{joining the steps with equation (5)} \\
\sim & \mathtt{K}\langle s\{x\backslash\!\!\backslash\mu_1\}^\flat,\dots,s\{x\backslash\!\!\backslash\mu_n\}^\flat\rangle \\
 & \quad \text{by Lem. 126}
\end{array}
$$

◀

▶ **Lemma 128** (Congruence for $\sim$ below abstraction). *Let* $\mathtt{K}_1\langle\mu_1,\dots,\mu_n\rangle \sim \mathtt{K}_2\langle\nu_1,\dots,\nu_m\rangle$. *Then:*

$$\mathtt{K}_1\langle(\lambda x.\mu_1)^\flat,\dots,(\lambda x.\mu_n)^\flat\rangle \sim \mathtt{K}_2\langle(\lambda x.\nu_1)^\flat,\dots,(\lambda x.\nu_m)^\flat\rangle$$

Note. *The multisteps μ_i and ν_i are in $\overset{\flat}{\mapsto}$-normal form because the $\sim$ relation only relates flat rewrites. But observe that $\lambda x.\mu_i$ and $\lambda x.\nu_i$ may not necessarily be in $\overset{\flat}{\mapsto}$-normal form because there may be an $\mathcal{F}$-EtaM redex at the root.*

Proof. If ρ is a flat rewrite, we define $\lambda\!\!\lambda x.\rho$ as follows:

$$
\begin{array}{rcl}
\lambda\!\!\lambda x.\mu & \overset{\text{def}}{=} & (\lambda x.\mu)^\flat \\
\lambda\!\!\lambda x.(\rho \,;\, \sigma) & \overset{\text{def}}{=} & (\lambda\!\!\lambda x.\rho) \,;\, (\lambda\!\!\lambda x.\sigma)
\end{array}
$$

Another way to state this lemma is to say that $\rho \sim \sigma$ implies $\lambda\!\!\lambda x.\rho \sim \lambda\!\!\lambda x.\sigma$. The proof proceeds by induction on the derivation of $\rho \sim \sigma$. The reflexivity, symmetry, and transitivity cases are immediate. We analyze the cases in which an axiom is applied at the root, as well as closure under composition contexts:

1. **Rule $\sim$-Assoc.** Let $\rho = ((\rho_1 \,;\, \rho_2) \,;\, \rho_3) \sim (\rho_1 \,;\, (\rho_2 \,;\, \rho_3)) = \sigma$. Then $\lambda\!\!\lambda x.\rho = ((\lambda\!\!\lambda x.\rho_1 \,;\, \lambda\!\!\lambda x.\rho_2) \,;\, \lambda\!\!\lambda x.\rho_3) \sim (\lambda\!\!\lambda x.\rho_1 \,;\, (\lambda\!\!\lambda x.\rho_2 \,;\, \lambda\!\!\lambda x.\rho_3)) = \lambda\!\!\lambda x.\sigma$ can be derived applying the $\sim$-Assoc rule.

2. **Rule $\sim$-Perm.** Let $\mu \sim \mu_1^\flat \,;\, \mu_2^\flat$ be derived from $\mu \Leftrightarrow \mu_1 \,;\, \mu_2$. Then note that $\lambda x.\mu \Leftrightarrow \lambda x.\mu_1 \,;\, \lambda x.\mu_2$ holds by the SAbs rule. Hence by the generalized $\sim$-Perm rule (Prop. 121) we have that $(\lambda x.\mu)^\flat \sim (\lambda x.\mu_1)^\flat \,;\, (\lambda x.\mu_2)^\flat$. Moreover, by confluence of flattening (Prop. 96), we have that $(\lambda x.\mu)^\flat \sim (\lambda x.\mu_1^\flat)^\flat \,;\, (\lambda x.\mu_2^\flat)^\flat$.
3. **Congruence (left of a composition).** Let $\rho = (\rho' \,;\, \tau) \sim (\sigma' \,;\, \tau) = \sigma$ be derived from $\rho' \sim \sigma'$. Then by IH we have that $\lambda\!\!\lambda x.\rho' \sim \lambda\!\!\lambda x.\sigma'$, so $\lambda\!\!\lambda x.\rho = (\lambda\!\!\lambda x.\rho' \,;\, \lambda\!\!\lambda x.\tau) \sim (\lambda\!\!\lambda x.\sigma' \,;\, \lambda\!\!\lambda x.\tau) = \lambda\!\!\lambda x.\sigma$.
4. **Congruence (right of a composition).** Similar to the previous case.

◀

▶ **Lemma 129** (Congruence for $\sim$ below application). *Let $\mathtt{K}_1\langle\mu_1, \ldots, \mu_n\rangle \sim \mathtt{K}_2\langle\nu_1, \ldots, \nu_m\rangle$ and let s be an arbitrary term. Then:*
1. $\mathtt{K}_1\langle(\mu_1\,\underline{s})^\flat, \ldots, (\mu_n\,\underline{s})^\flat\rangle \sim \mathtt{K}_2\langle(\nu_1\,\underline{s})^\flat, \ldots, (\nu_m\,\underline{s})^\flat\rangle$
2. $\mathtt{K}_1\langle(\underline{s}\,\mu_1)^\flat, \ldots, (\underline{s}\,\mu_n)^\flat\rangle \sim \mathtt{K}_2\langle(\underline{s}\,\nu_1)^\flat, \ldots, (\underline{s}\,\nu_m)^\flat\rangle$

Proof. We only prove item 1. (item 2. is symmetric). If ρ is a flat rewrite, we define $\rho@s$ as follows:

$$\begin{array}{rcl} \mu@s & \stackrel{\text{def}}{=} & (\mu\,\underline{s})^\flat \\ (\rho\,;\,\sigma)@s & \stackrel{\text{def}}{=} & (\rho@s)\,;\,(\sigma@s) \end{array}$$

Another way to state item 1. is to say that $\rho \sim \sigma$ implies $\rho@s \sim \sigma@s$. The proof proceeds by induction on the derivation of $\rho \sim \sigma$. The reflexivity, symmetry, and transitivity cases are immediate. We analyze the cases in which an axiom is applied at the root, as well as closure under composition contexts:
1. **Rule $\sim$-Assoc.** Let $\rho = ((\rho_1 \,;\, \rho_2) \,;\, \rho_3) \sim (\rho_1 \,;\, (\rho_2 \,;\, \rho_3)) = \sigma$. Then $\rho@s = ((\rho_1@s \,;\, \rho_2@s) \,;\, \rho_3@s) \sim (\rho_1@s \,;\, (\rho_2@s \,;\, \rho_3@s)) = \sigma@s$ can be derived applying the $\sim$-Assoc rule.
2. **Rule $\sim$-Perm.** Let $\mu \sim \mu_1^\flat \,;\, \mu_2^\flat$ be derived from $\mu \Leftrightarrow \mu_1 \,;\, \mu_2$. Then note that $\mu\,\underline{s} \Leftrightarrow \mu_1\,\underline{s} \,;\, \mu_2\,\underline{s}$ holds by the SApp rule, also using the straightforward fact that $\underline{s} \Leftrightarrow \underline{s} \,;\, \underline{s}$, given that s is a term, *i.e.* it has no occurrences of rule symbols. Then by the generalized $\sim$-Perm rule (Prop. 121) we have that $(\mu\,\underline{s})^\flat \sim (\mu_1\,\underline{s})^\flat \,;\, (\mu_2\,\underline{s})^\flat$. Moreover, by confluence of flattening (Prop. 96), we have that $(\mu\,\underline{s})^\flat \sim (\mu_1^\flat\,\underline{s})^\flat \,;\, (\mu_2^\flat\,\underline{s})^\flat$, as required.
3. **Congruence (left of a composition).** Let $\rho = (\rho' \,;\, \tau) \sim (\sigma' \,;\, \tau) = \sigma$ be derived from $\rho' \sim \sigma'$. Then by IH we have that $\rho'@s \sim \sigma'@s$, so $\rho@s = (\rho'@s \,;\, \tau@s) \sim (\sigma'@s \,;\, \tau@s) = \sigma@s$.
4. **Congruence (right of a composition).** Similar to the previous case.

◀

▶ **Theorem 130** (Soundness and completeness of flat permutation equivalence). *Let $\Gamma \vdash \rho : s \twoheadrightarrow t : A$ and $\Gamma \vdash \sigma : s' \twoheadrightarrow t' : A$. The following are equivalent:*
1. $\rho \approx \sigma$
2. $\rho^\flat \sim \sigma^\flat$

Proof. The implication ($2 \implies 1$) is immediate, given that reduction $\stackrel{\flat}{\mapsto}$ in the flattening system $\mathcal{F}$ is included in permutation equivalence (Lem. 100) and, similarly, flat permutation equivalence is included in permutation equivalence (Lem. 114).

For the implication ($1 \implies 2$), we proceed by induction on the derivation of $\rho \approx \sigma$. In the proof, sometimes we implicitly use the fact that $\stackrel{\flat}{\mapsto}$ is strongly normalizing (Prop. 95) and confluent (Prop. 96). In particular, note that $(\rho \,;\, \sigma)^\flat = \rho^\flat \,;\, \sigma^\flat$ and more in general $\mathtt{K}\langle\rho_1, \ldots, \rho_n\rangle^\flat = \mathtt{K}\langle\rho_1^\flat, \ldots, \rho_n^\flat\rangle$. In the inductive proof, the cases for reflexivity, symmetry, and transitivity are immediate. We analyze the cases when a rule is applied at the root, as well as congruence closure under rewrite constructors:

1. $\approx$**-IdL.** Let $\underline{\rho^{\mathsf{src}}}\,;\rho \approx \rho$. Let $\rho^\flat = \mathtt{K}\langle \mu_1, \dots, \mu_n\rangle$. Note that $(\rho^\flat)^{\mathsf{src}} = \mu_1^{\mathsf{src}}$. Then:

$$\begin{array}{rcll}
(\underline{\rho^{\mathsf{src}}}\,;\rho)^\flat & = & (\underline{\rho^{\mathsf{src}}})^\flat\,;\rho^\flat & \\
& = & (\underline{(\rho^\flat)^{\mathsf{src}}})^\flat\,;\rho^\flat & \text{since by Lem. 105 } \underline{(\rho^\flat)^{\mathsf{src}}} \overset{\flat}{\mapsto}{}^* (\underline{\rho^{\mathsf{src}}})^\flat \\
& = & (\underline{\mu_1^{\mathsf{src}}})^\flat\,;\rho^\flat & \text{since } (\rho^\flat)^{\mathsf{src}} = \mu_1^{\mathsf{src}} \\
& = & (\underline{\mu_1^{\mathsf{src}}})^\flat\,;\mathtt{K}\langle \mu_1, \dots, \mu_n\rangle & \\
& = & \mathtt{K}\langle ((\underline{\mu_1^{\mathsf{src}}})^\flat\,;\mu_1), \dots, \mu_n\rangle & \text{by Lem. 115} \\
& \sim & \mathtt{K}\langle \mu_1, \dots, \mu_n\rangle & \text{since } \mu_1 \Leftrightarrow \underline{\mu_1^{\mathsf{src}}}\,;\mu_1 \text{ by Lem. 116} \\
& = & \rho^\flat &
\end{array}$$

2. $\approx$**-IdR.** Similar to the previous case. Let $\rho\,;\underline{\rho^{\mathsf{tgt}}} \approx \rho$. Let $\rho^\flat = \mathtt{K}\langle \mu_1, \dots, \mu_n\rangle$. Note that $(\rho^\flat)^{\mathsf{tgt}} = \mu_n^{\mathsf{tgt}}$.

$$\begin{array}{rcll}
\rho\,;\underline{\rho^{\mathsf{tgt}}}^\flat & = & \rho^\flat\,;(\underline{\rho^{\mathsf{tgt}}})^\flat & \\
& = & \rho^\flat\,;(\underline{(\rho^\flat)^{\mathsf{tgt}}})^\flat & \text{since by Lem. 105 } \underline{(\rho^\flat)^{\mathsf{tgt}}} \overset{\flat}{\mapsto}{}^* (\underline{\rho^{\mathsf{tgt}}})^\flat \\
& = & \rho^\flat\,;(\underline{\mu_n^{\mathsf{tgt}}})^\flat & \text{since } (\rho^\flat)^{\mathsf{tgt}} = \mu_n^{\mathsf{tgt}} \\
& = & \mathtt{K}\langle \mu_1, \dots, \mu_n\rangle\,;(\underline{\mu_n^{\mathsf{tgt}}})^\flat & \text{since } (\rho^\flat)^{\mathsf{tgt}} = \mu_n^{\mathsf{tgt}} \\
& \sim & \mathtt{K}\langle \mu_1, \dots, (\mu_n\,;(\underline{\mu_n^{\mathsf{tgt}}})^\flat)\rangle & \text{by Lem. 115} \\
& \sim & \mathtt{K}\langle \mu_1, \dots, \mu_n\rangle & \text{since } \mu_n \Leftrightarrow \mu_n\,;\underline{\mu_n^{\mathsf{tgt}}} \text{ by Lem. 116} \\
& = & \rho^\flat &
\end{array}$$

3. $\approx$**-Assoc.** Let $(\rho\,;\sigma)\,;\tau \approx \rho\,;(\sigma\,;\tau)$. Then

$$\begin{array}{rcll}
((\rho\,;\sigma)\,;\tau)^\flat & = & (\rho^\flat\,;\sigma^\flat)\,;\tau^\flat & \\
& \sim & \rho^\flat\,;(\sigma^\flat\,;\tau^\flat) & \text{by } \sim\text{-}\mathsf{Assoc} \\
& = & (\rho\,;(\sigma\,;\tau))^\flat &
\end{array}$$

4. $\approx$**-Abs.** Let $(\lambda x.\rho)\,;(\lambda x.\sigma) \approx \lambda x.(\rho\,;\sigma)$. It suffices to show that $((\lambda x.\rho)\,;(\lambda x.\sigma))^\flat = \lambda x.(\rho\,;\sigma)^\flat$. Indeed:

$$\begin{array}{rcll}
\lambda x.(\rho\,;\sigma) & \overset{\flat}{\mapsto} & (\lambda x.\rho)\,;(\lambda x.\sigma) & \text{by } \mathcal{F}\text{-}\mathsf{Abs} \\
& \overset{\flat}{\mapsto}{}^* & (\lambda x.\rho^\flat)\,;(\lambda x.\sigma^\flat) & \\
& = & ((\lambda x.\rho)\,;(\lambda x.\sigma))^\flat &
\end{array}$$

5. $\approx$**-App.** Let $(\rho_1\,\rho_2)\,;(\sigma_1\,\sigma_2) \approx (\rho_1\,;\sigma_1)\,(\rho_2\,;\sigma_2)$. Consider the $\overset{\flat}{\mapsto}$-normal forms of each rewrite:

$$\begin{array}{ll}
\rho_1^\flat = \mathtt{K}_1\langle \mu_1, \dots, \mu_n\rangle & \rho_2^\flat = \mathtt{K}_2\langle \nu_1, \dots, \nu_m\rangle \\
\sigma_1^\flat = \tilde{\mathtt{K}}_1\langle \hat{\mu}_1, \dots, \hat{\mu}_p\rangle & \sigma_2^\flat = \tilde{\mathtt{K}}_2\langle \hat{\nu}_1, \dots, \hat{\nu}_q\rangle
\end{array}$$

Before going on, we make the following claim:

$$\mu_n^{\blacktriangleright} = \hat{\mu}_1^{\blacktriangleleft} \qquad \text{and} \qquad \nu_m^{\blacktriangleright} = \hat{\nu}_1^{\blacktriangleleft} \qquad (\star)$$

For the first equality, note that ρ_1 and σ_1 are composable, so $\rho_1^{\mathsf{tgt}} =_{\beta\eta} \sigma_2^{\mathsf{src}}$ are $\beta\eta$-equivalent terms. Moreover, by Rem. 84 and Lem. 87 we have that $(\rho_1^\flat)^{\mathsf{tgt}} =_{\beta\eta} \rho_1^{\mathsf{tgt}}$ and $(\sigma_1^\flat)^{\mathsf{src}} =_{\beta\eta} \sigma_1^{\mathsf{src}}$. This means that $\mu_n^{\mathsf{tgt}} =_{\beta\eta} \hat{\mu}_1^{\mathsf{src}}$, so by confluence and strong normalization of flattening $\mu_n^{\blacktriangleright} = \hat{\mu}_1^{\blacktriangleleft}$. Similarly, for the second equality, since ρ_2 and σ_2 are composable, we have that $\nu_m^{\blacktriangleright} = \hat{\nu}_1^{\blacktriangleleft}$.
Furthermore, we claim that the two following conditions hold:

(I) $\rho_1\,\rho_2 \overset{\flat}{\mapsto}{}^*\!\sim \mathtt{K}_1\langle(\mu_1\,\nu_1^{\blacktriangleleft})^\flat,\dots,(\mu_n\,\nu_1^{\blacktriangleleft})^\flat\rangle\,;\,\mathtt{K}_2\langle(\hat{\mu}_1^{\blacktriangleleft}\,\nu_1)^\flat,\dots,(\hat{\mu}_1^{\blacktriangleleft}\,\nu_m)^\flat\rangle$

(II) $\sigma_1\,\sigma_2 \overset{\flat}{\mapsto}{}^*\!\sim \tilde{\mathtt{K}}_1\langle(\hat{\mu}_1\,\nu_m^{\blacktriangleright})^\flat,\dots,(\hat{\mu}_p\,\nu_m^{\blacktriangleright})^\flat\rangle\,;\,\tilde{\mathtt{K}}_2\langle(\hat{\mu}_p^{\blacktriangleright}\,\hat{\nu}_1)^\flat,\dots,(\hat{\mu}_p^{\blacktriangleright}\,\hat{\nu}_q)^\flat\rangle$

To prove **(I)**, note that:

$$\begin{array}{lll} & \rho_1\,\rho_2 & \\ \overset{\flat}{\mapsto}{}^*\!\sim & \mathtt{K}_1\langle(\mu_1\,\nu_1^{\blacktriangleleft})^\flat,\dots,(\mu_n\,\nu_1^{\blacktriangleleft})^\flat\rangle\,;\,\mathtt{K}_2\langle(\mu_n^{\blacktriangleright}\,\nu_1)^\flat,\dots,(\mu_n^{\blacktriangleright}\,\nu_m)^\flat\rangle & \text{by Lem. 124} \\ = & \mathtt{K}_1\langle(\mu_1\,\nu_1^{\blacktriangleleft})^\flat,\dots,(\mu_n\,\nu_1^{\blacktriangleleft})^\flat\rangle\,;\,\mathtt{K}_2\langle(\hat{\mu}_1^{\blacktriangleleft}\,\nu_1)^\flat,\dots,(\hat{\mu}_1^{\blacktriangleleft}\,\nu_m)^\flat\rangle & \text{by the claim } (\star) \text{ above} \end{array}$$

The proof of **(II)** is symmetric to the proof of **(I)**.

To conclude the proof of the $\sim$-App case, let us rewrite the left-hand side. We use the associativity rule ($\sim$-Assoc) implicitly:

$$\begin{array}{ll} & (\rho_1\,\rho_2)\,;\,(\sigma_1\,\sigma_2) \\ \overset{\flat}{\mapsto}{}^*\!\sim & \mathtt{K}_1\langle(\mu_1\,\nu_1^{\blacktriangleleft})^\flat,\dots,(\mu_n\,\nu_1^{\blacktriangleleft})^\flat\rangle\,;\,\mathtt{K}_2\langle(\hat{\mu}_1^{\blacktriangleleft}\,\nu_1)^\flat,\dots,(\hat{\mu}_1^{\blacktriangleleft}\,\nu_m)^\flat\rangle\,; \\ & \tilde{\mathtt{K}}_1\langle(\hat{\mu}_1\,\nu_m^{\blacktriangleright})^\flat,\dots,(\hat{\mu}_p\,\nu_m^{\blacktriangleright})^\flat\rangle\,;\,\tilde{\mathtt{K}}_2\langle(\hat{\mu}_p^{\blacktriangleright}\,\hat{\nu}_1)^\flat,\dots,(\hat{\mu}_p^{\blacktriangleright}\,\hat{\nu}_q)^\flat\rangle \\ & \quad \text{by claims } \textbf{(I)} \text{ and } \textbf{(II)} \\ \sim & \mathtt{K}_1\langle(\mu_1\,\nu_1^{\blacktriangleleft})^\flat,\dots,(\mu_n\,\nu_1^{\blacktriangleleft})^\flat\rangle\,;\,\tilde{\mathtt{K}}_1\langle(\hat{\mu}_1\,\nu_1^{\blacktriangleleft})^\flat,\dots,(\hat{\mu}_p\,\nu_1^{\blacktriangleleft})^\flat\rangle\,; \\ & \mathtt{K}_2\langle(\hat{\mu}_p^{\blacktriangleright}\,\nu_1)^\flat,\dots,(\hat{\mu}_p^{\blacktriangleright}\,\nu_m)^\flat\rangle\,;\,\tilde{\mathtt{K}}_2\langle(\hat{\mu}_p^{\blacktriangleright}\,\hat{\nu}_1)^\flat,\dots,(\hat{\mu}_p^{\blacktriangleright}\,\hat{\nu}_q)^\flat\rangle \\ & \quad \text{by Lem. 123} \end{array}$$

On the other hand, rewriting the right-hand side:

$$\begin{array}{ll} & (\rho_1\,;\,\sigma_1)\,(\rho_2\,;\,\sigma_2) \\ \overset{\flat}{\mapsto}{}^* & (\mathtt{K}_1\langle\mu_1,\dots,\mu_n\rangle\,;\,\tilde{\mathtt{K}}_1\langle\hat{\mu}_1,\dots,\hat{\mu}_p\rangle)\,(\mathtt{K}_2\langle\nu_1,\dots,\nu_m\rangle\,;\,\tilde{\mathtt{K}}_2\langle\hat{\nu}_1,\dots,\hat{\nu}_q\rangle) \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle(\mu_1\,\underline{\nu_1^{\mathsf{src}}}),\dots,(\mu_n\,\underline{\nu_1^{\mathsf{src}}})\rangle\,;\,\tilde{\mathtt{K}}_1\langle(\hat{\mu}_1\,\underline{\nu_1^{\mathsf{src}}}),\dots,(\hat{\mu}_p\,\underline{\nu_1^{\mathsf{src}}})\rangle\,; \\ & \mathtt{K}_2\langle(\underline{\hat{\mu}_p^{\mathsf{tgt}}}\,\nu_1),\dots,(\underline{\hat{\mu}_p^{\mathsf{tgt}}}\,\nu_m)\rangle\,;\,\tilde{\mathtt{K}}_2\langle(\underline{\hat{\mu}_p^{\mathsf{tgt}}}\,\hat{\nu}_1),\dots,(\underline{\hat{\mu}_p^{\mathsf{tgt}}}\,\hat{\nu}_q)\rangle \\ & \quad \text{by generalized } \mathcal{F}\text{-App3 (Lem. 106)} \\ \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle(\mu_1\,\nu_1^{\blacktriangleleft})^\flat,\dots,(\mu_n\,\nu_1^{\blacktriangleleft})^\flat\rangle\,;\,\tilde{\mathtt{K}}_1\langle(\hat{\mu}_1\,\nu_1^{\blacktriangleleft})^\flat,\dots,(\hat{\mu}_p\,\nu_1^{\blacktriangleleft})^\flat\rangle\,; \\ & \mathtt{K}_2\langle(\hat{\mu}_p^{\blacktriangleright}\,\nu_1)^\flat,\dots,(\hat{\mu}_p^{\blacktriangleright}\,\nu_m)^\flat\rangle\,;\,\tilde{\mathtt{K}}_2\langle(\hat{\mu}_p^{\blacktriangleright}\,\hat{\nu}_1)^\flat,\dots,(\hat{\mu}_p^{\blacktriangleright}\,\hat{\nu}_q)^\flat\rangle \end{array}$$

6. **$\approx$-BetaTR.** Let $(\lambda x.\underline{s})\,\rho \approx s\{x\backslash\!\backslash\rho\}$, and suppose that $\rho^\flat = \mathtt{K}\langle\mu_1,\dots,\mu_n\rangle$. First note that, $(\lambda x.\underline{s})\,\rho \overset{\flat}{\mapsto}{}^* \mathtt{K}\langle((\lambda x.\underline{s})\,\mu_1),\dots,((\lambda x.\underline{s})\,\mu_n)\rangle$. Indeed, if $n = 1$ this is immediate, and if $n > 1$ this is a consequence of the generalized $\mathcal{F}$-App2 rule (Lem. 106). Hence:

$$\begin{array}{llll} (\lambda x.\underline{s})\,\rho & \overset{\flat}{\mapsto}{}^* & \mathtt{K}\langle((\lambda x.\underline{s})\,\mu_1),\dots,((\lambda x.\underline{s})\,\mu_n)\rangle & \\ & \overset{\flat}{\mapsto}{}^* & \mathtt{K}\langle s\{x\backslash\mu_1\},\dots,s\{x\backslash\mu_n\}\rangle & \text{by } \mathcal{F}\text{-BetaM } (n \text{ times}) \\ & = & \mathtt{K}\langle s\{x\backslash\!\backslash\mu_1\},\dots,s\{x\backslash\!\backslash\mu_n\}\rangle & \text{by Rem. 14} \\ & \overset{\flat}{\mapsto}{}^* & \mathtt{K}\langle s\{x\backslash\!\backslash\mu_1\},\dots,s\{x\backslash\!\backslash\mu_n\}\rangle^\flat & \\ & \sim\overset{\flat}{\leftarrow}{}^* & s\{x\backslash\!\backslash\mathtt{K}\langle\mu_1,\dots,\mu_n\rangle\} & \text{by Lem. 127} \\ & \overset{\flat}{\leftarrow}{}^* & s\{x\backslash\!\backslash\rho\} & \text{by Lem. 108} \end{array}$$

7. **$\approx$-BetaRT.** Let $(\lambda x.\rho)\,\underline{s} \approx \rho\{x\backslash s\}$, and suppose that $\rho^\flat = \mathtt{K}\langle\mu_1,\dots,\mu_n\rangle$. First note that $(\lambda x.\rho)\,\underline{s} \overset{\flat}{\mapsto}{}^* \mathtt{K}\langle(\lambda x.\mu_1)\,\underline{s},\dots,(\lambda x.\mu_n)\,\underline{s}\rangle$. Indeed, if $n = 1$ this is immediate, and if $n > 1$ this is a consequence of the generalized $\mathcal{F}$-Abs and $\mathcal{F}$-App1 rules (Lem. 106). Hence:

$$\begin{array}{llll} (\lambda x.\rho)\,\underline{s} & \overset{\flat}{\mapsto}{}^* & \mathtt{K}\langle(\lambda x.\mu_1)\,\underline{s},\dots,(\lambda x.\mu_n)\,\underline{s}\rangle & \\ & \overset{\flat}{\mapsto}{}^* & \mathtt{K}\langle\mu_1\{x\backslash\underline{s}\},\dots,\mu_n\{x\backslash\underline{s}\}\rangle & \text{by } \mathcal{F}\text{-BetaM } (n \text{ times}) \\ & = & \mathtt{K}\langle\mu_1\{x\backslash s\},\dots,\mu_n\{x\backslash s\}\rangle & \text{by Rem. 14} \\ & = & \mathtt{K}\langle\mu_1,\dots,\mu_n\rangle\{x\backslash s\} & \\ & \overset{\flat}{\leftarrow}{}^* & \rho\{x\backslash s\} & \text{by Lem. 109} \end{array}$$

8. **≈-Eta.** Let $\lambda x.\rho\, x \approx \rho$ where $x \notin \mathsf{fv}(\rho)$. Let $\rho^\flat = \mathtt{K}\langle \mu_1, \ldots, \mu_n \rangle$. It suffices to note that $(\lambda x.\rho\, x)^\flat = \rho^\flat$. Indeed:

$$\begin{array}{lll} \lambda x.\rho\, x & \overset{\flat}{\mapsto}{}^* & \lambda x.\rho^\flat\, x \\ & = & \lambda x.\mathtt{K}\langle \mu_1, \ldots, \mu_n \rangle\, x \\ & \overset{\flat}{\mapsto}{}^* & \lambda x.\mathtt{K}\langle (\mu_1\, x), \ldots, (\mu_n\, x) \rangle \quad \text{by generalized } \mathcal{F}\text{-}\mathsf{App1} \text{ (Lem. 106)} \\ & \overset{\flat}{\mapsto}{}^* & \mathtt{K}\langle \lambda x.(\mu_1\, x), \ldots, \lambda x.(\mu_n\, x) \rangle \quad \text{by generalized } \mathcal{F}\text{-}\mathsf{Abs} \text{ (Lem. 106)} \\ & \overset{\flat}{\mapsto}{}^* & \mathtt{K}\langle \mu_1, \ldots, \mu_n \rangle \quad \text{by } \mathcal{F}\text{-}\mathsf{EtaM} \text{ } (n \text{ times}) \\ & = & \rho^\flat \end{array}$$

Note that we may apply the $\mathcal{F}$-EtaM rule because, for each $1 \le i \le n$, we have that $x \notin \mathsf{fv}(\mu_i)$. This in turn is justified by noting that $x \notin \mathsf{fv}(\mathtt{K}\langle \mu_1, \ldots, \mu_n \rangle) = \mathsf{fv}(\rho^\flat)$, which is a consequence of the fact that flattening does not create free variables.

9. **Congruence under an abstraction.** Let $\lambda x.\rho \approx \lambda x.\sigma$ be derived from $\rho \approx \sigma$. Consider their $\overset{\flat}{\mapsto}$-normal forms, $\rho^\flat = \mathtt{K}_1\langle \mu_1, \ldots, \mu_n \rangle$ and $\sigma^\flat = \mathtt{K}_2\langle \nu_1, \ldots, \nu_m \rangle$. By IH we have that $\rho^\flat \sim \sigma^\flat$. Then:

$$\begin{array}{lll} \lambda x.\rho & \overset{\flat}{\mapsto}{}^* & \lambda x.\mathtt{K}_1\langle \mu_1, \ldots, \mu_n \rangle \\ & \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle \lambda x.\mu_1, \ldots, \lambda x.\mu_n \rangle \quad \text{by generalized } \mathcal{F}\text{-}\mathsf{Abs} \text{ (Lem. 106)} \\ & \overset{\flat}{\mapsto}{}^* & \mathtt{K}_1\langle (\lambda x.\mu_1)^\flat, \ldots, (\lambda x.\mu_n)^\flat \rangle \\ & \sim & \mathtt{K}_2\langle (\lambda x.\nu_1)^\flat, \ldots, (\lambda x.\nu_m)^\flat \rangle \quad \text{by Lem. 128, as } \rho^\flat \sim \sigma^\flat \\ & \overset{\flat}{\leftarrow\!\shortmid}{}^* & \mathtt{K}_2\langle \lambda x.\nu_1, \ldots, \lambda x.\nu_m \rangle \\ & \overset{\flat}{\leftarrow\!\shortmid}{}^* & \lambda x.\mathtt{K}_2\langle \nu_1, \ldots, \nu_m \rangle \quad \text{by generalized } \mathcal{F}\text{-}\mathsf{Abs} \text{ (Lem. 106)} \\ & \overset{\flat}{\leftarrow\!\shortmid}{}^* & \lambda x.\sigma \end{array}$$

10. **Congruence under an application.** Let $\rho_1\, \rho_2 \approx \sigma_1\, \sigma_2$ be derived from $\rho_1 \approx \sigma_1$ and $\rho_2 \approx \sigma_2$. Consider the $\overset{\flat}{\mapsto}$-normal forms of each rewrite:

$$\begin{array}{ll} \rho_1^\flat = \mathtt{K}_1\langle \mu_1, \ldots, \mu_n \rangle & \rho_2^\flat = \mathtt{K}_2\langle \nu_1, \ldots, \nu_m \rangle \\ \sigma_1^\flat = \tilde{\mathtt{K}}_1\langle \hat{\mu}_1, \ldots, \hat{\mu}_p \rangle & \sigma_2^\flat = \tilde{\mathtt{K}}_2\langle \hat{\nu}_1, \ldots, \hat{\nu}_q \rangle \end{array}$$

By IH we have that $\rho_1^\flat \sim \sigma_1^\flat$ and $\rho_2^\flat \sim \sigma_2^\flat$. Before going on, we make the following claim:

$$\nu_1^{◀} = \hat{\nu}_1^{◀} \qquad \text{and} \qquad \mu_n^{◀} = \hat{\mu}_p^{◀} \qquad (\star)$$

For the first equality, note that $\rho_2 \approx \sigma_2$, so $\rho_2^{\mathsf{src}} =_{\beta\eta} \sigma_2^{\mathsf{src}}$ are $\beta\eta$-equivalent terms by Lem. 52. Moreover, by Rem. 84 and Lem. 87 we have that $(\rho_2^\flat)^{\mathsf{src}} =_{\beta\eta} \rho_2^{\mathsf{src}}$ and $(\sigma_2^\flat)^{\mathsf{src}} =_{\beta\eta} \sigma_2^{\mathsf{src}}$. This means that $\nu_1^{\mathsf{src}} =_{\beta\eta} \hat{\nu}_1^{\mathsf{src}}$, so by confluence and strong normalization of flattening $\nu_1^{◀} = \hat{\nu}_1^{◀}$. Similarly, for the second equality, since $\rho_1 \approx \sigma_1$, we have that $\mu_n^{▶} = \hat{\mu}_p^{▶}$. Then:

$$\begin{array}{lll} & \rho_1\, \rho_2 & \\ \overset{\flat}{\mapsto}{}^*\!\sim & \mathtt{K}_1\langle (\mu_1\, \nu_1^{◀})^\flat, \ldots, (\mu_n\, \nu_1^{◀})^\flat \rangle \,;\, \mathtt{K}_2\langle (\mu_n^{▶}\, \nu_1)^\flat, \ldots, (\mu_n^{▶}\, \nu_m)^\flat \rangle & \text{by Lem. 124} \\ \sim & \tilde{\mathtt{K}}_1\langle (\hat{\mu}_1\, \nu_1^{◀})^\flat, \ldots, (\hat{\mu}_p\, \nu_1^{◀})^\flat \rangle \,;\, \mathtt{K}_2\langle (\mu_n^{▶}\, \nu_1)^\flat, \ldots, (\mu_n^{▶}\, \nu_m)^\flat \rangle & \text{by Lem. 129 as } \rho_1^\flat \sim \sigma_1^\flat \\ \sim & \tilde{\mathtt{K}}_1\langle (\hat{\mu}_1\, \nu_1^{◀})^\flat, \ldots, (\hat{\mu}_p\, \nu_1^{◀})^\flat \rangle \,;\, \tilde{\mathtt{K}}_2\langle (\mu_n^{▶}\, \hat{\nu}_1)^\flat, \ldots, (\mu_n^{▶}\, \hat{\nu}_q)^\flat \rangle & \text{by Lem. 129 as } \rho_2^\flat \sim \sigma_2^\flat \\ = & \tilde{\mathtt{K}}_1\langle (\hat{\mu}_1\, \hat{\nu}_1^{◀})^\flat, \ldots, (\hat{\mu}_p\, \hat{\nu}_1^{◀})^\flat \rangle \,;\, \tilde{\mathtt{K}}_2\langle (\hat{\mu}_p^{▶}\, \hat{\nu}_1)^\flat, \ldots, (\hat{\mu}_p^{▶}\, \hat{\nu}_q)^\flat \rangle & \text{by the claim } (\star) \text{ above} \\ \overset{\flat}{\leftarrow\!\shortmid}{}^* & \sigma_1\, \sigma_2 & \text{by Lem. 124} \end{array}$$

11. **Congruence under a composition.** Let $\rho_1 \mathbin{;} \rho_2 \approx \sigma_1 \mathbin{;} \sigma_2$ be derived from $\rho_1 \approx \sigma_1$ and $\rho_2 \approx \sigma_2$. Then:

$$\begin{array}{rcll} (\rho_1 \mathbin{;} \rho_2)^\flat & = & \rho_1^\flat \mathbin{;} \rho_2^\flat & \\ & \sim & \sigma_1^\flat \mathbin{;} \sigma_2^\flat & \text{by IH} \\ & = & (\sigma_1 \mathbin{;} \sigma_2)^\flat & \end{array}$$

◀

E Projection for Flat Rewrites

E.1 Projection for multisteps

▶ **Lemma 131** (Matching). *If* $(\varrho^{\mathsf{src}}\, \mu_1 \ldots \mu_n)^\flat = (\varrho^{\mathsf{src}}\, \nu_1 \ldots \nu_n)^\flat$ *then* $\mu_i^\flat = \nu_i^\flat$ *for all* $1 \leq i \leq n$.

Proof. We begin with a few auxiliary definitions. Let $\mathtt{N}$ be the set of n-hole term contexts generated by the following grammar:

$$\mathtt{N} ::= \Box \mid \lambda x.\mathtt{N} \mid x\, \mathtt{N}_1 \ldots \mathtt{N}_n \mid \mathbf{c}\, \mathtt{N}_1 \ldots \mathtt{N}_n$$

Moreover, if $\vec{y} = y_1, \ldots, y_n$ is a sequence of variables, we write $x\,\vec{y}$ for the application $x\, y_1 \ldots y_n$. We prove an auxiliary result.

Claim. Let V be a set of variables, and let $\mathtt{N}\langle x_1\vec{y_1}, \ldots, x_n\vec{y_n}\rangle$ be a term such that, for all $1 \leq i \leq n$, the variable x_i is not in V nor bound by the context $\mathtt{N}$, and furthermore all the variables in the sequence $\vec{y_i}$ are either in V or bound by the context $\mathtt{N}$. Suppose moreover that $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_m$ are multisteps whose free variables are disjoint from V, and such that $\mathtt{N}\langle \mu_1\vec{y_1}, \ldots, \mu_n\vec{y_n}\rangle^\flat = \mathtt{N}\langle \nu_1\vec{y_1}, \ldots, \nu_n\vec{y_n}\rangle^\flat$. Then $\mu_i^\flat = \nu_i^\flat$ for all $1 \leq i \leq n$.
Proof of the claim. By induction on $\mathtt{N}$:

1. $\mathtt{N} = \Box$: Then $n = 1$ and by hypothesis $(\mu_1\, \vec{y_1})^\flat = (\nu_1\, \vec{y_1})^\flat$. Note that the variables in $\vec{y_1}$ do not occur free in μ_1 nor in ν_2. Then:

$$\begin{array}{rcll} \mu_1 & \overset{\flat}{\hookleftarrow}{}^* & \lambda\vec{y_1}.\mu_1\, \vec{y_1} & \text{repeatedly applying } \mathcal{F}\text{-}\mathsf{EtaM} \\ & \overset{\flat}{\mapsto}{}^* & \lambda\vec{y_1}.(\mu_1\, \vec{y_1})^\flat & \\ & = & \lambda\vec{y_1}.(\nu_1\, \vec{y_1})^\flat & \text{by hypothesis} \\ & \overset{\flat}{\hookleftarrow}{}^* & \lambda\vec{y_1}.\nu_1\, \vec{y_1} & \\ & \overset{\flat}{\mapsto}{}^* & \nu_1 & \text{repeatedly applying } \mathcal{F}\text{-}\mathsf{EtaM} \end{array}$$

   Hence by confluence of flattening (Prop. 96) we have that $\mu_1^\flat = \nu_1^\flat$, as required.

2. $\mathtt{N} = \lambda z.\mathtt{N}'$: By hypothesis, we have that $(\lambda z.\mathtt{N}'\langle \mu_1\, \vec{y_1}, \ldots, \mu_n\, \vec{y_n}\rangle)^\flat = (\lambda z.\mathtt{N}'\langle \mu_1\, \vec{y_1}, \ldots, \mu_n\, \vec{y_n}\rangle)^\flat$. Then:

$$\begin{array}{rcll} \mathtt{N}'\langle \mu_1\, \vec{y_1}, \ldots, \mu_n\, \vec{y_n}\rangle & \overset{\flat}{\hookleftarrow} & (\lambda z.\mathtt{N}'\langle \mu_1\, \vec{y_1}, \ldots, \mu_n\, \vec{y_n}\rangle)\, z & \text{by } \mathcal{F}\text{-}\mathsf{BetaM} \\ & \overset{\flat}{\mapsto}{}^* & (\lambda z.\mathtt{N}'\langle \mu_1\, \vec{y_1}, \ldots, \mu_n\, \vec{y_n}\rangle)^\flat\, z & \\ & = & (\lambda z.\mathtt{N}'\langle \nu_1\, \vec{y_1}, \ldots, \nu_n\, \vec{y_n}\rangle)^\flat\, z & \text{by hypothesis} \\ & \overset{\flat}{\hookleftarrow}{}^* & (\lambda z.\mathtt{N}'\langle \nu_1\, \vec{y_1}, \ldots, \nu_n\, \vec{y_n}\rangle)\, z & \\ & \overset{\flat}{\mapsto} & \mathtt{N}'\langle \nu_1\, \vec{y_1}, \ldots, \nu_n\, \vec{y_n}\rangle & \text{by } \mathcal{F}\text{-}\mathsf{BetaM} \end{array}$$

   Hence by confluence of flattening (Prop. 96) we have that:

$$\mathtt{N}'\langle \mu_1\, \vec{y_1}, \ldots, \mu_n\, \vec{y_n}\rangle^\flat = \mathtt{N}'\langle \nu_1\, \vec{y_1}, \ldots, \nu_n\, \vec{y_n}\rangle^\flat$$

   Applying the IH with the set of variables $V \cup \{z\}$ we conclude that $\mu_i^\flat = \nu_i^\flat$ for all $1 \leq i \leq n$.

3. $\mathtt{N} = z\,\mathtt{N}_1 \ldots \mathtt{N}_m$: Then there exist non-negative integers $0 = i_0 \leq i_1 \leq i_2 \leq \ldots \leq i_m = n$ such that $z\,\mathtt{N}_1 \ldots \mathtt{N}_k$ has i_k holes for every $1 \leq k \leq m$. In particular:

$$\mathtt{N}\langle \mu_1\,\vec{y_1}, \ldots, \mu_n\,\vec{y_n}\rangle^\flat = z\,\mathtt{N}_1\langle \mu_{i_0}\,\vec{y_{i_0}}, \ldots, \mu_{i_1}\,\vec{y_{i_1}}\rangle^\flat \ldots \mathtt{N}_m\langle \mu_{i_{(m-1)}}\,\vec{y_{i_{(m-1)}}}, \ldots, \mu_{i_m}\,\vec{y_{i_m}}\rangle^\flat$$

And similarly:

$$\mathtt{N}\langle \nu_1\,\vec{y_1}, \ldots, \nu_n\,\vec{y_n}\rangle^\flat = z\,\mathtt{N}_1\langle \nu_{i_0}\,\vec{y_{i_0}}, \ldots, \nu_{i_1}\,\vec{y_{i_1}}\rangle^\flat \ldots \mathtt{N}_m\langle \nu_{i_{(m-1)}}\,\vec{y_{i_{(m-1)}}}, \ldots, \nu_{i_m}\,\vec{y_{i_m}}\rangle^\flat$$

So for each $1 \leq k \leq m$ we have that

$$\mathtt{N}_k\langle \mu_{i_{(k-1)}}\,\vec{y_{i_{(k-1)}}}, \ldots, \mu_{i_k}\,\vec{y_{i_k}}\rangle^\flat = \mathtt{N}_k\langle \nu_{i_{(k-1)}}\,\vec{y_{i_{(k-1)}}}, \ldots, \nu_{i_k}\,\vec{y_{i_k}}\rangle^\flat$$

Applying the IH for each value of k, we conclude $\mu_i^\flat = \nu_i^\flat$ for all $1 \leq i \leq n$.

4. $\mathtt{N} = \mathbf{c}\,\mathtt{N}_1 \ldots \mathtt{N}_m$: Similar to the previous case.

To conclude the proof of the lemma using the claim, recall that ϱ^{src} is in normal form and a rule-pattern, so it can be written as $\varrho^{\mathsf{src}} = \lambda x_1 \ldots x_n.\mathtt{N}\langle (x_1\,\vec{y_1}), \ldots, (x_n\,\vec{y_n})\rangle$. Note that all of the variables in $\vec{y_i}$ are bound by $\mathtt{N}$ because ϱ^{src} is a rule-pattern. Moreover note that $(\varrho^{\mathsf{src}}\,\mu_1 \ldots \mu_n)^\flat = \mathtt{N}\langle \mu_1\,\vec{y_1}, \ldots, \mu_n\,\vec{y_n}\rangle^\flat = \mathtt{N}\langle \nu_1\,\vec{y_1}, \ldots, \nu_n\,\vec{y_n}\rangle^\flat = (\varrho^{\mathsf{src}}\,\nu_1 \ldots \nu_n)^\flat$. By the claim, taking the set $V = \varnothing$, we obtain that $\mu_i^\flat = \nu_i^\flat$ for all $1 \leq i \leq n$, as required. ◀

▶ **Lemma 132** (Projection for compatible multisteps). *Let $\mu \uparrow \nu$. Then there exists a unique ξ such that $\mu /\!\!/\!\!/ \nu \Rightarrow \xi$.*

Proof. Straightforward by induction on the derivation of $\mu \uparrow \nu$. ◀

▶ **Lemma 133** (Projection of variables). *Let μ and ν be multisteps such that $\mu^\flat = \nu^\flat = x$ and $\mu \uparrow \nu$. Moreover, let $\mu /\!\!/\!\!/ \nu \Rightarrow \xi$. Then $\xi^\flat = x$.*

Proof. By induction on the derivation of $\mu \uparrow \nu$. Note that this judgment can only be derived by the CVar rule, given that, in any other case, the head of μ is either a constant $\mathbf{c}$ or a rule symbol ϱ, which in turn implies that the head of $\mu^\flat$ is either a constant or a rule symbol, contradicting the fact that $\mu^\flat = x$.

This means that $\mu = \lambda y_1 \ldots y_n.z\,\mu_1 \ldots \mu_m$ and $\nu = \lambda y_1 \ldots y_n.z\,\nu_1 \ldots \nu_m$ where $\mu_i \uparrow \nu_i$ for all $1 \leq i \leq n$. Note that the head of $\mu^\flat$ must be z, so $z = x$. Moreover, by confluence of flattening (Prop. 96), $\mu^\flat = (\lambda y_1 \ldots y_n.x\,\mu_1^\flat \ldots \mu_m^\flat)^\flat$. The term $\lambda y_1 \ldots y_n.x\,\mu_1^\flat \ldots \mu_m^\flat$ is already in $\mathcal{F}$-BetaM-normal form so, given that the contraction of an $\mathcal{F}$-EtaM redex does not create $\mathcal{F}$-BetaM redexes, there is a reduction sequence $\lambda y_1 \ldots y_n.x\,\mu_1^\flat \ldots \mu_m^\flat \overset{\flat}{\mapsto}{}^*_{\mathcal{F}\text{-EtaM}}\, x$ using only the $\mathcal{F}$-EtaM rule. Given that each $\mathcal{F}$-EtaM-reduction step erases exactly one abstraction and exactly one application, it must be the case that $n = m$ and $\mu_i^\flat = y_i$ for all $1 \leq i \leq n$. Using a symmetric argument, we have that $\nu_i^\flat = y_i$ for all $1 \leq i \leq n$.

To sum up, the situation is that $\mu = \lambda y_1 \ldots y_n.x\,\mu_1 \ldots \mu_n$ and $\nu = \lambda y_1 \ldots y_n.x\,\nu_1 \ldots \nu_n$ with $\mu_i^\flat = \nu_i^\flat = y_i$ for all $1 \leq i \leq n$. Note that $\mu /\!\!/\!\!/ \nu \Rightarrow \xi$ where ξ must be of the form $\lambda y_1 \ldots y_n.x\,\xi_1 \ldots \xi_n$ and $\mu_i /\!\!/\!\!/ \nu_i \Rightarrow \xi_i$ for all $1 \leq i \leq n$. By IH, for each $1 \leq i \leq n$ we have that $\xi_i^\flat = y_i$. To conclude, note that:

$$\begin{array}{rcll} \xi^\flat & = & (\lambda y_1 \ldots y_n.x\,\xi_1 \ldots \xi_n)^\flat & \\ & = & (\lambda y_1 \ldots y_n.x\,y_1 \ldots y_n)^\flat & \text{by confluence of flattening (Prop. 96)} \\ & = & x & \text{using the } \mathcal{F}\text{-EtaM rule } (n \text{ times}) \end{array}$$

◀

▶ **Definition 134** (Compatibility). *We give explicit names for the compatibility rules given in the body:*

$$\frac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda\vec{x}.y\,\vec{\mu} \uparrow \lambda\vec{x}.y\,\vec{\nu}}\,\mathsf{CVar} \quad \frac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda\vec{x}.\mathbf{c}\,\vec{\mu} \uparrow \lambda\vec{x}.\mathbf{c}\,\vec{\nu}}\,\mathsf{CCon} \quad \frac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda\vec{x}.\varrho\,\vec{\mu} \uparrow \lambda\vec{x}.\varrho\,\vec{\nu}}\,\mathsf{CRule}$$

$$\frac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda\vec{x}.\varrho\,\vec{\mu} \uparrow \lambda\vec{x}.\varrho^{\mathsf{src}}\,\vec{\nu}}\,\mathsf{CRuleL} \quad \frac{(\mu_i \uparrow \nu_i)_{i=1}^m}{\lambda\vec{x}.\varrho^{\mathsf{src}}\,\vec{\mu} \uparrow \lambda\vec{x}.\varrho\,\vec{\nu}}\,\mathsf{CRuleR}$$

▶ **Lemma 135** (Coherence of projection). *Let $\mu_1, \nu_1, \mu_2, \nu_2$ be multisteps such that:*

1. $\mu_1 \uparrow \nu_1$ *and* $\mu_2 \uparrow \nu_2$
2. $\mu_1^\flat = \mu_2^\flat$ *and* $\nu_1^\flat = \nu_2^\flat$
3. $\mu_1 /\!/\!/ \nu_1 \Rightarrow \xi_1$ *and* $\mu_2 /\!/\!/ \nu_2 \Rightarrow \xi_2$

Then $\xi_1^\flat = \xi_2^\flat$.

Proof. During this proof we use the following notion of **arity**: a multistep μ is said to be n*-ary* if its type, under the corresponding typing context, is of the form $A_1 \to \ldots \to A_n \to \alpha$ where α is a base type.

The proof proceeds by induction on the derivation of $\mu_1 \uparrow \nu_1$. Note that the last rule used to derive $\mu_2 \uparrow \nu_2$ must be the same as the last rule used to derive $\mu_1 \uparrow \nu_1$. For instance, it cannot be the case that $\mu_1 \uparrow \nu_1$ is derived using the CVar rule and $\mu_2 \uparrow \nu_2$ is derived using the CCon rule, because this would mean that $\mu_1 = \lambda x_1 \ldots x_n.y\,\mu_{11} \ldots \mu_{1m}$ and $\mu_2 = \lambda x_1 \ldots x_n.\mathbf{c}\,\mu_{21} \ldots \mu_{2p}$ and furthermore, by hypothesis, $\mu_1^\flat = \mu_2^\flat$, which is impossible, given that the head of $\mu_1^\flat$ is a variable y whereas the head of $\mu_2^\flat$ is a constant $\mathbf{c}$. Recall that the left-hand side of a rule must be headed by a constant, so using similar arguments this can be shown for all the other cases as well.

The only remaining possibilities are when $\mu_1 \uparrow \nu_1$ and $\mu_2 \uparrow \nu_2$ are derived using the same rule. The core of the argument in these cases is to apply the IH using the matching lemma (Lem. 131). The difficulty is that, in the equalities $\mu_1^\flat = \mu_2^\flat$ and $\nu_1^\flat = \nu_2^\flat$, flattening includes $\mathcal{F}$-EtaM reduction, so we need to provide extra arguments to be able "align" the normal forms.

We only study the CRuleL case (the remaining cases are similar). That is, suppose that $\mu_1 \uparrow \nu_1$ and $\mu_2 \uparrow \nu_2$ are derived by the CRuleL rule. Then:

- We have that $\mu_1 = \lambda x_1 \ldots x_n.\varrho\,\mu_{11} \ldots \mu_{1m}$ and $\nu_1 = \lambda x_1 \ldots x_n.\varrho^{\mathsf{src}}\,\nu_{11} \ldots \nu_{1m}$ where $\mu_{1i} \uparrow \nu_{1i}$ for all $1 \leq i \leq n$.
- We have that $\mu_2 = \lambda x_1 \ldots x_p.\vartheta\,\mu_{21} \ldots \mu_{2q}$ and $\nu_2 = \lambda x_1 \ldots x_p.\vartheta^{\mathsf{src}}\,\nu_{21} \ldots \nu_{2q}$ where $\mu_{2i} \uparrow \nu_{2i}$ for all $1 \leq i \leq q$.
- Note that $\mu_1 /\!/\!/ \nu_1 \Rightarrow \lambda x_1 \ldots x_n.\varrho\,\xi_{11} \ldots \xi_{1m}$ where $\mu_{1i} /\!/\!/ \nu_{1i} \Rightarrow \xi_{1i}$ for all $1 \leq i \leq n$.
- Note that $\mu_2 /\!/\!/ \nu_2 \Rightarrow \lambda x_1 \ldots x_p.\varrho\,\xi_{21} \ldots \xi_{2q}$ where $\mu_{2i} /\!/\!/ \nu_{2i} \Rightarrow \xi_{2i}$ for all $1 \leq i \leq q$.

Given that, by hypothesis, $\mu_1^\flat = \mu_2^\flat$, we know that $\varrho = \vartheta$. Suppose that μ_1 is N-ary for some $N \geq 0$ and that ϱ is M-ary for some $M \geq 0$. Then, since they all have the same types as μ_1, we know that ν_1, μ_2, ν_2 must also be N-ary. Similarly, since ϱ^{src} has the same type as ϱ, we know that it is M-ary. Note that $n, p \leq N$ and $m, q \leq M$. Moreover, note that μ_1 has n explicit abstractions, it is N-ary, and its body is $(M - m)$-ary, so $n + M - m \leq N$. Similarly, $p + M - q \leq N$. Consider the multisteps $\mu_1', \nu_1', \mu_2', \nu_2'$ that result from $\mu_1, \nu_1, \mu_2, \nu_2$ by performing η-expansions in order to "complete" the number of arguments in the body, to match the arity M:

- $\mu_1' = \lambda x_1 \ldots x_n x_{(n+1)} \ldots x_{(n+M-m)}.\varrho\,\mu_{11} \ldots \mu_{1m}\,x_{(n+1)} \ldots x_{(n+M-m)}$
- $\nu_1' = \lambda x_1 \ldots x_n x_{(n+1)} \ldots x_{(n+M-m)}.\varrho^{\mathsf{src}}\,\nu_{11} \ldots \nu_{1m}\,x_{(n+1)} \ldots x_{(n+M-m)}$
- $\mu_2' = \lambda x_1 \ldots x_p x_{(p+1)} \ldots x_{(p+M-q)}.\varrho\,\mu_{21} \ldots \mu_{2q}\,x_{(p+1)} \ldots x_{(p+M-q)}$

- $\nu_2' = \lambda x_1 \dots x_p x_{(p+1)} \dots x_{(p+M-q)}.\varrho^{\mathsf{src}}\, \nu_{21} \dots \nu_{2q}\, x_{(p+1)} \dots x_{(p+M-q)}$

Note that by defining:

$$\begin{array}{lcll} \mu_{1i} = \nu_{1i} & \overset{\text{def}}{=} & x_{n+i-m} & \text{for all } m+1 \le i \le M \\ \mu_{2i} = \nu_{2i} & \overset{\text{def}}{=} & x_{p+i-q} & \text{for all } q+1 \le i \le M \end{array}$$

we have that the following equalities hold:

$$\begin{array}{rcl} \mu_1' & = & \lambda x_1 \dots x_{(n+M-m)}.\rho\, \mu_{11} \dots \mu_{1M} \\ \nu_1' & = & \lambda x_1 \dots x_{(n+M-m)}.\rho^{\mathsf{src}}\, \nu_{11} \dots \nu_{1M} \\ \mu_2' & = & \lambda x_1 \dots x_{(p+M-q)}.\rho\, \mu_{21} \dots \mu_{2M} \\ \nu_2' & = & \lambda x_1 \dots x_{(p+M-q)}.\rho^{\mathsf{src}}\, \nu_{21} \dots \nu_{2M} \end{array}$$

Before going on, we need to establish two auxiliary facts:

[A] Note that $(\mu_1')^\flat = \mu_1^\flat = \mu_2^\flat = (\mu_2')^\flat$. Then we have that:

$$\begin{array}{lll} & \rho\, \mu_{11} \dots \mu_{1M}\, x_{(n+M-m+1)} \dots x_N & \\ \overset{\flat}{\hookleftarrow}^* & \mu_1'\, x_1 \dots x_N & \text{by } \mathcal{F}\text{-}\mathsf{BetaM}\ (N \text{ times}) \\ \overset{\flat}{\mapsto}^* & (\mu_1')^\flat\, x_1 \dots x_N & \\ = & (\mu_2')^\flat\, x_1 \dots x_N & \text{as remarked} \\ \overset{\flat}{\mapsto}^* & \rho\, \mu_{21} \dots \mu_{2M}\, x_{(p+M-q+1)} \dots x_N & \text{by } \mathcal{F}\text{-}\mathsf{BetaM}\ (N \text{ times}) \end{array}$$

By confluence of flattening (Prop. 96) this implies that:

$$(\rho\, \mu_{11} \dots \mu_{1M}\, x_{(n+M-m+1)} \dots x_N)^\flat = (\rho\, \mu_{21} \dots \mu_{2M}\, x_{(p+M-q+1)} \dots x_N)^\flat$$

Since ϱ is a "rigid" symbol, from this we obtain that $\mu_{1i}^\flat = \mu_{2i}^\flat$ holds for all $1 \le i \le M$.

[B] On the other hand, note that $(\nu_1')^\flat = \nu_1^\flat = \nu_2^\flat = (\nu_2')^\flat$ so we have that:

$$\begin{array}{lll} & \rho^{\mathsf{src}}\, \nu_{11} \dots \nu_{1M}\, x_{(n+M-m+1)} \dots x_N & \\ \overset{\flat}{\hookleftarrow}^* & \nu_1'\, x_1 \dots x_N & \text{by } \mathcal{F}\text{-}\mathsf{BetaM}\ (N \text{ times}) \\ \overset{\flat}{\mapsto}^* & (\nu_1')^\flat\, x_1 \dots x_N & \\ = & (\nu_2')^\flat\, x_1 \dots x_N & \text{as remarked} \\ \overset{\flat}{\mapsto}^* & \rho^{\mathsf{src}}\, \nu_{21} \dots \nu_{2M}\, x_{(p+M-q+1)} \dots x_N & \text{by } \mathcal{F}\text{-}\mathsf{BetaM}\ (N \text{ times}) \end{array}$$

By confluence of flattening (Prop. 96) this implies that:

$$(\rho^{\mathsf{src}}\, \nu_{11} \dots \nu_{1M}\, x_{(n+M-m+1)} \dots x_N)^\flat = (\rho^{\mathsf{src}}\, \nu_{21} \dots \nu_{2M}\, x_{(p+M-q+1)} \dots x_N)^\flat$$

By the matching lemma (Lem. 131), from this we obtain that $\nu_{1i}^\flat = \nu_{2i}^\flat$ holds for all $1 \le i \le M$.

To conclude the proof, we consider two symmetric cases, depending on whether $m \le q$ or $q \le m$. Without loss of generality, suppose that $m \le q$. Note that:

[C] For every i such that $1 \le i \le m$ all of the following conditions hold:

$$\begin{array}{lll} \mu_{1i} \uparrow \nu_{1i} & \mu_{2i} \uparrow \nu_{2i} & \\ \mu_{1i}^\flat = \mu_{2i}^\flat & \nu_{1i}^\flat = \nu_{2i}^\flat & \text{by } [\mathbf{A}] \text{ and } [\mathbf{B}] \\ \mu_{1i} /\!\!/\!\!/ \nu_{1i} \Rightarrow \xi_{1i} & \mu_{2i} /\!\!/\!\!/ \nu_{2i} \Rightarrow \xi_{2i} & \end{array}$$

so by IH we have that $\xi_{1i}^\flat = \xi_{2i}^\flat$.

[D] For every i such that $m < i \leq q$ we have that $\mu^\flat_{1i} = \nu^\flat_{1i} = x_{(n+i-m)}$ by definition, and therefore $\mu^\flat_{2i} = \nu^\flat_{2i} = x_{(n+i-m)}$ by **[A]** and **[B]**. Hence by Lem. 133 we have that $\xi^\flat_{2i} = x_{(n+i-m)}$.

[E] Thus we may build the following chain of reductions:

$$\begin{array}{lll}
 & (\lambda x_1 \dots x_n.\varrho\,\xi_{11} \dots \xi_{1m})\,x_1 \dots x_N & \\
\overset{\flat}{\mapsto}{}^* & \varrho\,\xi_{11} \dots \xi_{1m}\,x_{(n+1)} \dots x_N & \text{by } \mathcal{F}\text{-BetaM } (n \text{ times}) \\
\overset{\flat}{\mapsto}{}^* & \varrho\,\xi^\flat_{11} \dots \xi^\flat_{1m}\,x_{(n+1)} \dots x_N & \\
\overset{\flat}{\mapsto}{}^* & \varrho\,\xi^\flat_{21} \dots \xi^\flat_{2m}\,x_{(n+1)} \dots x_N & \text{by } [\mathbf{C}] \\
= & \varrho\,\xi^\flat_{21} \dots \xi^\flat_{2m}\,\xi^\flat_{2(m+1)} \dots \xi^\flat_{2q}\,x_{(p+1)} \dots x_N & \text{by } [\mathbf{D}] \\
\overset{\flat}{\leftarrow\!\!\dashv}{}^* & \varrho\,\xi_{21} \dots \xi_{2q}\,x_{(p+1)} \dots x_N & \\
\overset{\flat}{\leftarrow\!\!\dashv}{}^* & (\lambda x_1 \dots x_p.\varrho\,\xi_{21} \dots \xi_{2q})\,x_1 \dots x_N & \text{by } \mathcal{F}\text{-BetaM } (p \text{ times})
\end{array}$$

Finally, we have:

$$\begin{array}{rcll}
\xi^\flat_1 & = & (\lambda x_1 \dots x_n.\varrho\,\xi_{11} \dots \xi_{1m})^\flat & \\
 & = & (\lambda x_1 \dots x_N.(\lambda x_1 \dots x_n.\varrho\,\xi_{11} \dots \xi_{1m})\,x_1 \dots x_N)^\flat & \text{by confluence using } \mathcal{F}\text{-EtaM} \\
 & = & (\lambda x_1 \dots x_N.(\lambda x_1 \dots x_p.\varrho\,\xi_{21} \dots \xi_{2q})\,x_1 \dots x_N)^\flat & \text{by confluence and } [\mathbf{E}] \\
 & = & (\lambda x_1 \dots x_p.\varrho\,\xi_{21} \dots \xi_{2q})^\flat & \text{by confluence using } \mathcal{F}\text{-EtaM} \\
 & = & \xi^\flat_2 &
\end{array}$$

◀

▶ **Definition 136** (η-expanded source). *If μ is a multistep in $\overline{\eta}$-normal form, we write $\mu^{<\overline{\eta}}$ for the source of μ in which the sources of rule symbols are also η-expanded. More precisely:*

$$\begin{array}{rcll}
x^{<\overline{\eta}} & \stackrel{\text{def}}{=} & x & \\
\mathbf{c}^{<\overline{\eta}} & \stackrel{\text{def}}{=} & \mathbf{c} & \\
\varrho^{<\overline{\eta}} & \stackrel{\text{def}}{=} & s' & \text{if } \varrho : s \multimap t : A \in \mathcal{R} \text{ and } s' \text{ is the } \overline{\eta}\text{-normal form of } s \\
(\lambda x.\mu)^{<\overline{\eta}} & \stackrel{\text{def}}{=} & \lambda x.s^{<\overline{\eta}} & \\
(\mu\,\nu)^{<\overline{\eta}} & \stackrel{\text{def}}{=} & \mu^{<\overline{\eta}}\,\nu^{<\overline{\eta}} &
\end{array}$$

Note that $\mu^{<\overline{\eta}} \overset{\flat}{\mapsto}{}^* \mu^{\mathsf{src}}$ so, in particular, $(\mu^{<\overline{\eta}})^\flat = (\mu^{\mathsf{src}})^\flat = \mu^\blacktriangleleft$.

▶ **Lemma 137** (Constructor/rule matching). *Let $\mu = \lambda x_1 \dots x_n.\mathbf{c}\,\mu_1 \dots \mu_m$ and $\nu = \lambda y_1 \dots y_p.\varrho\,\nu_1 \dots \nu_q$. Let $\varrho : l_0 \multimap r_0 : A \in \mathcal{R}$ and let l, r be the $\to_{\overline{\eta}}$-normal forms of l_0, r_0 respectively. Suppose that $\mu^{\mathsf{src}} =_{\beta\eta} \nu^{\mathsf{src}}$ and, moreover, that μ and ν are in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form. Then there exist multisteps $\xi_1, \dots, \xi_q$ such that:*

$$\mu = (\lambda x_1 \dots x_n.\underline{l}\,\xi_1 \dots \xi_q)^\circ$$

and $(\xi_i^{<\overline{\eta}})^\circ = (\nu_i^{<\overline{\eta}})^\circ$ for all $1 \leq i \leq q$.
Furthermore, the multisteps ξ_i can be chosen in such a way that they are in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form, and the number of applications in each multistep ξ_i, for all $1 \leq i \leq k$, is strictly less than the number of applications in μ.

Proof. Note that the condition $\mu^{\mathsf{src}} =_{\beta\eta} \nu^{\mathsf{src}}$ implies that $(\mu^{\mathsf{src}})^\flat = (\nu^{\mathsf{src}})^\flat$. Then also $(\mu^{<\overline{\eta}})^\flat = (\nu^{<\overline{\eta}})^\flat$. By Lem. 110 this implies that:

$$(\mu^{<\overline{\eta}})^\circ = (\nu^{<\overline{\eta}})^\circ \tag{6}$$

Suppose $l = \lambda z_1.\dots.\lambda z_k.\mathbf{d}\, r_1 \dots r_{k'}$. From (6) we have:

$$(\lambda x_1 \dots x_n.\mathbf{c}\, \mu_1^{<\overline{\eta}} \dots \mu_m^{<\overline{\eta}})^\circ = (\lambda y_1 \dots y_p.l\, \nu_1^{<\overline{\eta}} \dots \nu_q^{<\overline{\eta}})^\circ$$

Since $\mathbf{c}\, \mu_1^{<\overline{\eta}} \dots \mu_m^{<\overline{\eta}}$ and $l\, \nu_1^{<\overline{\eta}} \dots \nu_q^{<\overline{\eta}}$ have base types, then $n = p$ and we may assume that $x_i = y_i$, for all $1 \leq i \leq p$. Moreover,

$$(\mathbf{c}\, \mu_1^{<\overline{\eta}} \dots \mu_m^{<\overline{\eta}})^\circ = (l\, \nu_1^{<\overline{\eta}} \dots \nu_q^{<\overline{\eta}})^\circ$$

Therefore $\mathbf{d} = \mathbf{c}$, $q = k$ and $m = k'$, and:

$$(\mu_i^{<\overline{\eta}})^\circ = (r_i\{z_1\backslash\nu_1^{<\overline{\eta}}\} \dots \{z_k\backslash\nu_k^{<\overline{\eta}}\})^\circ$$

for all $1 \leq i \leq m$. Note that each r_i is part of the pattern of the rewrite rule l. By orthogonality of $\mathcal{R}$, there exist $\xi_1, \dots, \xi_k$ such that:

$$\mu_i = (r_i\{z_1\backslash\xi_1\} \dots \{z_k\backslash\xi_k\})^\circ \quad \text{for all } 1 \leq i \leq m$$

and

$$(\xi_i^{<\overline{\eta}})^\circ = (\nu_i^{<\overline{\eta}})^\circ \quad \text{for all } 1 \leq i \leq k$$

Therefore:

$$\mu = \lambda x_1 \dots x_n.\mathbf{c}\, \mu_1 \dots \mu_m = (\lambda x_1 \dots x_n.l\, \xi_1 \dots \xi_k)^\circ$$

Furthermore, we claim that each ξ_i, for $1 \leq i \leq k$, can be chosen in such a way that it is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form and with strictly less applications than μ. First, note that if the ξ_i are not in $\overset{\circ}{\mapsto}$-normal form, then we may take $\hat{\xi}_i := \xi_i^\circ$ instead, and we still have that $\mu = (\lambda x_1 \dots x_n.l\, \xi_1 \dots \xi_k)^\circ = (\lambda x_1 \dots x_n.l\, \hat{\xi}_1 \dots \hat{\xi}_k)^\circ$ by confluence of flattening (Prop. 96). So let us assume that the ξ_i are in $\overset{\circ}{\mapsto}$-normal form. Furthermore, suppose that ξ_i is of arity N, *i.e.* its type is of the form $A_1 \to \dots \to A_N \to \alpha$ with α a base type, and suppose that $\xi_i = \lambda w_1 \dots w_M.\xi_i'$ where ξ_i' is not a λ-abstraction. Note that $M \leq N$. Take $\hat{\xi}_i := \lambda w_1 \dots w_M w_{M+1} \dots w_N.\xi_i'\, w_{M+1} \dots w_N$. Note that each $\hat{\xi}_i$ is in $\overset{\circ}{\mapsto}$-normal form.

To conclude, we claim that $\mu = (\lambda x_1 \dots x_n.\underline{l}\, \hat{\xi}_1 \dots \hat{\xi}_k)^\circ$ and that each $\hat{\xi}_i$ has strictly less applications than μ. To see this, note that each variable $z_1, \dots, z_k$. occurs free exactly once in the body of l, applied to different bound variables. More precisely, for a fixed index $1 \leq i_0 \leq k$ there is exactly one $1 \leq j_0 \leq m$ such that z_{i_0} occurs free in r_j. Then the multistep that results from substituting in r_j each z_i by $\hat{\xi}_i$ other than for $i = i_0$ contains exactly one occurrence of z_{i_0} applied to N variables, where N is the arity of $\hat{\xi}_{i_0}$, that is, it is of the form:

$$r_j\{z_1\backslash\xi_1\} \dots \{z_{(i_0-1)}\backslash\xi_{(i_0-1)}\}\{z_{(i_0+1)}\backslash\xi_{(i_0+1)}\} \dots \{z_k\backslash\xi_k\} = \mathtt{C}\langle z_{i_0}\, w_1 \dots w_N\rangle$$

where $w_1, \dots, w_N$ are bound by $\mathtt{C}$ and the hole of $\mathtt{C}$ is not applied. As a consequence, we have that:

$$\begin{array}{rcl} \mu & = & \lambda x_1 \dots x_n.\mathbf{c}\mu_1 \dots \mu_{i_0-1}\, \mathtt{C}\langle(\hat{\xi}_{i_0}\, w_1 \dots\, w_N)^\circ\rangle\, \mu_{i_0+1} \dots \mu_m \\ & = & \lambda x_1 \dots x_n.\mathbf{c}\mu_1 \dots \mu_{i_0-1}\, \mathtt{C}\langle((\lambda w_1 \dots w_N.\xi_{i_0}\, w_{M+1} \dots w_N)\, w_1 \dots\, w_N)^\circ\rangle\, \mu_{i_0+1} \dots \mu_m \\ & = & \lambda x_1 \dots x_n.\mathbf{c}\mu_1 \dots \mu_{i_0-1}\, \mathtt{C}\langle\xi_{i_0}\, w_{M+1} \dots w_N\rangle\, \mu_{i_0+1} \dots \mu_m \end{array}$$

Hence, since μ is in $\overline{\eta}$-normal form, and $\xi_{i_0}\, w_{M+1} \dots w_N$ is a subterm of μ which is not applied, we know that $\xi_{i_0}\, w_{M+1} \dots w_N$ is in $\overline{\eta}$-normal form. Hence $\hat{\xi}_{i_0} = \lambda w_1 \dots w_n.\xi_{i_0}\, w_{M+1} \dots w_N$ is also in $\overline{\eta}$-normal form.

Finally, note that $(\lambda x_1 \ldots x_n.\underline{l}\,\hat{\xi}_1 \ldots \hat{\xi}_k)^\circ$ is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form and that $\mu^\flat = (\lambda x_1 \ldots x_n.\underline{l}\,\xi_1 \ldots \xi_k)^\flat = (\lambda x_1 \ldots x_n.\underline{l}\,\hat{\xi}_1 \ldots \hat{\xi}_k)^\flat$ so by Lem. 110 $\mu^\circ = (\lambda x_1 \ldots x_n.\underline{l}\,\hat{\xi}_1 \ldots \hat{\xi}_k)^\circ$. Moreover, for each $1 \leq i_0 \leq k$ the multistep $\hat{\xi}_{i_0} = \lambda x_1 \ldots x_N.\xi_{i_0}\,x_{M+1} \ldots x_N$ has the same number of applications as $\xi_{i_0}\,x_{M+1} \ldots x_M$, which is a subterm of μ, and has strictly less applications than μ. ◀

▶ **Lemma 138** (Compatibilization of coinitial multisteps). *Let μ, ν be multisteps such that $\mu^\blacktriangleleft = \nu^\blacktriangleleft$. Then there exist multisteps $\hat{\mu}, \hat{\nu}$ such that $\hat{\mu} \uparrow \hat{\nu}$ and moreover $\hat{\mu}^\flat = \mu^\flat$ and $\hat{\nu}^\flat = \nu^\flat$.*

Proof. We begin with a few observations. Recall that if $\mu \overset{\flat}{\mapsto}{}^* \mu'$ then $\mu^{\mathsf{src}} =_{\beta\eta} \mu'^{\mathsf{src}}$, which is a consequence of Rem. 84 and Lem. 87. This means that, without loss of generality, we may assume that μ, ν are in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form. Note that, given this assumption, we have that $\mu^{<\overline{\eta}}$ and $\mu^{<\overline{\eta}}$ are in $\overline{\eta}$-normal form. Moreover, $(\mu^{<\overline{\eta}})^\flat = (\mu^{\mathsf{src}})^\flat = (\nu^{\mathsf{src}})^\flat = (\nu^{<\overline{\eta}})^\flat$. From this, by Lem. 110, we have that $(\mu^{<\overline{\eta}})^\circ = (\nu^{<\overline{\eta}})^\circ$.

The proof proceeds by induction on the sum of the number of applications in μ and ν, using the characterization of flat multisteps (Lem. 101). We consider three cases, depending on whether the head of μ is a variable, a constant, or a rule symbol:

1. **Variable, $\mu = \lambda x_1 \ldots x_n.y\,\mu_1 \ldots \mu_m$.** Note that $(\mu^{<\overline{\eta}})^\circ = \lambda x_1 \ldots x_n.y\,(\mu_1^{<\overline{\eta}})^\circ \ldots (\mu_m^{<\overline{\eta}})^\circ$. Since $(\mu^{<\overline{\eta}})^\circ = (\nu^{<\overline{\eta}})^\circ$, the head of ν cannot be a constant or a rule symbol. Hence the only possibility is that $\nu = \lambda x_1 \ldots x_p.y\,\nu_1 \ldots \nu_q$. Furthermore, it can only be the case that $p = n$ and $q = m$ and $(\mu_i^{<\overline{\eta}})^\circ = (\nu_i^{<\overline{\eta}})^\circ$ for all $1 \leq i \leq m$. This in turn implies that $\mu_i^\blacktriangleleft = \nu_i^\blacktriangleleft$ for all $1 \leq i \leq m$. Hence, by IH, for each $1 \leq i \leq m$, there exist multisteps such that $\hat{\mu}_i \uparrow \hat{\nu}_i$ where $\hat{\mu}_i^\flat = \mu_i^\flat$ and $\hat{\nu}_i^\flat = \nu_i^\flat$. To conclude, note that taking $\hat{\mu} := \lambda x_1 \ldots x_n.y\,\hat{\mu}_1 \ldots \hat{\mu}_m$ and $\hat{\nu} := \lambda x_1 \ldots x_n.y\,\hat{\nu}_1 \ldots \hat{\nu}_m$ we have that $\hat{\mu} \uparrow \hat{\nu}$ by the CVar rule, and moreover $\hat{\mu}^\flat = \mu^\flat$ and $\hat{\nu}^\flat = \nu^\flat$.
2. **Constant, $\mu = \lambda x_1 \ldots x_n.\mathbf{c}\,\mu_1 \ldots \mu_m$.** Note that $(\mu^{<\overline{\eta}})^\circ = \lambda x_1 \ldots x_n.\mathbf{c}\,(\mu_1^{<\overline{\eta}})^\circ \ldots (\mu_m^{<\overline{\eta}})^\circ$. Since $(\nu^{<\overline{\eta}})^\circ = (\mu^{<\overline{\eta}})^\circ$, the head of ν cannot be a variable. We consider two cases, depending on whether the head of ν is a constant or a rule symbol:
   - **2.1 Constant, $\nu = \lambda x_1 \ldots x_p.\mathbf{c}\,\nu_1 \ldots \nu_q$.** The proof of this case proceeds similarly as for case 1, when the heads of μ and ν are both variables.
   - **2.2 Rule symbol, $\nu = \lambda x_1 \ldots x_p.\varrho\,\nu_1 \ldots \nu_q$.** Recall that we assume that μ and ν are in $\overline{\eta}$-normal form. This implies that $n = p$. Then $(\nu^{<\overline{\eta}})^\circ = (\lambda x_1 \ldots x_n.\varrho^{<\overline{\eta}}\,\nu_1^{<\overline{\eta}} \ldots \nu_q^{<\overline{\eta}})^\circ$. By Lem. 137, there exist multisteps $\xi_1, \ldots, \xi_q$ such that $\mu = (\lambda x_1 \ldots x_n.\varrho^{<\overline{\eta}}\,\xi_1 \ldots \xi_q)^\circ$ and $(\xi_i^{<\overline{\eta}})^\circ = (\nu_i^{<\overline{\eta}})^\circ$ for all $1 \leq i \leq q$. Moreover, each ξ_i has less applications than μ. So by IH, for each $1 \leq i \leq q$ there exist multisteps such that $\hat{\xi}_i \uparrow \hat{\nu}_i$ where $\hat{\xi}_i^\flat = \xi_i^\flat$. and $\hat{\nu}_i^\flat = \nu_i^\flat$. Taking $\hat{\mu} := \lambda x_1 \ldots x_n.\varrho^{<\overline{\eta}}\,\hat{\xi}_1 \ldots \hat{\xi}_q$ and $\hat{\nu} := \lambda x_1 \ldots x_n.\varrho\,\hat{\nu}_1 \ldots \hat{\nu}_q$ we have that $\hat{\mu} \uparrow \hat{\nu}$. It is easy to note that $\hat{\nu}^\flat = \nu^\flat$. Moreover:

$$
\begin{array}{rcll}
\hat{\mu}^\flat & = & (\lambda x_1 \ldots x_n.\varrho^{<\overline{\eta}}\,\hat{\xi}_1 \ldots \hat{\xi}_q)^\flat & \\
& = & (\lambda x_1 \ldots x_n.\varrho^{<\overline{\eta}}\,\xi_1 \ldots \xi_q)^\flat & \text{by confluence of flattening (Prop. 96)} \\
& = & ((\lambda x_1 \ldots x_n.\varrho^{<\overline{\eta}}\,\xi_1 \ldots \xi_q)^\circ)^\flat & \text{by confluence of flattening (Prop. 96)} \\
& = & \mu^\flat &
\end{array}
$$

3. **Rule symbol, $\mu = \lambda x_1 \ldots x_n.\varrho\,\mu_1 \ldots \mu_m$.** Note that $(\mu^{<\overline{\eta}})^\circ = \lambda x_1 \ldots x_n.(\varrho^{<\overline{\eta}}\,\mu_1^{<\overline{\eta}} \ldots \mu_m^{<\overline{\eta}})^\circ$. Since $(\nu^{<\overline{\eta}})^\circ = (\mu^{<\overline{\eta}})^\circ$, the head of ν cannot be a variable. We consider two cases, depending on whether the head of ν is a constant or a rule symbol:
   - **3.1 Constant, $\nu = \lambda x_1 \ldots x_p.\mathbf{c}\,\nu_1 \ldots \nu_q$.** The proof of this case proceeds symmetrically as for case 2.2, when the head of μ is a constant and the head of ν is a rule symbol.
   - **3.2 Rule symbol, $\nu = \lambda x_1 \ldots x_p.\vartheta\,\nu_1 \ldots \nu_q$.** Recall that we assume that μ and ν are in $\overline{\eta}$-normal form. This implies that $n = p$. Then $(\nu^{<\overline{\eta}})^\circ = \lambda x_1 \ldots x_n.(\vartheta^{<\overline{\eta}}\,\nu_1^{<\overline{\eta}} \ldots \nu_q^{<\overline{\eta}})^\circ$.

Note that this implies that $(\varrho^{<\overline{\eta}}\,\mu_1^{<\overline{\eta}}\ldots\nu_m^{<\overline{\eta}})^\circ = (\vartheta^{<\overline{\eta}}\,\nu_1^{<\overline{\eta}}\ldots\nu_q^{<\overline{\eta}})^\circ$. By orthogonality, this means that $\varrho = \vartheta$ and $m = q$, and moreover $(\mu_i^{<\overline{\eta}})^\circ = (\nu_i^{<\overline{\eta}})^\circ$ for all $1 \leq i \leq m$. By IH, for each $1 \leq i \leq m$, there exist multisteps such that $\hat{\mu}_i \uparrow \hat{\nu}_i$, where $\hat{\mu}_i^\flat = \mu_i^\flat$ and $\hat{\nu}_i^\flat = \nu_i^\flat$. Taking $\hat{\mu} = \lambda x_1 \ldots x_n.\varrho\,\hat{\mu}_1 \ldots \hat{\mu}_m$ and $\hat{\nu} = \lambda x_1 \ldots x_n.\varrho\,\hat{\nu}_1 \ldots \hat{\nu}_m$ it is then easy to check that $\hat{\mu} \uparrow \hat{\nu}$, and moreover $\hat{\mu}^\flat = \mu^\flat$ and $\hat{\nu}^\flat = \nu^\flat$.

◀

▶ **Lemma 139** (Projection of a substitution). *Let $\mu_1 /\!\!/\!\!/ \nu_1 \Rightarrow \xi_1$ and $\mu_2 /\!\!/\!\!/ \nu_2 \Rightarrow \xi_2$. Then $\mu_1\{x\backslash\mu_2\} /\!\!/\!\!/ \nu_1\{x\backslash\nu_2\} \Rightarrow \xi_1\{x\backslash\xi_2\}$*

Proof. By induction on the derivation of $\mu_1 /\!\!/\!\!/ \nu_1 \Rightarrow \xi_1$:

1. ProjVar: Let $y /\!\!/\!\!/ y \Rightarrow y$. If $y \neq x$, it is immediate. If $y = x$, note that $\mu_2 /\!\!/\!\!/ \nu_2 \Rightarrow \xi_2$ by hypothesis.
2. ProjCon: Let $\mathbf{c} /\!\!/\!\!/ \mathbf{c} \Rightarrow \mathbf{c}$. This case is immediate, as $\mathbf{c}\{x\backslash\mu_2\} = \mathbf{c}\{x\backslash\nu_2\} = \mathbf{c}\{x\backslash\xi_2\} = \mathbf{c}$.
3. ProjRule: Let $\varrho /\!\!/\!\!/ \varrho \Rightarrow \varrho^{\mathsf{tgt}}$. This case is immediate, as ϱ^{tgt} is closed, so $\varrho\{x\backslash\mu_2\} = \varrho\{x\backslash\nu_2\} = \varrho$ and $\varrho^{\mathsf{tgt}}\{x\backslash\xi_2\} = \varrho^{\mathsf{tgt}}$.
4. ProjRuleL: Let $\varrho /\!\!/\!\!/ \varrho^{\mathsf{src}} \Rightarrow \varrho$. This case is immediate, as ϱ^{src} is closed, so $\varrho\{x\backslash\mu_2\} = \varrho\{x\backslash\xi_2\} = \varrho$ and $\varrho^{\mathsf{src}}\{x\backslash\nu_2\} = \varrho^{\mathsf{src}}$.
5. ProjRuleR: Let $\varrho^{\mathsf{src}} /\!\!/\!\!/ \varrho \Rightarrow \varrho^{\mathsf{tgt}}$. This case is immediate, as ϱ^{src} and ϱ^{tgt} are closed, so $\varrho^{\mathsf{src}}\{x\backslash\mu_2\} = \varrho^{\mathsf{src}}$ and $\varrho\{x\backslash\nu_2\} = \varrho$ and $\varrho^{\mathsf{tgt}}\{x\backslash\xi_2\} = \varrho^{\mathsf{tgt}}$.
6. ProjAbs: Let $\lambda y.\mu_1 /\!\!/\!\!/ \lambda y.\nu_1 \Rightarrow \lambda y.\xi_1$ be derived from $\mu_1 /\!\!/\!\!/ \nu_1 \Rightarrow \xi_1$. By IH $\mu_1\{x\backslash\mu_2\} /\!\!/\!\!/ \nu_1\{x\backslash\nu_2\} \Rightarrow \xi_1\{x\backslash\xi_2\}$, so applying the ProjAbs rule $\lambda y.\mu_1\{x\backslash\mu_2\} /\!\!/\!\!/ \lambda y.\nu_1\{x\backslash\nu_2\} \Rightarrow \lambda y.\xi_1\{x\backslash\xi_2\}$.
7. ProjApp: Let $\mu_{11}\,\mu_{12} /\!\!/\!\!/ \nu_{11}\,\nu_{12} \Rightarrow \xi_{11}\,\xi_{12}$ be derived from $\mu_{11} /\!\!/\!\!/ \nu_{11} \Rightarrow \xi_{11}$ and $\mu_{12} /\!\!/\!\!/ \nu_{12} \Rightarrow \xi_{12}$. Then by IH $\mu_{11}\{x\backslash\mu_2\} /\!\!/\!\!/ \nu_{11}\{x\backslash\nu_2\} \Rightarrow \xi_{11}\{x\backslash\xi_2\}$ and $\mu_{12}\{x\backslash\mu_2\} /\!\!/\!\!/ \nu_{12}\{x\backslash\nu_2\} \Rightarrow \xi_{12}\{x\backslash\xi_2\}$, so applying the ProjApp rule $(\mu_{11}\,\mu_{12})\{x\backslash\mu_2\} /\!\!/\!\!/ (\nu_{11}\,\nu_{12})\{x\backslash\nu_2\} \Rightarrow (\xi_{11}\,\xi_{12})\{x\backslash\xi_2\}$.

◀

▶ **Lemma 140** (Compatibilization of projection). *Let $\mu /\!\!/\!\!/ \nu \Rightarrow \xi$. Then there exist multisteps $\hat{\mu}, \hat{\nu}, \hat{\xi}$ such that:*

1. $\mu \overset{\flat}{\mapsto}{}^* \hat{\mu}$ *and* $\nu \overset{\flat}{\mapsto}{}^* \hat{\nu}$ *and* $\xi \overset{\flat}{\mapsto}{}^* \hat{\xi}$
2. $\hat{\mu} \uparrow \hat{\nu}$
3. $\hat{\mu} /\!\!/\!\!/ \hat{\nu} \Rightarrow \hat{\xi}$

Proof. If $\mu \uparrow \nu$ holds, we are done. By strong normalization of flattening (Prop. 95) it suffices to show that if $\mu /\!\!/\!\!/ \nu \Rightarrow \xi$ and $\mu \uparrow \nu$ does *not* hold, then there exist steps $\mu \overset{\flat}{\mapsto} \hat{\mu}$ and $\nu \overset{\flat}{\mapsto} \hat{\nu}$ and $\xi \overset{\flat}{\mapsto} \hat{\xi}$ such that $\hat{\mu} /\!\!/\!\!/ \hat{\nu} \Rightarrow \hat{\xi}$. The proof proceeds by induction on the derivation of $\mu /\!\!/\!\!/ \nu \Rightarrow \xi$. Note that rules ProjVar, ProjCon, ProjRule, ProjRuleL, and ProjRuleR cannot apply, as then we would have that $\mu \uparrow \nu$. The remaining possibilities are:

1. ProjAbs: Let $\lambda x.\mu /\!\!/\!\!/ \lambda x.\nu \Rightarrow \lambda x.\xi$ be derived from $\mu /\!\!/\!\!/ \nu \Rightarrow \xi$. By IH there exist steps $\mu \overset{\flat}{\mapsto} \hat{\mu}$ and $\nu \overset{\flat}{\mapsto} \hat{\nu}$ and $\nu \overset{\flat}{\mapsto} \hat{\xi}$ such that $\hat{\mu} /\!\!/\!\!/ \hat{\nu} \Rightarrow \hat{\xi}$. Applying the ProjAbs rule, we obtain that $\lambda x.\hat{\mu} /\!\!/\!\!/ \lambda x.\hat{\nu} \Rightarrow \lambda x.\hat{\xi}$ as required.
2. ProjApp: Let $\mu_1\,\mu_2 /\!\!/\!\!/ \nu_1\,\nu_2 \Rightarrow \xi_1\,\xi_2$ be derived from $\mu_1 /\!\!/\!\!/ \nu_1 \Rightarrow \xi_1$ and $\mu_2 /\!\!/\!\!/ \nu_2 \Rightarrow \xi_2$. We consider three subcases:
   - 2.1 **If $\mu_1 \uparrow \nu_1$ does not hold.** Then by IH there exist steps $\mu_1 \overset{\flat}{\mapsto} \hat{\mu}_1$ and $\nu_1 \overset{\flat}{\mapsto} \hat{\nu}_1$ and $\xi_1 \overset{\flat}{\mapsto} \hat{\xi}_1$ such that $\hat{\mu}_1 /\!\!/\!\!/ \hat{\nu}_1 \Rightarrow \hat{\xi}_1$. Then we have that $\mu_1\,\mu_2 \overset{\flat}{\mapsto} \hat{\mu}_1\,\mu_2$ and $\nu_1\,\nu_2 \overset{\flat}{\mapsto} \hat{\nu}_1\,\nu_2$ and $\xi_1\,\xi_2 \overset{\flat}{\mapsto} \hat{\xi}_1\,\xi_2$ and, applying the ProjApp rule, $\hat{\mu}_1\,\mu_2 /\!\!/\!\!/ \hat{\nu}_1\,\nu_2 \Rightarrow \hat{\xi}_1\,\xi_2$.
   - 2.2 **If $\mu_1 \uparrow \nu_1$ holds and $\mu_2 \uparrow \nu_2$ does not hold.** Similar to the previous case.

2.3 **If both $\mu_1 \uparrow \nu_1$ and $\mu_2 \uparrow \nu_2$ hold.** If μ_1 and ν_1 do not start with a λ-abstraction, then it can be checked by case analysis on the rules defining the judgment $\mu_1 \uparrow \nu_1$ that $\mu_1\,\mu_2 \uparrow \nu_1\,\nu_2$ holds, contradicting the hypothesis. For example, if $\mu_1 \uparrow \nu_1$ is derived using the CVar rule, with the assumption that μ_1 does not start with a λ-abstraction, we would have that $\mu_1 = y\,\mu_{11} \ldots \mu_{1n}$ and $\nu_1 = y\,\nu_{11} \ldots \nu_{1n}$ with $\mu_{1i} \uparrow \nu_{1i}$ for all $1 \leq i \leq n$. And from this we would have, using the CVar rule, that $y\,\mu_{11} \ldots \mu_{1n}\,\mu_2 \uparrow y\,\nu_{11} \ldots \nu_{1n}\,\nu_2$. The argument is similar for the other rules besides CVar.
This means that $\mu_1 = \lambda x.\mu_1'$ and $\nu_1 = \lambda x.\nu_1'$. Note, moreover, that the judgment $\mu_1 /\!\!/\!\!/ \nu_1 \Rightarrow \xi_1$ can only be derived by the ProjAbs rule, so $\xi_1 = \lambda x.\xi_1'$ with $\mu_1' /\!\!/\!\!/ \nu_1' \Rightarrow \xi_1'$. To conclude note that by taking $\hat{\mu} := \mu_1'\{x\backslash\mu_2\}$ and $\hat{\nu} := \nu_1'\{x\backslash\nu_2\}$ and $\hat{\xi} := \xi_1'\{x\backslash\xi_2\}$ we have that $\mu = (\lambda x.\mu_1')\,\mu_2 \overset{\flat}{\mapsto}{}^* \hat{\mu}$ and $\nu = (\lambda x.\nu_1')\,\nu_2 \overset{\flat}{\mapsto}{}^* \hat{\nu}$ and $\xi = (\lambda x.\xi_1')\,\xi_2 \overset{\flat}{\mapsto}{}^* \hat{\xi}$. Moreover, $\hat{\mu} /\!\!/\!\!/ \hat{\nu} \Rightarrow \hat{\xi}$ is a consequence of Lem. 139.

◀

▶ **Proposition 141** (Existence and uniqueness of projection)**.** *Let μ, ν be such that $\mu^{\mathsf{src}} =_{\beta\eta} \nu^{\mathsf{src}}$. Then:*

1. **Existence.** *There exist $\hat{\mu}, \hat{\nu}, \hat{\xi}$ such that $\hat{\mu}^\flat = \mu^\flat$ and $\hat{\nu}^\flat = \nu^\flat$ and $\hat{\mu} /\!\!/\!\!/ \hat{\nu} \Rightarrow \hat{\xi}$.*
2. **Compatibility.** *Furthermore, $\hat{\mu}$ and $\hat{\nu}$ can be chosen in such a way that $\hat{\mu} \uparrow \hat{\nu}$.*
3. **Uniqueness.** *If $(\hat{\mu}')^\flat = \mu^\flat$ and $(\hat{\nu}')^\flat = \nu^\flat$ and $\hat{\mu}' /\!\!/\!\!/ \hat{\nu}' \Rightarrow \hat{\xi}'$ then $(\hat{\xi}')^\flat = \xi^\flat$.*

Proof. Note that $\mu^{◀} = \nu^{◀}$, so by Lem. 138 there exist multisteps $\hat{\mu}, \hat{\nu}$ such that $\hat{\mu} \uparrow \hat{\nu}$ and moreover $\hat{\mu}^\flat = \mu^\flat$ and $\hat{\nu}^\flat = \nu^\flat$. By Lem. 132 this implies that there exists a multistep $\hat{\xi}$ such that $\hat{\mu} /\!\!/\!\!/ \hat{\nu} \Rightarrow \hat{\xi}$. This proves items 1. and 2.

For item 3., suppose that $\hat{\mu}', \hat{\nu}', \hat{\xi}'$ are such that $(\hat{\mu}')^\flat = \mu^\flat$ and $(\hat{\nu}')^\flat = \nu^\flat$ and $\hat{\mu}' /\!\!/\!\!/ \hat{\nu}' \Rightarrow \hat{\xi}'$. By Lem. 140 this implies that there exist multisteps $\hat{\mu}'', \hat{\nu}'', \hat{\xi}''$ such that $\hat{\mu}' \overset{\flat}{\mapsto}{}^* \hat{\mu}''$ and $\hat{\nu}' \overset{\flat}{\mapsto}{}^* \hat{\nu}''$ and $\hat{\xi}' \overset{\flat}{\mapsto}{}^* \hat{\xi}''$, and moreover $\hat{\mu}'' \uparrow \hat{\nu}''$ and $\hat{\mu}'' /\!\!/\!\!/ \hat{\nu}'' \Rightarrow \hat{\xi}''$. Note that $\hat{\mu}'^\flat = \hat{\mu}''^\flat$ given that $\hat{\mu}' \overset{\flat}{\mapsto}{}^* \hat{\mu}''$. Similarly, $\hat{\nu}'^\flat = \hat{\nu}''^\flat$ and $\hat{\xi}'^\flat = \hat{\xi}''^\flat$. Hence by Lem. 135 we may conclude that $\hat{\xi}^\flat = \hat{\xi}'^\flat$, as required. ◀

▶ **Proposition 142** (Properties of the projection operator)**.**

1. $\mu/\nu = (\mu/\nu)^\flat$
2. $\mu/\nu = \mu^\flat/\nu^\flat$
3. $\mu/\mu = \mu^{▶}$ *and, as particular cases:*

$$\underline{s}/\underline{s} = \underline{s}^\flat \qquad x/x = x \qquad \mathbf{c}/\mathbf{c} = \mathbf{c} \qquad \varrho/\varrho = \varrho^{▶}$$

4. $\mu/\mu^{◀} = \mu^\flat$ *and, as a particular case, $\varrho/\varrho^{◀} = \varrho$*
5. $\mu^{◀}/\mu = \mu^{▶}$ *and, as a particular case, $\varrho^{◀}/\varrho = \varrho^{▶}$*
6. $(\lambda x.\mu)/(\lambda x.\nu) = (\lambda x.(\mu/\nu))^\flat$
7. $(\mu_1\,\mu_2)/(\nu_1\,\nu_2) = ((\mu_1/\nu_1)\,(\mu_2/\nu_2))^\flat$ *provided that μ_1/ν_1 and μ_2/ν_2 are defined.*

Proof. We prove each item:

1. $\mu/\nu = (\mu/\nu)^\flat$: By definition there exist $\hat{\mu}, \hat{\nu}, \hat{\xi}$ such that $\hat{\mu}^\flat = \mu^\flat$ and $\hat{\nu}^\flat = \nu^\flat$ and $\hat{\mu} /\!\!/\!\!/ \hat{\nu} \Rightarrow \hat{\xi}$ where $\mu/\nu = \hat{\xi}^\flat$. Then $\mu/\nu = \hat{\xi}^\flat = (\hat{\xi}^\flat)^\flat = (\mu/\nu)^\flat$.
2. $\mu/\nu = \mu^\flat/\nu^\flat$: By definition of μ/ν there exist $\hat{\mu}, \hat{\nu}, \hat{\xi}$ such that $\hat{\mu}^\flat = \mu^\flat$ and $\hat{\nu}^\flat = \nu^\flat$ and $\hat{\mu} /\!\!/\!\!/ \hat{\nu} \Rightarrow \hat{\xi}$ where $\mu/\nu = \hat{\xi}^\flat$. Then since $\mu^\flat = (\mu^\flat)^\flat$ and $\nu^\flat = (\nu^\flat)^\flat$, the triple $\hat{\mu}, \hat{\nu}, \hat{\xi}$ also fulfills the conditions for the definition of $\mu^\flat/\nu^\flat$. But Prop. 141 ensures uniqueness, so $\mu/\nu = \hat{\xi}^\flat = \mu^\flat/\nu^\flat$.
3. $\mu/\mu = \mu^{▶}$: It suffices to note that $\mu /\!\!/\!\!/ \mu \Rightarrow \mu^{\mathsf{tgt}}$ holds, as can be checked by induction on μ.

4. $\mu/\mu^{\blacktriangleleft} = \mu^{\flat}$: It suffices to note that $\mu /\!\!/\!\!/ \mu^{\mathsf{src}} \Rightarrow \mu$, as can be checked by induction on μ.
5. $\mu^{\blacktriangleleft}/\mu = \mu^{\blacktriangleright}$: It suffices to note that $\mu^{\mathsf{src}} /\!\!/\!\!/ \mu \Rightarrow \mu^{\mathsf{tgt}}$, as can be checked by induction on μ.
6. $(\lambda x.\mu)/(\lambda x.\nu) = (\lambda x.(\mu/\nu))^{\flat}$: Observe that the left-hand side of the equation is defined if and only if the right-hand side is defined, given that $(\lambda x.\mu)^{\mathsf{src}} =_{\beta\eta} (\lambda x.\nu)^{\mathsf{src}}$ if and only if $\mu^{\mathsf{src}} =_{\beta\eta} \nu^{\mathsf{src}}$, which is easy to check.
   By definition of μ/ν there exist $\hat{\mu}, \hat{\nu}, \hat{\xi}$ such that $\hat{\mu}^{\flat} = \mu^{\flat}$ and $\hat{\nu}^{\flat} = \nu^{\flat}$ and $\hat{\mu} /\!\!/\!\!/ \hat{\nu} \Rightarrow \hat{\xi}$ where $\mu/\nu = \hat{\xi}^{\flat}$. Then note, by confluence of flattening (Prop. 96), that $(\lambda x.\hat{\mu})^{\flat} = (\lambda x.\hat{\mu}^{\flat})^{\flat} = (\lambda x.\mu^{\flat})^{\flat} = (\lambda x.\mu)^{\flat}$ and, similarly, $(\lambda x.\hat{\nu})^{\flat} = (\lambda x.\nu)^{\flat}$. Moreover, by the ProjAbs rule, $\lambda x.\hat{\mu} /\!\!/\!\!/ \lambda x.\hat{\nu} \Rightarrow \lambda x.\hat{\xi}$. By uniqueness of projection (Prop. 141) this means that $(\lambda x.\mu)/(\lambda x.\nu) = (\lambda x.\hat{\xi})^{\flat} = (\lambda x.\hat{\xi}^{\flat})^{\flat} = (\lambda x.(\mu/\nu))^{\flat}$.
7. $(\mu_1\,\mu_2)/(\nu_1\,\nu_2) = ((\mu_1/\nu_1)\,(\mu_2/\nu_2))^{\flat}$: Observe that, if the right-hand side of the equation is defined, then the left-hand side is also defined, given that if $\mu_1^{\mathsf{src}} =_{\beta\eta} \nu_1^{\mathsf{src}}$ and $\mu_2^{\mathsf{src}} =_{\beta\eta} \nu_2^{\mathsf{src}}$ then $(\mu_1\,\mu_2)^{\mathsf{src}} =_{\beta\eta} (\nu_1\,\nu_2)^{\mathsf{src}}$.
   Note that, by hypothesis, the right-hand side of the equation is defined. By definition of μ_1/ν_1 there exist $\hat{\mu}_1, \hat{\nu}_1, \hat{\xi}_1$ such that $\hat{\mu}_1^{\flat} = \mu_1^{\flat}$ and $\hat{\nu}_1^{\flat} = \nu_1^{\flat}$ and $\hat{\mu}_1 /\!\!/\!\!/ \hat{\nu}_1 \Rightarrow \hat{\xi}_1$ where $\mu_1/\nu_1 = \hat{\xi}_1^{\flat}$. Similarly, by definition of μ_2/ν_2 there exist $\hat{\mu}_2, \hat{\nu}_2, \hat{\xi}_2$ such that $\hat{\mu}_2^{\flat} = \mu_2^{\flat}$ and $\hat{\nu}_2^{\flat} = \nu_2^{\flat}$ and $\hat{\mu}_2 /\!\!/\!\!/ \hat{\nu}_2 \Rightarrow \hat{\xi}_2$ where $\mu_2/\nu_2 = \hat{\xi}_2^{\flat}$.
   Note, by confluence of flattening (Prop. 96), $(\hat{\mu}_1\,\hat{\mu}_2)^{\flat} = (\hat{\mu}_1^{\flat}\,\hat{\mu}_2^{\flat})^{\flat} = (\mu_1^{\flat}\,\mu_2^{\flat})^{\flat} = (\mu_1\,\mu_2)^{\flat}$ and, similarly, $(\hat{\nu}_1\,\hat{\nu}_2)^{\flat} = (\nu_1\,\nu_2)^{\flat}$. Moreover, by the ProjApp rule, $\hat{\mu}_1\,\hat{\mu}_2 /\!\!/\!\!/ \hat{\nu}_1\,\hat{\nu}_2 \Rightarrow \hat{\xi}_1\,\hat{\xi}_2$. By uniqueness of projection Prop. 141 this means that $\mu_1\,\mu_2/\nu_1\,\nu_2 = (\hat{\xi}_1\,\hat{\xi}_2)^{\flat} = (\hat{\xi}_1^{\flat}\,\hat{\xi}_2^{\flat})^{\flat} = ((\mu_1/\nu_1)\,(\mu_2/\nu_2))^{\flat}$.

◀

▶ **Example 143.** Let $\varrho : \lambda x.\mathbf{c}\,x \to \lambda x.\mathbf{d}\,x$. Then:

$$\begin{array}{rcl}(\lambda x.(\lambda x.\mathbf{c}\,x)\,x)/(\lambda x.\varrho\,x) &=& (\lambda x.((\lambda x.\mathbf{c}\,x)\,x)/(\varrho\,x))^{\flat}\\ &=& (\lambda x.(((\lambda x.\mathbf{c}\,x)/\varrho)(x/x))^{\flat})^{\flat}\\ &=& (\lambda x.((\lambda x.\mathbf{d}\,x)\,x)^{\flat})^{\flat}\\ &=& (\lambda x.\mathbf{d}\,x)^{\flat}\\ &=& \mathbf{d}\end{array}$$

▶ **Lemma 144** (Cube lemma). *Given multisteps μ, ν, ξ such that $\mu^{\mathsf{src}} =_{\beta\eta} \nu^{\mathsf{src}} =_{\beta\eta} \xi^{\mathsf{src}}$, the following equality holds:*

$$(\mu/\nu)/(\xi/\nu) = (\mu/\xi)/(\nu/\xi)$$

Proof. The proof proceeds by induction on the number of applications in the $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form of μ. Note by Prop. 142 that, without loss of generality, we may assume that μ, ν, and ξ are in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form. The head of μ may be a variable, a constant, or a rule symbol. We consider three cases:

1. **μ headed by a variable, $\mu = \lambda\vec{x}.y\,\mu_1 \ldots \mu_n$.** Then since μ and ν are coinitial, the head of ν must be y, and since we assume that the multisteps are in $\overline{\eta}$-normal form, $\nu = \lambda\vec{x}.y\,\nu_1 \ldots \nu_n$. Similarly, since μ and ξ are coinitial, $\xi = \lambda\vec{x}.y\,\xi_1 \ldots \xi_n$. Then:

$$\begin{array}{rcll}(\mu/\nu)/(\xi/\nu) &=& (\lambda\vec{x}.y\,((\mu_1/\nu_1)/(\xi_1/\nu_1))\ldots((\mu_n/\nu_n)/(\xi_n/\nu_n)))^{\flat} & \text{by Prop. 142}\\ &=& (\lambda\vec{x}.y\,((\mu_1/\xi_1)/(\nu_1/\xi_1))\ldots((\mu_n/\xi_n)/(\nu_n/\xi_n)))^{\flat} & \text{by IH}\\ &=& (\mu/\xi)/(\nu/\xi) &\end{array}$$

2. **μ headed by a constant, $\mu = \lambda\vec{x}.\mathbf{c}\,\mu_1 \ldots \mu_n$.** Then since μ and ν are coinitial, the head of ν may be a constant or a rule symbol. We consider are two subcases:

2.1 ν **headed by a constant.** Then the head of ν must be the constant $\mathbf{c}$, and since the multisteps are in $\overline{\eta}$-normal form, $\nu = \lambda\vec{x}.\mathbf{c}\,\nu_1 \ldots \nu_n$. Note that, in turn, as μ and ξ are coinitial, the head of ξ may be a constant or a rule symbol. So we consider two further subcases:

2.1.1 ξ **headed by a constant.** Then the head of ξ must be the constant $\mathbf{c}$, and since the multisteps are in $\overline{\eta}$-normal form, $\xi = \lambda\vec{x}.\mathbf{c}\,\xi_1 \ldots \xi_n$. The proof of this case is similar to case 1, when the heads of all three multisteps are variables.

2.1.2 ξ **headed by a rule symbol.** Then since the multisteps are in $\overline{\eta}$-normal form, $\xi = \lambda\vec{x}.\varrho\,\xi_1 \ldots \xi_q$. Since μ and ξ are coinitial, by Lem. 137 there exist multisteps $\mu'_1, \ldots, \mu'_q$ in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form such that $\mu = (\lambda\vec{x}.\varrho^{\mathsf{src}}\,\mu'_1 \ldots \mu'_q)^\circ$ where each μ'_i has strictly less applications than μ, and such that μ'_i and ξ_i are coinitial. Similarly, since ν and ξ are coinitial, by Lem. 137 there exist multisteps $\nu'_1, \ldots, \nu'_q$ in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form such that $\nu = (\lambda\vec{x}.\varrho^{\mathsf{src}}\,\nu'_1 \ldots \nu'_q)^\circ$ and such that, for each $1 \leq i \leq q$, ν'_i and ξ_i are coinitial.
Before proceeding, note that, using Prop. 142, we have:

$$\begin{array}{rcl}
(\varrho^{\mathsf{src}}/\varrho^{\mathsf{src}})/(\varrho/\varrho^{\mathsf{src}}) & = & (\varrho^{\blacktriangleleft}/\varrho^{\blacktriangleleft})/(\varrho/\varrho^{\blacktriangleleft}) \\
 & = & \varrho^{\blacktriangleleft}/\varrho \\
 & = & \varrho^{\blacktriangleright} \\
 & = & \varrho^{\blacktriangleright}/\varrho^{\blacktriangleright} \\
 & = & (\varrho^{\blacktriangleleft}/\varrho)/(\varrho^{\blacktriangleleft}/\varrho) \\
 & = & (\varrho^{\mathsf{src}}/\varrho)/(\varrho^{\mathsf{src}}/\varrho)
\end{array}$$

Then:

$$\begin{array}{rl}
 & (\mu/\nu)/(\xi/\nu) \\
= & ((\lambda\vec{x}.\varrho^{\mathsf{src}}\,\mu'_1 \ldots \mu'_q)/(\lambda\vec{x}.\varrho^{\mathsf{src}}\,\nu'_1 \ldots \nu'_q))/((\lambda\vec{x}.\varrho\,\xi_1 \ldots \xi_q)/(\lambda\vec{x}.\varrho^{\mathsf{src}}\,\nu'_1 \ldots \nu'_q)) \\
 & \quad \text{by Prop. 142} \\
= & (\lambda\vec{x}.((\varrho^{\mathsf{src}}/\varrho^{\mathsf{src}})/(\varrho/\varrho^{\mathsf{src}}))\,((\mu'_1/\nu'_1)/(\xi_1/\nu'_1)) \ldots ((\mu'_q/\nu'_q)/(\xi_q/\nu'_q)))^\flat \\
 & \quad \text{by Prop. 142} \\
= & (\lambda\vec{x}.((\varrho^{\mathsf{src}}/\varrho)/(\varrho^{\mathsf{src}}/\varrho))\,((\mu'_1/\nu'_1)/(\xi_1/\nu'_1)) \ldots ((\mu'_q/\nu'_q)/(\xi_q/\nu'_q)))^\flat \\
 & \quad \text{by the preceding remark} \\
= & (\lambda\vec{x}.((\varrho^{\mathsf{src}}/\varrho)/(\varrho^{\mathsf{src}}/\varrho))\,((\mu'_1/\xi_1)/(\nu'_1/\xi_1)) \ldots ((\mu'_q/\xi_q)/(\nu'_q/\xi_q)))^\flat \\
 & \quad \text{by IH} \\
= & ((\lambda\vec{x}.\varrho^{\mathsf{src}}\,\mu'_1 \ldots \mu'_q)/(\lambda\vec{x}.\varrho\,\xi_1 \ldots \xi_q))/((\lambda\vec{x}.\varrho^{\mathsf{src}}\,\nu'_1 \ldots \nu'_q)/(\lambda\vec{x}.\varrho\,\xi_1 \ldots \xi_q)) \\
 & \quad \text{by Prop. 142} \\
= & (\mu/\xi)/(\nu/\xi) \\
 & \quad \text{by Prop. 142}
\end{array}$$

2.2 ν **headed by a rule symbol.** Then since the multisteps are in $\overline{\eta}$-normal form, $\nu = \lambda\vec{x}.\varrho\,\nu_1 \ldots \nu_q$ Note that, in turn, as μ and ξ are coinitial, the head of ξ may be a constant or a rule symbol. If the head of ξ is a constant, this case is symmetric to case 2.1.2, when ν is headed by a constant and ξ by a rule symbol. If the head of ξ is a rule symbol, then since the multisteps are in $\overline{\eta}$-normal form, $\xi = \lambda\vec{x}.\varrho\,\xi_1 \ldots \xi_q$. Since μ and ν are coinitial, by Lem. 137 there exist multisteps $\mu'_1, \ldots, \mu'_q$ in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form such that $\mu = (\lambda\vec{x}.\varrho^{\mathsf{src}}\,\mu'_1 \ldots \mu'_q)^\circ$ where each μ'_i has strictly less applications than μ, and such that μ'_i and ν_i are coinitial. Then:

$$\begin{array}{rl}
& (\mu/\nu)/(\xi/\nu) \\
= & ((\lambda\vec{x}.\varrho^{\mathsf{src}}\,\mu'_1\ldots\mu'_q)/(\lambda\vec{x}.\varrho\,\nu_1\ldots\nu_q))/((\lambda\vec{x}.\varrho\,\xi_1\ldots\xi_q)/(\lambda\vec{x}.\varrho\,\nu_1\ldots\nu_q)) \\
& \quad\text{by Prop. 142} \\
= & (\lambda\vec{x}.((\varrho^{\mathsf{src}}/\varrho)/(\varrho/\varrho))\,((\mu'_1/\nu_1)/(\xi_1/\nu_1))\ldots((\mu'_q/\nu_q)/(\xi_q/\nu_q)))^\flat \\
& \quad\text{by Prop. 142} \\
= & (\lambda\vec{x}.((\varrho^{\mathsf{src}}/\varrho)/(\varrho/\varrho))\,((\mu'_1/\xi_1)/(\nu_1/\xi_1))\ldots((\mu'_q/\xi_q)/(\nu_q/\xi_q)))^\flat \\
& \quad\text{by IH} \\
= & ((\lambda\vec{x}.\varrho^{\mathsf{src}}\,\mu'_1\ldots\mu'_q)/(\lambda\vec{x}.\varrho\,\xi_1\ldots\xi_q))/((\lambda\vec{x}.\varrho\,\nu_1\ldots\nu_q)/(\lambda\vec{x}.\varrho\,\xi_1\ldots\xi_q)) \\
& \quad\text{by Prop. 142} \\
= & (\mu/\xi)/(\nu/\xi) \\
& \quad\text{by Prop. 142}
\end{array}$$

3. μ **headed by a rule symbol,** $\mu = \lambda\vec{x}.\varrho\,\mu_1\ldots\mu_n$**.** Then since μ and ν are coinitial, the head of ν may be a constant or a rule symbol. We consider two subcases:

3.1 ν **headed by a constant.** Then since the multisteps are in $\overline{\eta}$-normal form, $\nu = \lambda\vec{x}.\mathbf{c}\,\nu_1\ldots\nu_m$. By Lem. 137 there exist multisteps $\nu'_1,\ldots,\nu'_n$ in $\overset{\circ}{\mapsto},\overline{\eta}$-normal form such that $\nu = (\lambda\vec{x}.\varrho^{\mathsf{src}}\,\nu'_1\ldots\nu'_n)^\circ$ where μ_i and ν'_i are coinitial. Moreover, since μ and ξ are coinitial, the head of ξ may be a constant or a rule symbol. We consider two further subcases:

3.1.1 ξ **headed by a constant.** Then since the multisteps are in $\overline{\eta}$-normal form, $\xi = \lambda\vec{x}.\mathbf{c}\,\xi_1\ldots\xi_m$. By Lem. 137 there exist multisteps $\xi'_1,\ldots,\xi'_n$ in $\overset{\circ}{\mapsto},\overline{\eta}$-normal form such that $\xi = (\lambda\vec{x}.\varrho^{\mathsf{src}}\,\xi'_1\ldots\xi'_n)^\circ$ where μ_i and ξ'_i are coinitial. Then the proof proceeds similarly as for case 2.1.2.

3.1.2 ξ **headed by a rule symbol.** Then since the multisteps are in $\overline{\eta}$-normal form and by orthogonality, $\xi = \lambda\vec{x}.\varrho\,\xi_1\ldots\xi_n$. Then the proof proceeds similarly as for case 2.1.2, noting that, by Prop. 142:

$$\begin{array}{rcl}
(\varrho/\varrho^{\mathsf{src}})/(\varrho/\varrho^{\mathsf{src}}) & = & \varrho^\flat/\varrho^\flat \\
& = & \varrho^{\blacktriangleright} \\
& = & \varrho^{\blacktriangleright}/\varrho^{\blacktriangleright} \\
& = & (\varrho/\varrho)/(\varrho^{\mathsf{src}}/\varrho)
\end{array}$$

3.2 ν **headed by a rule symbol.** Then since the multisteps are in $\overline{\eta}$-normal form, the head of ξ may be a constant or a rule symbol. If the head of ξ is a constant, this case is symmetric to case 3.1.2, when ν is headed by a constant and ξ by a rule symbol. If the head of ξ is a rule symbol, then by orthogonality, $\xi = \lambda\vec{x}.\varrho\,\xi_1\ldots\xi_n$. Then the proof proceeds similarly as for case 2.1.2.

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E.2 Projection for rewrites

▶ **Lemma 145** (Rewrite/rewrite generalizes rewrite/multistep and multistep/rewrite projections)**.** *If μ is a flat multistep and ρ a coinitial flat rewrite then:*

1. $\mu /^1 \rho = \mu /^3 \rho$
2. $\rho /^2 \mu = \rho /^3 \mu$

Proof. Item 2. is immediate by definition of $\rho /^3 \mu$. For item 1, we proceed by induction on ρ:

1. **Multistep,** $\rho = \nu$**.** Then $\mu /^1 \nu = \mu/\nu = \mu /^2 \nu = \mu /^3 \nu$.

2. **Composition,** $\rho = \rho_1 \mathbin{;} \rho_2$. Then:

$$\begin{array}{rcll} \mu /^1 (\rho_1 \mathbin{;} \rho_2) & = & (\mu /^1 \rho_1) /^1 \rho_2 & \\ & = & (\mu /^1 \rho_1) /^3 \rho_2 & \text{by IH} \\ & = & (\mu /^3 \rho_1) /^3 \rho_2 & \text{by IH} \\ & = & \mu /^3 (\rho_1 \mathbin{;} \rho_2) & \end{array}$$

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▶ **Definition 146** (Size of a flat rewrite). *If ρ is a flat rewrite, we write $|\rho|$ for the number of multisteps in ρ. More precisely:*

$$\begin{array}{rcl} |\mu| & \stackrel{\text{def}}{=} & 1 \\ |\rho \mathbin{;} \sigma| & \stackrel{\text{def}}{=} & |\rho| + |\sigma| \end{array}$$

▶ **Lemma 147** (Size of a projection).
1. $|\mu /^1 \rho| = |\mu|$
2. $|\rho /^2 \mu| = |\rho|$
3. $|\rho /^3 \sigma| = |\rho|$

Proof.
1. By induction on ρ. If $\rho = \nu$, then $|\mu /^1 \nu| = |\mu/\nu| = 1 = |\mu|$. If $\rho = \rho_1 \mathbin{;} \rho_2$, then by IH we have that $|\mu /^1 (\rho_1 \mathbin{;} \rho_2)| = |(\mu /^1 \rho_1) /^1 \rho_2| = |\mu /^1 \rho_1| = |\mu|$.
2. By induction on ρ. If $\rho = \nu$, then $|\nu /^2 \mu| = |\nu/\mu| = 1 = |\nu|$. If $\rho = \rho_1 \mathbin{;} \rho_2$, then by IH we have that $|(\rho_1 \mathbin{;} \rho_2) /^2 \mu| = |(\rho_1 /^2 \mu) \mathbin{;} (\rho_2 /^2 (\mu /^1 \rho_1))| = |\rho_1 /^2 \mu| + |\rho_2 /^2 (\mu /^1 \rho_1)| = |\rho_1| + |\rho_2| = |\rho_1 \mathbin{;} \rho_2|$.
3. By induction on σ. If $\sigma = \mu$, then by item 2. $|\rho /^3 \mu| = |\rho /^2 \mu| = |\rho|$. If $\sigma = \sigma_1 \mathbin{;} \sigma_2$ then by IH we have that $|\rho /^3 (\sigma_1 \mathbin{;} \sigma_2)| = |(\rho /^3 \sigma_1) /^3 \sigma_2| = |\rho /^3 \sigma_1| = |\rho|$.

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▶ **Lemma 148** (Rewrite/rewrite projection of a sequence). *If $(\rho_1 \mathbin{;} \rho_2)$ and σ are coinitial flat rewrites, then:*

$$(\rho_1 \mathbin{;} \rho_2) /^3 \sigma = (\rho_1 /^3 \sigma) \mathbin{;} (\rho_2 /^3 (\sigma /^3 \rho_1))$$

Proof. We proceed by induction on $|\rho_1| + |\rho_2| + |\sigma|$. We consider two cases, depending on whether σ is a multistep or a composition:
1. **Multistep,** $\sigma = \mu$**.** Then:

$$\begin{array}{rcll} (\rho_1 \mathbin{;} \rho_2) /^3 \mu & = & (\rho_1 \mathbin{;} \rho_2) /^2 \mu & \\ & = & (\rho_1 /^2 \mu) \mathbin{;} (\rho_2 /^2 (\mu /^1 \rho_1)) & \\ & = & (\rho_1 /^3 \mu) \mathbin{;} (\rho_2 /^3 (\mu /^3 \rho_1)) & \text{by Lem. 145} \end{array}$$

2. **Composition,** $\sigma = \sigma_1 \mathbin{;} \sigma_2$**.** Then:

$$\begin{array}{rll} & (\rho_1 \mathbin{;} \rho_2) /^3 (\sigma_1 \mathbin{;} \sigma_2) & \\ = & ((\rho_1 \mathbin{;} \rho_2) /^3 \sigma_1) /^3 \sigma_2 & \\ = & ((\rho_1 /^3 \sigma_1) \mathbin{;} (\rho_2 /^3 (\sigma_1 /^3 \rho_1))) /^3 \sigma_2 & \text{by IH} \\ = & ((\rho_1 /^3 \sigma_1) /^3 \sigma_2) \mathbin{;} ((\rho_2 /^3 (\sigma_1 /^3 \rho_1)) /^3 (\sigma_2 /^3 (\rho_1 /^3 \sigma_1))) & \text{by IH} \\ = & (\rho_1 /^3 (\sigma_1 \mathbin{;} \sigma_2)) \mathbin{;} (\rho_2 /^3 ((\sigma_1 /^3 \rho_1) \mathbin{;} (\sigma_2 /^3 (\rho_1 /^3 \sigma_1)))) & \text{by definition} \\ = & (\rho_1 /^3 (\sigma_1 \mathbin{;} \sigma_2)) \mathbin{;} (\rho_2 /^3 ((\sigma_1 \mathbin{;} \sigma_2) /^3 \rho_1)) & \text{by IH} \end{array}$$

To apply the IH the first time, note that $|\rho_1| + |\rho_2| + |\sigma_1| < |\rho_1| + |\rho_2| + |\sigma_1 \mathbin{;} \sigma_2|$. To apply the IH the second time, note that by Lem. 147 we have that $|\rho_1 /^3 \sigma_1| + |\rho_2 /^3 (\sigma_1 /^3 \rho_1)| + |\sigma_2| = |\rho_1| + |\rho_2| + |\sigma_2| < |\rho_1| + |\rho_2| + |\sigma_1 \mathbin{;} \sigma_2|$. To apply the IH the third time, note that $|\sigma_1| + |\sigma_2| + |\rho_1| < |\rho_1| + |\rho_2| + |\sigma_1 \mathbin{;} \sigma_2|$.

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▶ **Convention 1.** *From this point on, we overload ρ/σ to stand for $\rho\,/^1\,\sigma$ (if ρ is a multistep), $\rho\,/^2\,\sigma$ (if σ is a multistep), or $\rho\,/^3\,\sigma$ (in the general case). Note that Lem. 145 and Lem. 148 ensure that this overloading is "safe".*

F Properties of Projection for Flat Rewrites

▶ **Lemma 149** (Basic properties of rewrite projection). *Let ρ stand for a flat rewrite. Then the following hold:*

1. $\rho^{◀}/\rho = \rho^{▶}$
2. $\rho/\rho^{◀} = \rho$

Proof.

1. $\rho^{◀}/\rho = \rho^{▶}$: By induction on ρ. If $\rho = \mu$ is a multistep, then $\mu^{◀}/\mu = \mu^{▶}$ by Prop. 142. If $\rho = \rho_1 \mathbin{;} \rho_2$, then:

$$\begin{array}{rcll} (\rho_1 \mathbin{;} \rho_2)^{◀}/(\rho_1 \mathbin{;} \rho_2) & = & (\rho_1^{◀}/\rho_1)/\rho_2 & \text{by definition} \\ & = & \rho_1^{▶}/\rho_2 & \text{by IH} \\ & = & \rho_2^{◀}/\rho_2 & \text{as } \rho_1^{\mathsf{tgt}} =_{\beta\eta} \rho_2^{\mathsf{src}} \\ & = & \rho_2^{▶} & \text{by IH} \\ & = & (\rho_1 \mathbin{;} \rho_2)^{▶} & \end{array}$$

2. $\rho/\rho^{◀} = \rho$: By induction on ρ. If $\rho = \mu$ is a multistep, then $\mu/\mu^{◀} = \mu^{\flat} = \mu$ by Prop. 142, using the fact that ρ is flat by hypothesis. If $\rho = \rho_1 \mathbin{;} \rho_2$ then:

$$\begin{array}{rll} & (\rho_1 \mathbin{;} \rho_2)/(\rho_1 \mathbin{;} \rho_2)^{◀} & \\ = & (\rho_1/(\rho_1 \mathbin{;} \rho_2)^{◀}) \mathbin{;} (\rho_2/((\rho_1 \mathbin{;} \rho_2)^{◀}/\rho_1)) & \text{by Lem. 148} \\ = & (\rho_1/(\rho_1 \mathbin{;} \rho_2)^{◀}) \mathbin{;} (\rho_2/(\rho_1^{◀}/\rho_1)) & \\ = & (\rho_1/(\rho_1 \mathbin{;} \rho_2)^{◀}) \mathbin{;} (\rho_2/\rho_1^{▶}) & \text{by item 1. of this lemma} \\ = & (\rho_1/\rho_1^{◀}) \mathbin{;} (\rho_2/\rho_2^{◀}) & \text{as } \rho_1 \mathbin{;} \rho_2^{\mathsf{src}} = \rho_1^{\mathsf{src}} \text{ and } \rho_1^{\mathsf{tgt}} = \rho_2^{\mathsf{src}} \\ = & \rho_1 \mathbin{;} \rho_2 & \text{by IH} \end{array}$$

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▶ **Definition 150** (Splitting, up to flattening). *We write $\mu_1 \overset{\flat}{\Leftrightarrow} \mu_2 \mathbin{;} \mu_3$ if there exist multisteps μ_1', μ_2', μ_3' such that $\mu_1' \Leftrightarrow \mu_2' \mathbin{;} \mu_3'$, and moreover $(\mu_1')^{\flat} = \mu_1$ and $(\mu_2')^{\flat} = \mu_2$ and $(\mu_3')^{\flat} = \mu_3$.*

▶ **Lemma 151** (Projection of a splitting over a multistep). *If $\mu_1 \Leftrightarrow \mu_2 \mathbin{;} \mu_3$ and ν is an arbitrary multistep coinitial to μ_1, then $(\mu_1/\nu) \overset{\flat}{\Leftrightarrow} (\mu_2/\nu) \mathbin{;} (\mu_3/(\nu/\mu_2))$.*

Proof. Suppose that $\mu_1 \Leftrightarrow \mu_2 \mathbin{;} \mu_3$. Recall from Prop. 142 that μ_1/ν does not depend on the representative of μ_1 (up to flattening), and similarly for μ_2/ν and $\mu_3/(\nu/\mu_2)$. Hence by Lem. 120 we may assume without loss of generality that μ_1 is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form. We may also assume without loss of generality that ν is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form. We show that $(\mu_1/\nu) \overset{\flat}{\Leftrightarrow} (\mu_2/\nu) \mathbin{;} (\mu_3/(\nu/\mu_2))$ holds, by induction on the number of applications in μ_1 and by case analysis on the head of μ_1, using the fact that it is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form:

1. **μ_1 headed by a variable.** Then $\mu_1 = \lambda\vec{x}.y\,\mu_{11}\ldots\mu_{1n}$. Note that $\mu_1 \Leftrightarrow \mu_2 \mathbin{;} \mu_3$ must be derived by a number of instances of the SAbs rule, followed by n instances of SApp rule, followed by an instance of the SVar rule at the head. Hence $\mu_2 = \lambda\vec{x}.y\,\mu_{21}\ldots\mu_{2n}$ and $\mu_3 = \lambda\vec{x}.y\,\mu_{31}\ldots\mu_{3n}$ with $\mu_{1i} \Leftrightarrow \mu_{2i} \mathbin{;} \mu_{3i}$ for all $1 \leq i \leq n$. Moreover, since μ_1 and

ν are coinitial and in $\overset{\circ}{\mapsto},\overline{\eta}$-normal form, we have that $\nu = \lambda\vec{x}.y\,\nu_1 \ldots \nu_n$ where μ_{1i} and ν_i are coinitial for each $1 \leq i \leq n$. Then by Prop. 142 we have:

$$\begin{array}{rcl} \mu_1/\nu & = & (\lambda\vec{x}.y\,(\mu_{11}/\nu_1)\ldots(\mu_{1n}/\nu_n))^\flat \\ \mu_2/\nu & = & (\lambda\vec{x}.y\,(\mu_{21}/\nu_1)\ldots(\mu_{2n}/\nu_n))^\flat \\ \mu_3/(\nu/\mu_2) & = & (\lambda\vec{x}.y\,(\mu_{31}/(\nu_1/\mu_{21}))\ldots(\mu_{3n}/(\nu_n/\mu_{2n})))^\flat \end{array}$$

Note that, by IH we have that $\mu_{1i}/\nu_i \overset{\flat}{\Leftrightarrow} \mu_{2i}/\nu_i \,;\, \mu_{3i}/(\nu_i/\mu_{2i})$. Hence:

$$\begin{array}{rcl} \lambda\vec{x}.y\,(\mu_{11}/\nu_1)\ldots(\mu_{1n}/\nu_n) & \overset{\flat}{\Leftrightarrow} & \lambda\vec{x}.y\,(\mu_{21}/\nu_1)\ldots(\mu_{2n}/\nu_n) \\ & ; & \lambda\vec{x}.y\,(\mu_{31}/(\nu_1/\mu_{21}))\ldots(\mu_{3n}/(\nu_n/\mu_{2n})) \end{array}$$

This in turn implies that $(\mu_1/\nu) \overset{\flat}{\Leftrightarrow} (\mu_2/\nu) \,;\, (\mu_3/(\nu/\mu_2))$.

2. μ_1 **headed by a constant.** Then $\mu_1 = \lambda\vec{x}.\mathbf{c}\,\mu_{11}\ldots\mu_{1n}$. Note that $\mu_1 \Leftrightarrow \mu_2 \,;\, \mu_3$ must be derived by a number of instances of the SAbs rule, followed by n instances of SApp rule, followed by an instance of the SConst rule at the head. Hence $\mu_2 = \lambda\vec{x}.\mathbf{c}\,\mu_{21}\ldots\mu_{2n}$ and $\mu_3 = \lambda\vec{x}.\mathbf{c}\,\mu_{31}\ldots\mu_{3n}$ where $\mu_{1i} \Leftrightarrow \mu_{2i} \,;\, \mu_{3i}$ for all $1 \leq i \leq n$. Moreover, since μ_1 and ν are coinitial and in $\overset{\circ}{\mapsto},\overline{\eta}$-normal form, the head of ν must be either a constant or a rule symbol. We consider two subcases:

   2.1 ν **headed by a constant.** Then $\nu = \lambda\vec{x}.\mathbf{c}\,\nu_1\ldots\nu_n$. The proof proceeds similarly as for case 1, when the heads of μ_1, μ_2, μ_3, ν are all variables.

   2.2 ν **headed by a rule symbol.** Then $\nu = \lambda\vec{x}.\varrho\,\nu_1\ldots\nu_q$. By Lem. 137 we have that $\mu_1 = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_{11}\ldots\mu'_{1q})^\circ$ and $\mu_2 = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_{21}\ldots\mu'_{2q})^\circ$ and $\mu_3 = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_{31}\ldots\mu'_{3q})^\circ$ where μ'_{1i} and ν_i are coinitial for all $1 \leq i \leq q$, and each μ'_{1i} has strictly less applications than μ_1. By Prop. 142 we have that:

$$\begin{array}{rcl} \mu_1/\nu & = & (\lambda\vec{x}.(\varrho^{\blacktriangleleft}/\varrho)\,(\mu'_{11}/\nu_1)\ldots(\mu'_{1q}/\nu_q))^\flat \\ \mu_2/\nu & = & (\lambda\vec{x}.(\varrho^{\blacktriangleleft}/\varrho)\,(\mu'_{21}/\nu_1)\ldots(\mu'_{2q}/\nu_q))^\flat \\ \mu_3/(\nu/\mu_2) & = & (\lambda\vec{x}.(\varrho^{\blacktriangleleft}/(\varrho/\varrho^{\blacktriangleleft}))\,(\mu'_{31}/(\nu_1/\mu'_{21}))\ldots(\mu'_{3q}/(\nu_q/\mu'_{21})))^\flat \end{array}$$

   Furthermore, note that, since $\varrho^{<\overline{\eta}}$ is a rule-pattern, μ'_{1i} is of the form $\lambda y.\mu''_{1i}$ and a renaming of μ''_{1i} occurs in μ_1 in a certain position. Similarly, μ'_{2i} and μ'_{3i} are of the form $\lambda y.\mu''_{2i}$ and $\lambda y.\mu''_{3i}$ respectively, and they occur in μ_2 and μ_3 respectively, in the same position. Then, since $\mu_1 \Leftrightarrow \mu_2 \,;\, \mu_3$, we have that $\mu'_{1i} \Leftrightarrow \mu'_{2i} \,;\, \mu'_{3i}$ holds for all $1 \leq i \leq q$. So we may apply the IH to conclude that $(\mu'_{1i}/\nu_i) \overset{\flat}{\Leftrightarrow} (\mu'_{2i}/\nu_i) \,;\, (\mu'_{3i}/(\nu_i/\mu'_{2i}))$ holds for all $1 \leq i \leq q$. Moreover, note that $(\varrho^{\blacktriangleleft}/\varrho) \Leftrightarrow (\varrho^{\blacktriangleleft}/\varrho) \,;\, (\varrho^{\blacktriangleleft}/(\varrho/\varrho^{\blacktriangleright}))$, given that $\varrho^{\blacktriangleleft}/\varrho = \varrho^{\blacktriangleright}$ and $\varrho^{\blacktriangleleft}/(\varrho/\varrho^{\blacktriangleleft}) = \varrho^{\blacktriangleleft}/\varrho = \varrho^{\blacktriangleright}$ by Prop. 142, and $\varrho^{\blacktriangleright} \Leftrightarrow \varrho^{\blacktriangleright} \,;\, \varrho^{\blacktriangleright}$, which is immediate given that $\varrho^{\blacktriangleright}$ has no rule symbols. Hence:

$$\begin{array}{rcl} \lambda\vec{x}.(\varrho^{\blacktriangleleft}/\varrho)\,(\mu'_{11}/\nu_1)\ldots(\mu'_{1q}/\nu_q) & \overset{\flat}{\Leftrightarrow} & \lambda\vec{x}.(\varrho^{\blacktriangleleft}/\varrho)\,(\mu'_{21}/\nu_1)\ldots(\mu'_{2q}/\nu_q) \\ & ; & \lambda\vec{x}.(\varrho^{\blacktriangleleft}/(\varrho/\varrho^{\blacktriangleright}))\,(\mu'_{31}/(\nu_1/\mu'_{21}))\ldots(\mu'_{3q}/(\nu_q/\mu'_{21})) \end{array}$$

   This in turn implies that $(\mu_1/\nu) \overset{\flat}{\Leftrightarrow} (\mu_2/\nu) \,;\, (\mu_3/(\nu/\mu_2))$.

3. μ_1 **headed by a rule symbol.** Then $\mu_1 = \lambda\vec{x}.\varrho\,\mu_{11}\ldots\mu_{1n}$. Note that $\mu_1 \Leftrightarrow \mu_2 \,;\, \mu_3$ must be derived must be derived by a number of instances of the SAbs rule, followed by n instances of SApp rule, followed by an instance of either SRuleL or SRuleL at the head. We consider two subcases:

   3.1 SRuleL: Then $\mu_2 = \lambda\vec{x}.\varrho\,\mu_{21}\ldots\mu_{2n}$ and $\mu_3 = \lambda\vec{x}.\varrho^{\mathsf{tgt}}\,\mu_{31}\ldots\mu_{3n}$ where $\mu_{1i} \Leftrightarrow \mu_{2i} \,;\, \mu_{3i}$ for all $1 \leq i \leq n$. Since μ_1 and ν are coinitial, the head of ν must be a constant or a rule symbol. We consider two further subcases:

3.1.1 ν **headed by a constant.** By Lem. 137 we have that $\nu = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\nu_1 \ldots \nu_n)^\circ$, where μ_{1i} and ν_i are coinitial for all $1 \leq i \leq n$. By Prop. 142 we have that:

$$\begin{array}{rcl} \mu_1/\nu & = & (\lambda x.(\varrho/\varrho^{\mathsf{src}})(\mu_{11}/\nu_1)\ldots(\mu_{1n}/\nu_n))^\flat \\ \mu_2/\nu & = & (\lambda x.(\varrho/\varrho^{\mathsf{src}})(\mu_{21}/\nu_1)\ldots(\mu_{2n}/\nu_n))^\flat \\ \mu_3/(\nu/\mu_2) & = & (\lambda x.(\varrho^{\mathsf{tgt}}/(\varrho^{\mathsf{src}}/\varrho))(\mu_{31}/(\nu_1/\mu_{21}))\ldots(\mu_{3n}/(\nu_n/\mu_{2n})))^\flat \end{array}$$

By IH we know that $(\mu_{1i}/\nu_i) \overset{\flat}{\Leftrightarrow} (\mu_{2i}/\nu_i)\,;\,(\mu_{3i}/(\nu_i/\mu_{2i}))$ for all $1 \leq i \leq n$. Moreover, note that $\varrho/\varrho^{\blacktriangleleft} = \varrho$ and $\varrho^{\blacktriangleright}/(\varrho^{\blacktriangleleft}/\varrho) = \varrho^{\blacktriangleright}/\varrho^{\blacktriangleright} = \varrho^{\blacktriangleright}$ by Prop. 142, and that $\varrho \Leftrightarrow \varrho\,;\,\varrho^{\mathsf{tgt}}$. Hence:

$$\begin{array}{rcl} \lambda\vec{x}.\varrho\,(\mu_{11}/\nu_1)\ldots(\mu_{1n}/\nu_n) & \overset{\flat}{\Leftrightarrow} & \lambda\vec{x}.\varrho\,(\mu_{21}/\nu_1)\ldots(\mu_{2n}/\nu_n) \\ & ; & \lambda\vec{x}.\varrho^{\mathsf{tgt}}\,(\mu_{31}/(\nu_1/\mu_{21}))\ldots(\mu_{3n}/(\nu_n/\mu_{2n})) \end{array}$$

This in turn implies that $\mu_1/\nu \overset{\flat}{\Leftrightarrow} \mu_2/\nu\,;\,\mu_3/(\nu/\mu_1)$.

3.1.2 ν **headed by a rule symbol.** Then by orthogonality $\nu = \lambda\vec{x}.\varrho\,\nu_1 \ldots \nu_n$. By Prop. 142 we have that:

$$\begin{array}{rcl} \mu_1/\nu & = & (\lambda x.(\varrho/\varrho)(\mu_{11}/\nu_1)\ldots(\mu_{1n}/\nu_n))^\flat \\ \mu_2/\nu & = & (\lambda x.(\varrho/\varrho)(\mu_{21}/\nu_1)\ldots(\mu_{2n}/\nu_n))^\flat \\ \mu_3/(\nu/\mu_2) & = & (\lambda x.(\varrho^{\mathsf{tgt}}/(\varrho/\varrho))(\mu_{31}/(\nu_1/\mu_{21}))\ldots(\mu_{3n}/(\nu_n/\mu_{2n})))^\flat \end{array}$$

Then the proof proceeds similarly as for case 3.1.1, when the head of ν is a constant, noting that $\varrho/\varrho = \varrho^{\blacktriangleright}$ and that $\varrho^{\blacktriangleright}/(\varrho/\varrho) = \varrho^{\blacktriangleright}/\varrho^{\blacktriangleright} = \varrho^{\blacktriangleright}$ by Prop. 142, and that $\varrho^{\mathsf{tgt}} \Leftrightarrow \varrho^{\mathsf{tgt}}\,;\,\varrho^{\mathsf{tgt}}$.

3.2 SRuleR: Then $\mu_2 = \lambda\vec{x}.\varrho^{\mathsf{src}}\,\mu_{21}\ldots\mu_{2n}$ and $\mu_3 = \lambda\vec{x}.\varrho\,\mu_{31}\ldots\mu_{3n}$ where $\mu_{1i} \Leftrightarrow \mu_{2i}\,;\,\mu_{3i}$ for all $1 \leq i \leq n$. Since μ_1 and ν are coinitial, the head of ν must be a constant or a rule symbol. We consider two further subcases:

3.2.1 ν **headed by a constant.** By Lem. 137 we have that $\nu = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\nu_1 \ldots \nu_n)^\circ$, where μ_{1i} and ν_i are coinitial for all $1 \leq i \leq n$. Hence:

$$\begin{array}{rcl} \mu_1/\nu & = & (\lambda x.(\varrho/\varrho^{\mathsf{src}})\,(\mu_{11}/\nu_1)\ldots(\mu_{1n}/\nu_n))^\flat \\ \mu_2/\nu & = & (\lambda x.(\varrho^{\mathsf{src}}/\varrho^{\mathsf{src}})\,(\mu_{21}/\nu_1)\ldots(\mu_{2n}/\nu_n))^\flat \\ \mu_3/(\nu/\mu_2) & = & (\lambda x.(\varrho/(\varrho^{\mathsf{src}}/\varrho^{\mathsf{src}}))\,(\mu_{31}/(\nu_1/\mu_{21}))\ldots(\mu_{3n}/(\nu_n/\mu_{2n})))^\flat \end{array}$$

Then the proof proceeds similarly as for case 3.1.1, noting that $\varrho/\varrho^{\blacktriangleleft} = \varrho$, that $\varrho^{\blacktriangleleft}/\varrho^{\blacktriangleleft} = \varrho^{\blacktriangleleft}$, and that $\varrho/(\varrho^{\blacktriangleleft}/\varrho^{\blacktriangleleft}) = \varrho/\varrho^{\blacktriangleleft} = \varrho$ by Prop. 142, and moreover that $\varrho \Leftrightarrow \varrho^{\mathsf{src}}\,;\,\varrho$.

3.2.2 ν **headed by a rule symbol.** Then by orthogonality $\nu = \lambda\vec{x}.\varrho\,\nu_1 \ldots \nu_n$. Hence:

$$\begin{array}{rcl} \mu_1/\nu & = & (\lambda x.(\varrho/\varrho)\,(\mu_{11}/\nu_1)\ldots(\mu_{1n}/\nu_n))^\flat \\ \mu_2/\nu & = & (\lambda x.(\varrho^{\mathsf{src}}/\varrho)\,(\mu_{21}/\nu_1)\ldots(\mu_{2n}/\nu_n))^\flat \\ \mu_3/(\nu/\mu_2) & = & (\lambda x.(\varrho/(\varrho/\varrho^{\mathsf{src}}))\,(\mu_{31}/(\nu_1/\mu_{21}))\ldots(\mu_{3n}/(\nu_n/\mu_{2n})))^\flat \end{array}$$

Then the proof proceeds similarly as for case 3.1.1, noting that $\varrho/\varrho = \varrho^{\blacktriangleright}$, that $\varrho^{\blacktriangleleft}/\varrho = \varrho^{\blacktriangleright}$, and that $\varrho/(\varrho/\varrho^{\blacktriangleleft}) = \varrho/\varrho = \varrho^{\blacktriangleright}$ by Prop. 142, and moreover that $\varrho^{\mathsf{tgt}} \Leftrightarrow \varrho^{\mathsf{tgt}}\,;\,\varrho^{\mathsf{tgt}}$.

◀

▶ **Lemma 152** (Projection of a splitting over a rewrite). *If $\mu_1 \Leftrightarrow \mu_2\,;\,\mu_3$ and ρ is an arbitrary flat rewrite coinitial to μ_1, then $(\mu_1/\rho) \overset{\flat}{\Leftrightarrow} (\mu_2/\rho)\,;\,(\mu_3/(\rho/\mu_2))$.*

Proof. By induction on ρ.

1. **Multistep, $\rho = \nu$.** Then this is an immediate consequence of Lem. 151.
2. **Composition, $\rho = \rho_1 \,;\, \rho_2$.** By IH, we have that: $\mu_1/\rho_1 \overset{\flat}{\Leftrightarrow} \mu_2/\rho_1 \,;\, \mu_3/(\rho_1/\mu_2)$, that is, there exist multisteps $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3$ such that $\hat{\mu}_1 \Leftrightarrow \hat{\mu}_2 \,;\, \hat{\mu}_3$ and such that $\hat{\mu}_1^\flat = \mu_1/\rho_1$ and $\hat{\mu}_2^\flat = \mu_2/\rho_1$ and $\hat{\mu}_3^\flat = \mu_3/(\rho_1/\mu_2)$. Applying the IH again, we have that $\hat{\mu}_1/\rho_2 \overset{\flat}{\Leftrightarrow} \hat{\mu}_2/\rho_2 \,;\, \hat{\mu}_3/(\rho_2/\hat{\mu}_2)$. Finally, recall that by Prop. 142, projection does not depend on the representative of the equivalence class (up to flattening), so we have the following equalities:

$$
\begin{array}{rclcl}
\mu_1/(\rho_1 \,;\, \rho_2) & = & (\mu_1/\rho_1)/\rho_2 & = & \hat{\mu}_1/\rho_2 \\
\mu_2/(\rho_1 \,;\, \rho_2) & = & (\mu_2/\rho_1)/\rho_2 & = & \hat{\mu}_2/\rho_2 \\
\mu_3/((\rho_1 \,;\, \rho_2)/\mu_2) & = & (\mu_3/(\rho_1/\mu_2))/(\rho_2/(\mu_2/\rho_1)) & = & \hat{\mu}_3/(\rho_2/\hat{\mu}_2)
\end{array}
$$

This means that $(\mu_1/(\rho_1 \,;\, \rho_2)) \overset{\flat}{\Leftrightarrow} (\mu_2/(\rho_1 \,;\, \rho_2)) \,;\, (\mu_3/((\rho_1 \,;\, \rho_2)/\mu_2))$ holds, as required. ◀

▶ **Lemma 153** (Projection of a multistep over a splitting). *If $\mu_1 \Leftrightarrow \mu_2 \,;\, \mu_3$ and ν is an arbitrary multistep coinitial to μ_1, then $\nu/\mu_1 = (\nu/\mu_2)/\mu_3$.*

Proof. Suppose that $\mu_1 \Leftrightarrow \mu_2 \,;\, \mu_3$. Recall from Prop. 142 that ν/μ_1 does not depend on the representative of μ_1 (up to flattening), and similarly for $(\mu/\mu_2)/\mu_3$. Hence by Lem. 120 we may assume without loss of generality that μ_1 is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form. We may also assume without loss of generality that ν is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form. We proceed by induction on the number of applications in μ_1 and by case analysis on the head of μ_1, using the fact that it is in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form.

1. **μ_1 headed by a variable.** Then $\mu_1 = \lambda\vec{x}.y\,\mu_{11}\ldots\mu_{1n}$. Note that $\mu_1 \Leftrightarrow \mu_2 \,;\, \mu_3$ must be derived by a number of instances of the SAbs rule, followed by n instances of the SApp rule, followed by an instance of the SVar rule at the head. Hence $\mu_2 = \lambda\vec{x}.y\,\mu_{21}\ldots\mu_{2n}$ and $\mu_3 = \lambda\vec{x}.y\,\mu_{31}\ldots\mu_{3n}$ with $\mu_{1i} \Leftrightarrow \mu_{2i} \,;\, \mu_{3i}$ for all $1 \le i \le n$. Moreover, since μ_1 and ν are coinitial and in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form, we have that $\nu = \lambda\vec{x}.y\,\nu_1\ldots\nu_n$ where μ_{1i} and ν_i are coinitial for all $1 \le i \le n$. Then by Prop. 142 we have that:

$$
\begin{array}{rcl}
\nu/\mu_1 & = & (\lambda\vec{x}.y\,(\nu_1/\mu_{11})\ldots(\nu_n/\mu_{1n}))^\flat \\
(\nu/\mu_2)/\mu_3 & = & (\lambda\vec{x}.y\,((\nu_1/\mu_{21})/\mu_{31})\ldots((\nu_n/\mu_{2n})/\mu_{3n}))^\flat
\end{array}
$$

   To conclude, note that, by IH we have that $\nu_i/\mu_{1i} = (\nu_i/\mu_{2i})/\mu_{3i}$.
2. **μ_1 headed by a constant.** Then $\mu_1 = \lambda\vec{x}.\mathbf{c}\,\mu_{11}\ldots\mu_{1n}$. Note that $\mu_1 \Leftrightarrow \mu_2 \,;\, \mu_3$ must be derived by a number of instances of the SAbs rule, followed by n instances of the SApp rule, followed by an instance of the SCons rule at the head. Hence $\mu_2 = \lambda\vec{x}.\mathbf{c}\,\mu_{21}\ldots\mu_{2n}$ and $\mu_3 = \lambda\vec{x}.\mathbf{c}\,\mu_{31}\ldots\mu_{3n}$ where $\mu_{1i} \Leftrightarrow \mu_{2i} \,;\, \mu_{3i}$ for all $1 \le i \le n$. Moreover, since μ_1 and ν are coinitial and in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form, the head of ν must be either a constant or a rule symbol. We consider two subcases:
   - **2.1** **ν headed by a constant.** Then $\nu = \lambda\vec{x}.\mathbf{c}\,\nu_1\ldots\nu_n$. The proof proceeds similarly as for case 1, when the heads of μ_1, μ_2, μ_3, ν are all variables.
   - **2.2** **ν headed by a rule symbol.** Then $\nu = \lambda\vec{x}.\varrho\,\nu_1\ldots\nu_q$. By Lem. 137 we have that $\mu_1 = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_{11}\ldots\mu'_{1q})^\circ$ and $\mu_2 = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_{21}\ldots\mu'_{2q})^\circ$ and $\mu_3 = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_{31}\ldots\mu'_{3q})^\circ$ where μ'_{1i} and ν_i are coinitial for all $1 \le i \le q$, and each μ'_{1i} has strictly less applications than μ_1. By Prop. 142 we have that:

$$
\begin{array}{rcl}
\nu/\mu_1 & = & (\lambda\vec{x}.(\varrho/\varrho^{\blacktriangleleft})\,(\nu_1/\mu'_{11})\ldots(\nu_q/\mu'_{1q}))^\flat \\
(\nu/\mu_2)/\mu_3 & = & (\lambda\vec{x}.((\varrho/\varrho^{\blacktriangleleft})/\varrho^{\blacktriangleleft})\,((\nu_1/\mu'_{21})/\mu'_{31})\ldots((\nu_q/\mu'_{2q})/\mu'_{3q}))^\flat
\end{array}
$$

To conclude, it suffices to note that $\varrho/\varrho^{◀} = (\varrho/\varrho^{◀})/\varrho^{◀}$ by Prop. 142 and that by IH we have that $\nu_i/\mu'_{1i} = (\nu_i/\mu'_{2i})/\mu'_{3i}$ holds for all $1 \leq i \leq q$. To be able to apply the IH, note that $\mu'_{1i} \Leftrightarrow \mu'_{2i}$ **;** μ'_{3i} holds given that ϱ^{src} is a rule-pattern. (The proof of this fact is discussed in the analogous case in the proof of Lem. 151).

3. μ_1 **headed by a rule symbol.** Then $\mu_1 = \lambda\vec{x}.\varrho\,\mu_{11}\ldots\mu_{1n}$. Note that $\mu_1 \Leftrightarrow \mu_2$ **;** μ_3 must be derived by a number of instances of the SAbs rule, followed by n instances of the SApp rule, followed by an instance of SRuleL or SRuleR at the head. We consider two subcases:

3.1 SRuleL: Then $\mu_2 = \lambda\vec{x}.\varrho\,\mu_{21}\ldots\mu_{2n}$ and $\mu_3 = \lambda\vec{x}.\varrho^{\mathsf{tgt}}\,\mu_{31}\ldots\mu_{3n}$ where $\mu_{1i} \Leftrightarrow \mu_{2i}$ **;** μ_{3i} for all $1 \leq i \leq n$. Since μ_1 and ν are coinitial, the head of ν must be either a constant or a rule symbol. We consider two further subcases:

3.1.1 ν **headed by a constant.** By Lem. 137 we have that $\nu = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\nu_1\ldots\nu_n)^{\circ}$. By Prop. 142 we have that:

$$\begin{array}{rcl} \nu/\mu_1 & = & (\lambda\vec{x}.(\varrho^{◀}/\varrho)\,(\nu_1/\mu_{11})\ldots(\nu_n/\mu_{1n}))^{\flat} \\ (\nu/\mu_2)/\mu_3 & = & (\lambda\vec{x}.((\varrho^{◀}/\varrho)/\varrho^{▶})\,((\nu_1/\mu_{21})/\mu_{31})\ldots((\nu_n/\mu_{2q})/\mu_{3n}))^{\flat} \end{array}$$

To conclude, it suffices to note that $\varrho^{◀}/\varrho = \varrho^{▶} = \varrho^{▶}/\varrho^{▶} = (\varrho^{◀}/\varrho)/\varrho^{▶}$ by Prop. 142 and that by IH we have that $\nu_i/\mu_{1i} = (\nu_i/\mu_{2i})/\mu_{3i}$ holds for all $1 \leq i \leq n$.

3.1.2 ν **headed by a rule symbol.** Then by orthogonality $\nu = \lambda x.\varrho\,\nu_1\ldots\nu_n$. By Prop. 142 we have that:

$$\begin{array}{rcl} \nu/\mu_1 & = & (\lambda\vec{x}.(\varrho/\varrho)\,(\nu_1/\mu_{11})\ldots(\nu_n/\mu_{1n}))^{\flat} \\ (\nu/\mu_2)/\mu_3 & = & (\lambda\vec{x}.((\varrho/\varrho)/\varrho^{▶})\,((\nu_1/\mu_{21})/\mu_{31})\ldots((\nu_n/\mu_{2q})/\mu_{3n}))^{\flat} \end{array}$$

To conclude, it suffices to note that $\varrho/\varrho = \varrho^{▶} = \varrho^{▶}/\varrho^{▶} = (\varrho/\varrho)/\varrho^{▶}$ by Prop. 142 and that by IH we have that $\nu_i/\mu_{1i} = (\nu_i/\mu_{2i})/\mu_{3i}$ holds for all $1 \leq i \leq n$.

3.2 SRuleR: Then $\mu_2 = \lambda\vec{x}.\varrho^{\mathsf{src}}\,\mu_{21}\ldots\mu_{2n}$ and $\mu_3 = \lambda\vec{x}.\varrho\,\mu_{31}\ldots\mu_{3n}$ where $\mu_{1i} \Leftrightarrow \mu_{2i}$ **;** μ_{3i} for all $1 \leq i \leq n$. Since μ_1 and ν are coinitial, the head of ν must be either a constant or a rule symbol. We consider two further subcases:

3.2.1 ν **headed by a constant.** By Lem. 137 we have that $\nu = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\nu_1\ldots\nu_n)^{\circ}$. By Prop. 142 we have that:

$$\begin{array}{rcl} \nu/\mu_1 & = & (\lambda\vec{x}.(\varrho^{◀}/\varrho)\,(\nu_1/\mu_{11})\ldots(\nu_n/\mu_{1n}))^{\flat} \\ (\nu/\mu_2)/\mu_3 & = & (\lambda\vec{x}.((\varrho^{◀}/\varrho^{◀})/\varrho)\,((\nu_1/\mu_{21})/\mu_{31})\ldots((\nu_n/\mu_{2q})/\mu_{3n}))^{\flat} \end{array}$$

To conclude, it suffices to note that $\varrho^{◀}/\varrho = \varrho^{▶} = \varrho^{◀}/\varrho = (\varrho^{◀}/\varrho^{◀})/\varrho$ by Prop. 142 and that by IH we have that $\nu_i/\mu_{1i} = (\nu_i/\mu_{2i})/\mu_{3i}$ holds for all $1 \leq i \leq n$.

3.2.2 ν **headed by a rule symbol.** Then by orthogonality $\nu = \lambda\vec{x}.\varrho\,\nu_1\ldots\nu_n$. By Prop. 142 we have that:

$$\begin{array}{rcl} \nu/\mu_1 & = & (\lambda\vec{x}.(\varrho/\varrho)\,(\nu_1/\mu_{11})\ldots(\nu_n/\mu_{1n}))^{\flat} \\ (\nu/\mu_2)/\mu_3 & = & (\lambda\vec{x}.((\varrho/\varrho^{◀})/\varrho)\,((\nu_1/\mu_{21})/\mu_{31})\ldots((\nu_n/\mu_{2q})/\mu_{3n}))^{\flat} \end{array}$$

To conclude, it suffices to note that $\varrho/\varrho = \varrho^{▶} = \varrho/\varrho = (\varrho/\varrho^{◀})/\varrho$ by Prop. 142 and that by IH we have that $\nu_i/\mu_{1i} = (\nu_i/\mu_{2i})/\mu_{3i}$ holds for all $1 \leq i \leq n$.

◀

▶ **Lemma 154** (Projection of a rewrite over a splitting). *If $\mu_1 \Leftrightarrow \mu_2$ **;** μ_3 and ρ is an arbitrary flat rewrite coinitial to μ_1, then $\rho/\mu_1 = (\rho/\mu_2)/\mu_3$.*

Proof. By induction on ρ.

1. **Multistep, $\rho = \nu$.** Then this is an immediate consequence of Lem. 153.
2. **Composition, $\rho = \rho_1 \,;\, \rho_2$.** Note by Lem. 152 that $(\mu_1/\rho_1) \overset{\flat}{\Leftrightarrow} (\mu_2/\rho_1) \,;\, (\mu_3/(\rho_1/\mu_2))$ holds, that is, there exist multisteps such that $\hat{\mu}_1 \Leftrightarrow \hat{\mu}_2 \,;\, \hat{\mu}_3$ and such that $\hat{\mu}_1^\flat = \mu_1/\rho_1$ and $\hat{\mu}_2^\flat = \mu_2/\rho_1$ and $\hat{\mu}_3^\flat = \mu_3/(\rho_1/\mu_2)$. Then:

$$\begin{array}{rcll}
(\rho_1 \,;\, \rho_2)/\mu_1 & = & (\rho_1/\mu_1) \,;\, (\rho_2/(\mu_1/\rho_1)) & \\
& = & ((\rho_1/\mu_2)/\mu_3) \,;\, (\rho_2/(\mu_1/\rho_1)) & \text{by IH} \\
& = & ((\rho_1/\mu_2)/\mu_3) \,;\, (\rho_2/\hat{\mu}_1) & \text{by Prop. 142} \\
& = & ((\rho_1/\mu_2)/\mu_3) \,;\, ((\rho_2/\hat{\mu}_2)/\hat{\mu}_3) & \text{by IH} \\
& = & ((\rho_1/\mu_2)/\mu_3) \,;\, ((\rho_2/(\mu_2/\rho_1))/(\mu_3/(\rho_1/\mu_2))) & \text{by Prop. 142} \\
& = & ((\rho_1/\mu_2) \,;\, (\rho_2/(\mu_2/\rho_1)))/\mu_3 & \\
& = & ((\rho_1 \,;\, \rho_2)/\mu_2)/\mu_3 &
\end{array}$$

◀

▶ **Proposition 155** (Congruence of $\sim$ with respect to projection). *Let $\rho \sim \sigma$, and let τ be an arbitrary flat rewrite coinitial to ρ. Then:*
1. $\tau/\rho = \tau/\sigma$
2. $\rho/\tau \sim \sigma/\tau$

Proof. We prove each item separately:
1. By induction on the derivation of $\rho \sim \sigma$. The reflexivity, symmetry, and transitivity cases are immediate. We analyze the cases in which an axiom is applied at the root, as well as congruence closure below composition contexts:

   1.1 Rule $\sim$-Assoc. Let $(\rho_1 \,;\, \rho_2) \,;\, \rho_3 \sim \rho_1 \,;\, (\rho_2 \,;\, \rho_3)$. Then:
$$\tau/((\rho_1 \,;\, \rho_2) \,;\, \rho_3) = ((\tau/\rho_1)/\rho_2)/\rho_3 = \tau/(\rho_1 \,;\, (\rho_2 \,;\, \rho_3))$$

   1.2 Rule $\sim$-Perm. Let $\mu_1 \sim \mu_2^\flat \,;\, \mu_3^\flat$ be derived from $\mu_1 \Leftrightarrow \mu_2 \,;\, \mu_3$. Then:
$$\begin{array}{rcll}
\tau/\mu_1 & = & (\tau/\mu_2)/\mu_3 & \text{by Lem. 154} \\
& = & (\tau/\mu_2^\flat)/\mu_3^\flat & \text{by Prop. 142}
\end{array}$$

   1.3 Congruence, left of a composition. Let $\rho \,;\, \upsilon \sim \sigma \,;\, \upsilon$ be derived from $\rho \sim \sigma$. Then:
$$\begin{array}{rcll}
\tau/(\rho \,;\, \upsilon) & = & (\tau/\rho)/\upsilon & \\
& = & (\tau/\sigma)/\upsilon & \text{by IH} \\
& = & \tau/(\sigma \,;\, \upsilon) &
\end{array}$$

   1.4 Congruence, right of a composition. Let $\upsilon \,;\, \rho \sim \upsilon \,;\, \sigma$ be derived from $\rho \sim \sigma$. Then:
$$\begin{array}{rcll}
\tau/(\upsilon \,;\, \rho) & = & (\tau/\upsilon)/\rho & \\
& = & (\tau/\upsilon)/\sigma & \text{by IH} \\
& = & \tau/(\upsilon \,;\, \sigma) &
\end{array}$$

2. By induction on the derivation of $\rho \sim \sigma$. The reflexivity, symmetry, and transitivity cases are immediate. We analyze the cases in which an axiom is applied at the root, as well as congruence closure below composition contexts:

2.1 Rule $\sim$-Assoc. Let $(\rho_1 \,;\, \rho_2) \,;\, \rho_3 \sim \rho_1 \,;\, (\rho_2 \,;\, \rho_3)$. Then:

$$\begin{array}{rcll}
((\rho_1 \,;\, \rho_2) \,;\, \rho_3)/\tau & = & ((\rho_1 \,;\, \rho_2)/\tau) \,;\, (\rho_3/(\tau/(\rho_1 \,;\, \rho_2))) & \text{by Lem. 148} \\
& = & ((\rho_1/\tau) \,;\, (\rho_2/(\tau/\rho_1))) \,;\, (\rho_3/(\tau/(\rho_1 \,;\, \rho_2))) & \text{by Lem. 148} \\
& \sim & (\rho_1/\tau) \,;\, ((\rho_2/(\tau/\rho_1)) \,;\, (\rho_3/(\tau/(\rho_1 \,;\, \rho_2)))) & \text{by } \sim\text{-Assoc} \\
& = & (\rho_1/\tau) \,;\, ((\rho_2/(\tau/\rho_1)) \,;\, (\rho_3/((\tau/\rho_1)/\rho_2)))) & \\
& = & (\rho_1/\tau) \,;\, ((\rho_2 \,;\, \rho_3)/(\tau/\rho_1)) & \text{by Lem. 148} \\
& = & (\rho_1 \,;\, (\rho_2 \,;\, \rho_3))/\tau & \text{by Lem. 148}
\end{array}$$

2.2 Rule $\sim$-Perm. Let $\mu_1 \sim \mu_2^\flat \,;\, \mu_3^\flat$ be derived from $\mu_1 \Leftrightarrow \mu_2 \,;\, \mu_3$. Then by Lem. 152 we have that $(\mu_1/\tau) \overset{\flat}{\Leftrightarrow} (\mu_2/\tau) \,;\, (\mu_3/(\tau/\mu_2))$, that is, there exist multisteps such that $\hat{\mu}_1 \Leftrightarrow \hat{\mu}_2 \,;\, \hat{\mu}_3$ and such that $\hat{\mu}_1^\flat = \mu_1/\tau$ and $\hat{\mu}_2^\flat = \mu_2/\tau$ and $\hat{\mu}_3^\flat = \mu_3/(\tau/\mu_2)$. Hence:

$$\begin{array}{rcll}
\mu_1/\tau & = & \hat{\mu}_1^\flat & \\
& \sim & \hat{\mu}_1^\flat \,;\, \hat{\mu}_2^\flat & \text{by Prop. 121, as } \hat{\mu}_1 \Leftrightarrow \hat{\mu}_2 \,;\, \hat{\mu}_3 \\
& = & (\mu_2/\tau) \,;\, (\mu_3/(\tau/\mu_2)) & \\
& = & (\mu_2^\flat/\tau) \,;\, (\mu_3^\flat/(\tau/\mu_2)) & \text{by Prop. 142} \\
& = & (\mu_2^\flat \,;\, \mu_3^\flat)/\tau &
\end{array}$$

2.3 Congruence, left of a composition. Let $\rho \,;\, \upsilon \sim \sigma \,;\, \upsilon$ be derived from $\rho \sim \sigma$. Then:

$$\begin{array}{rcll}
(\rho \,;\, \upsilon)/\tau & = & (\rho/\tau) \,;\, (\upsilon/(\tau/\rho)) & \text{by Lem. 148} \\
& \sim & (\sigma/\tau) \,;\, (\upsilon/(\tau/\rho)) & \text{by IH} \\
& = & (\sigma/\tau) \,;\, (\upsilon/(\tau/\sigma)) & \text{as } \tau/\rho = \tau/\sigma\text{, by item 1. of this proposition} \\
& = & (\sigma \,;\, \upsilon)/\tau & \text{by Lem. 148}
\end{array}$$

2.4 Congruence, right of a composition. Let $\upsilon \,;\, \rho \sim \upsilon \,;\, \sigma$ be derived from $\rho \sim \sigma$. Then:

$$\begin{array}{rcll}
(\upsilon \,;\, \rho)/\tau & = & (\upsilon/\tau) \,;\, (\rho/(\tau/\upsilon)) & \text{by Lem. 148} \\
& \sim & (\upsilon/\tau) \,;\, (\sigma/(\tau/\upsilon)) & \text{by IH} \\
& = & (\upsilon \,;\, \sigma)/\tau & \text{by Lem. 148}
\end{array}$$

◀

▶ **Lemma 156** (Self-erasure). $\rho/\rho \sim \rho^\blacktriangleright$

Proof. By induction on ρ. If $\rho = \mu$ is a multistep, then $\mu/\mu = \mu^\blacktriangleright$ by Prop. 142. If $\rho = \rho_1 \,;\, \rho_2$ then:

$$\begin{array}{rcll}
(\rho_1 \,;\, \rho_2)/(\rho_1 \,;\, \rho_2) & = & ((\rho_1 \,;\, \rho_2)/\rho_1)/\rho_2 & \\
& = & ((\rho_1/\rho_1) \,;\, (\rho_2/(\rho_1/\rho_1)))/\rho_2 & \text{by Lem. 148} \\
& = & ((\rho_1/\rho_1)/\rho_2) \,;\, ((\rho_2/(\rho_1/\rho_1))/(\rho_2/(\rho_1/\rho_1))) & \text{by Lem. 148} \\
& \sim & (\rho_1^\blacktriangleright/\rho_2) \,;\, ((\rho_2/(\rho_1/\rho_1))/(\rho_2/(\rho_1/\rho_1))) & \text{by IH and Prop. 155} \\
& = & (\rho_2^\blacktriangleleft/\rho_2) \,;\, ((\rho_2/(\rho_1/\rho_1))/(\rho_2/(\rho_1/\rho_1))) & \text{as } \rho_1^{\mathsf{tgt}} =_{\beta\eta} \rho_2^{\mathsf{src}} \\
& = & \rho_2^\blacktriangleright \,;\, ((\rho_2/(\rho_1/\rho_1))/(\rho_2/(\rho_1/\rho_1))) & \text{by Lem. 149} \\
& = & (\rho_2/(\rho_1/\rho_1))/(\rho_2/(\rho_1/\rho_1)) & \text{by } \approx\text{-}\mathsf{IdL} \text{ and Thm. 130} \\
& = & (\rho_2/\rho_1^\blacktriangleright)/(\rho_2/\rho_1^\blacktriangleright) & \text{by IH and Prop. 155} \\
& = & (\rho_2/\rho_2^\blacktriangleleft)/(\rho_2/\rho_2^\blacktriangleleft) & \text{as } \rho_1^{\mathsf{tgt}} =_{\beta\eta} \rho_2^{\mathsf{src}} \\
& = & \rho_2/\rho_2 & \text{by Lem. 149} \\
& \sim & \rho_2^\blacktriangleright & \text{by IH} \\
& = & (\rho_1 \,;\, \rho_2)^\blacktriangleright &
\end{array}$$

◀

▶ **Lemma 157** (Multistep permutation). *Let μ, ν be coinitial multisteps. Then:*

$$\mu \,;\, (\nu/\mu) \approx \nu \,;\, (\mu/\nu)$$

Proof. Recall that if $\mu^\flat = (\mu')^\flat$ and $\nu^\flat = (\nu')^\flat$, then $\mu/\nu = \mu'/\nu'$ by Prop. 142. Moreover, by soundness of flattening Lem. 100 if μ is a step and μ' is its $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form, then $\mu \approx \mu'$. Hence, to prove the statement of the lemma, it suffices to show that $\mu \,;\, (\nu/\mu) \approx \nu \,;\, (\mu/\nu)$ assuming that μ, ν are coinitial multisteps in $\overset{\circ}{\mapsto}, \overline{\eta}$-normal form.

The proof proceeds by induction on the number of applications in μ, and by case analysis on the head of μ:

1. **μ headed by a variable.** Then $\mu = \lambda\vec{x}.y\,\mu_1 \ldots \mu_n$. Since μ and ν are coinitial, the head of ν cannot be a constant or a rule symbol, and in fact it must also be y, that is $\nu = \lambda\vec{x}.y\,\nu_1 \ldots \nu_n$. Note that:

$$\begin{array}{lll} & \mu \,;\, (\nu/\mu) & \\ = & (\lambda\vec{x}.y\,\mu_1 \ldots \mu_n) \,;\, ((\lambda\vec{x}.y\,\nu_1 \ldots \nu_n)/(\lambda\vec{x}.y\,\mu_1 \ldots \mu_n)) & \\ = & (\lambda\vec{x}.y\,\mu_1 \ldots \mu_n) \,;\, (\lambda\vec{x}.y\,(\nu_1/\mu_1) \ldots (\nu_n/\mu_n))^\flat & \text{by Prop. 142} \\ \approx & (\lambda\vec{x}.y\,\mu_1 \ldots \mu_n) \,;\, (\lambda\vec{x}.y\,(\nu_1/\mu_1) \ldots (\nu_n/\mu_n)) & \text{by Lem. 100} \\ \approx & \lambda\vec{x}.y\,(\mu_1 \,;\, (\nu_1/\mu_1)) \ldots (\mu_n \,;\, (\nu_n/\mu_n)) & \text{by } \approx\text{-Abs}, \approx\text{-App} \end{array}$$

Similarly:

$$\nu \,;\, (\mu/\nu) \approx \lambda\vec{x}.y\,(\nu_1 \,;\, (\mu_1/\nu_1)) \ldots (\nu_n \,;\, (\mu_n/\nu_n))$$

Moreover, by IH we have that $\mu_i \,;\, (\nu_i/\mu_i) \approx \nu_i \,;\, (\mu_i/\nu_i)$ for all $1 \leq i \leq n$, so:

$$\lambda\vec{x}.y\,(\mu_1 \,;\, (\nu_1/\mu_1)) \ldots (\mu_n \,;\, (\nu_n/\mu_n)) \approx \lambda\vec{x}.y\,(\nu_1 \,;\, (\mu_1/\nu_1)) \ldots (\nu_n \,;\, (\mu_n/\nu_n))$$

2. **μ headed by a constant.** Then $\mu = \lambda\vec{x}.\mathbf{c}\,\mu_1 \ldots \mu_n$. Since μ and ν are coinitial, the head of ν can be either $\mathbf{c}$ or a rule symbol ϱ. We consider two subcases:

   2.1 **ν headed by a constant.** Then $\nu = \lambda\vec{x}.\mathbf{c}\,\nu_1 \ldots \nu_n$. The proof of this case proceeds similarly as for case 1, when the heads of μ and ν are both variables.

   2.2 **ν headed by a rule symbol.** Then $\nu = \lambda\vec{x}.\varrho\,\nu_1 \ldots \nu_q$. By Lem. 137 we have that $\mu_1 = (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_1 \ldots \mu'_q)^\circ$ where μ'_i and ν_i are coinitial for all $1 \leq i \leq q$, and each μ'_i has strictly less applications than μ. Note that:

$$\begin{array}{lll} & \mu \,;\, (\nu/\mu) & \\ = & (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_1 \ldots \mu'_q)^\circ \,;\, ((\lambda\vec{x}.\varrho\,\nu_1 \ldots \nu_q)/(\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_1 \ldots \mu'_q)^\circ) & \\ = & (\lambda\vec{x}.\varrho^{<\overline{\eta}}\,\mu'_1 \ldots \mu'_q)^\circ \,;\, (\lambda\vec{x}.(\varrho/\varrho^{\mathsf{src}})(\nu_1/\mu'_1) \ldots (\nu_q/\mu'_q))^\flat & \text{by Prop. 142} \\ \approx & (\lambda\vec{x}.\varrho^{\mathsf{src}}\,\mu'_1 \ldots \mu'_q) \,;\, (\lambda\vec{x}.(\varrho/\varrho^{\mathsf{src}})(\nu_1/\mu'_1) \ldots (\nu_q/\mu'_q)) & \text{by Lem. 100} \\ \approx & (\lambda\vec{x}.(\varrho^{\mathsf{src}} \,;\, (\varrho/\varrho^{\mathsf{src}}))(\mu'_1 \,;\, (\nu_1/\mu'_1)) \ldots (\mu'_q \,;\, (\nu_q/\mu'_q))) & \text{by } \approx\text{-Abs}, \approx\text{-App} \end{array}$$

   Similarly:

$$\nu \,;\, (\mu/\nu) \approx (\lambda\vec{x}.(\varrho \,;\, (\varrho^{\mathsf{src}}/\varrho))(\nu_1 \,;\, (\mu'_1/\nu_1)) \ldots (\nu_q \,;\, (\mu'_q/\nu_q)))$$

   Note that:

$$\begin{array}{rcll} \varrho^{\mathsf{src}} \,;\, (\varrho/\varrho^{\mathsf{src}}) & = & \varrho^{\mathsf{src}} \,;\, \varrho & \text{by Prop. 142} \\ & = & \varrho & \text{by } \approx\text{-IdL} \\ & = & \varrho \,;\, \varrho^{\mathsf{tgt}} & \text{by } \approx\text{-IdR} \\ & \approx & \varrho \,;\, \varrho^{\blacktriangleright} & \text{by Lem. 100} \\ & = & \varrho \,;\, (\varrho^{\mathsf{src}}/\varrho) & \end{array}$$

Moreover, by IH we know that $\mu'_i \,;\, (\nu_i/\mu'_i) \approx \nu_i \,;\, (\mu'_i/\nu_i)$ for all $1 \leq i \leq q$. Hence:

$$\begin{array}{ll} & \lambda\vec{x}.(\varrho^{\mathsf{src}} \,;\, (\varrho/\varrho^{\mathsf{src}}))(\mu'_1 \,;\, (\nu_1/\mu'_1)) \ldots (\mu'_q \,;\, (\nu_q/\mu'_q)) \\ \approx & \lambda\vec{x}.(\varrho \,;\, (\varrho^{\mathsf{src}}/\varrho))(\nu_1 \,;\, (\mu'_1/\nu_1)) \ldots (\nu_q \,;\, (\mu'_q/\nu_q)) \end{array}$$

3. μ **headed by a rule symbol.** Then $\mu = \lambda\vec{x}.\varrho\,\mu_1 \ldots \mu_n$. Since μ and ν are coinitial, the head of ν can be either $\mathbf{c}$ or a rule symbol ϱ. We consider two subcases:
   - **3.1** ν **headed by a constant.** Then the proof of this case proceeds similarly as the proof for the symmetric case 2.2, when the head of μ is a constant and the head of ν is a rule symbol.
   - **3.2** ν **headed by a rule symbol.** Then by orthogonality $\nu = \lambda\vec{x}.\varrho\,\nu_1 \ldots \nu_n$. Proceeding similarly as for case 1, we have that:

$$\begin{array}{lcl} \mu \,;\, (\nu/\mu) & \approx & \lambda\vec{x}.(\varrho/(\varrho/\varrho))\,(\mu_1 \,;\, (\nu_1/\mu_1)) \ldots (\mu_n \,;\, (\nu_n/\mu_n)) \\ \nu \,;\, (\mu/\nu) & \approx & \lambda\vec{x}.(\varrho/(\varrho/\varrho))\,(\nu_1 \,;\, (\mu_1/\nu_1)) \ldots (\nu_n \,;\, (\mu_n/\nu_n)) \end{array}$$

By IH we have that $\mu_i \,;\, (\nu_i/\mu_i) \approx \nu_i \,;\, (\mu_i/\nu_i)$ for all $1 \leq i \leq n$, so:

$$\begin{array}{ll} & \lambda\vec{x}.(\varrho/(\varrho/\varrho))\,(\mu_1 \,;\, (\nu_1/\mu_1)) \ldots (\mu_n \,;\, (\nu_n/\mu_n)) \\ \approx & \lambda\vec{x}.(\varrho/(\varrho/\varrho))\,(\nu_1 \,;\, (\mu_1/\nu_1)) \ldots (\nu_n \,;\, (\mu_n/\nu_n)) \end{array}$$

◀

▶ **Lemma 158** (Rewrite permutation). *If ρ, σ are coinitial flat rewrites then:*

$$\rho \,;\, (\sigma/\rho) \sim \sigma \,;\, (\rho/\sigma)$$

Proof. We proceed by induction on ρ:

1. **Multistep, $\rho = \mu$.** To prove $\mu \,;\, (\sigma/\mu) \sim \sigma \,;\, (\mu/\sigma)$ we proceed by a nested induction on σ:
   - **1.1 Multistep, $\sigma = \nu$.** By Lem. 157 we have that $\mu \,;\, (\nu/\mu) \approx \nu \,;\, (\mu/\nu)$. By Thm. 130 this implies that $\mu^\flat \,;\, (\nu/\mu)^\flat \sim \nu^\flat \,;\, (\mu/\nu)^\flat$. But μ and ν are flat, and by Prop. 142 we know that $\mu/\nu = (\mu/\nu)^\flat$ and $\nu/\mu = (\nu/\mu)^\flat$. Hence $\mu \,;\, (\nu/\mu) \sim \nu \,;\, (\mu/\nu)$.
   - **1.2 Composition, $\sigma = \sigma_1 \,;\, \sigma_2$.** To alleviate the notation, we work implicitly modulo the $\sim$-Assoc rule. Note that:

$$\begin{array}{lcll} \mu \,;\, ((\sigma_1 \,;\, \sigma_2)/\mu) & = & \mu \,;\, (\sigma_1/\mu) \,;\, (\sigma_2/(\mu/\sigma_1)) & \\ & \sim & \sigma_1 \,;\, (\mu/\sigma_1) \,;\, (\sigma_2/(\mu/\sigma_1)) & \text{by IH} \\ & \sim & \sigma_1 \,;\, \sigma_2 \,;\, ((\mu/\sigma_1)/\sigma_2) & \text{by IH} \\ & = & \sigma_1 \,;\, \sigma_2 \,;\, (\mu/(\sigma_1 \,;\, \sigma_2)) & \end{array}$$

2. **Composition, $\rho = \rho_1 \,;\, \rho_2$.** To alleviate the notation, we work implicitly modulo the $\sim$-Assoc rule. Note that:

$$\begin{array}{lcll} \rho_1 \,;\, \rho_2 \,;\, (\sigma/(\rho_1 \,;\, \rho_2)) & = & \rho_1 \,;\, \rho_2 \,;\, ((\sigma/\rho_1)/\rho_2) & \\ & \sim & \rho_1 \,;\, (\sigma/\rho_1) \,;\, (\rho_2/(\sigma/\rho_1)) & \text{by IH} \\ & \sim & \sigma \,;\, (\rho_1/\sigma) \,;\, (\rho_2/(\sigma/\rho_1)) & \text{by IH} \\ & = & \sigma \,;\, ((\rho_1 \,;\, \rho_2)/\sigma) & \text{by Lem. 148} \end{array}$$

◀

G Projection for Arbitrary Rewrites

In this subsection we prove properties of the generalized projection operator.

▶ **Lemma 159** (Projection of abstraction). *The following hold:*

1. *If μ is a flat multistep and ρ is a coinitial flat rewrite, then $(\lambda x.\mu)^\flat /^1 (\lambda x.\rho)^\flat = (\lambda x.(\mu /^1 \rho))^\flat$.*
2. *If ρ is a flat rewrite and μ is a coinitial flat multistep, then $(\lambda x.\rho)^\flat /^2 (\lambda x.\mu)^\flat = (\lambda x.(\rho /^2 \mu))^\flat$.*
3. *If ρ, σ are coinitial flat rewrites, then $(\lambda x.\rho)^\flat /^3 (\lambda x.\sigma)^\flat = (\lambda x.(\rho /^3 \sigma))^\flat$.*

Proof. We prove each item separately:

1. We proceed by induction on ρ:

1.1 If $\rho = \nu$, then:

$$\begin{array}{rcll} (\lambda x.\mu)^\flat /^1 (\lambda x.\nu)^\flat & = & (\lambda x.\mu)^\flat/(\lambda x.\nu)^\flat & \text{by definition} \\ & = & ((\lambda x.\mu)/(\lambda x.\nu))^\flat & \text{by Prop. 142} \\ & = & (\lambda x.(\mu/\nu))^\flat & \text{by Prop. 142} \\ & = & (\lambda x.(\mu /^1 \nu))^\flat & \text{by definition} \end{array}$$

1.2 If $\rho = \rho_1 \,;\, \rho_2$, then:

$$\begin{array}{rcll} (\lambda x.\mu)^\flat /^1 (\lambda x.(\rho_1 \,;\, \rho_2))^\flat & = & (\lambda x.\mu)^\flat /^1 ((\lambda x.\rho_1)^\flat \,;\, (\lambda x.\rho_2)^\flat) & \text{by Prop. 96} \\ & = & ((\lambda x.\mu)^\flat /^1 (\lambda x.\rho_1)^\flat) /^1 (\lambda x.\rho_2)^\flat & \text{by definition} \\ & = & (\lambda x.(\mu /^1 \rho_1))^\flat /^1 (\lambda x.\rho_2)^\flat & \text{by IH} \\ & = & (\lambda x.((\mu /^1 \rho_1) /^1 \rho_2))^\flat & \text{by IH} \\ & = & (\lambda x.(\mu /^1 (\rho_1 \,;\, \rho_2)))^\flat & \text{by definition} \end{array}$$

2. We proceed by induction on ρ:

2.1 If $\rho = \nu$, then:

$$\begin{array}{rcll} (\lambda x.\nu)^\flat /^2 (\lambda x.\mu)^\flat & = & (\lambda x.\nu)^\flat/(\lambda x.\mu)^\flat & \text{by definition} \\ & = & (\lambda x.(\nu/\mu))^\flat & \text{by Prop. 142} \\ & = & (\lambda x.(\nu /^2 \mu))^\flat & \text{by definition} \end{array}$$

2.2 If $\rho = \rho_1 \,;\, \rho_2$, then:

$$\begin{array}{cll} & (\lambda x.(\rho_1 \,;\, \rho_2))^\flat /^2 (\lambda x.\mu)^\flat & \\ = & ((\lambda x.\rho_1)^\flat \,;\, (\lambda x.\rho_2)^\flat) /^2 (\lambda x.\mu)^\flat & \text{by Prop. 96} \\ = & ((\lambda x.\rho_1)^\flat /^2 (\lambda x.\mu)^\flat) \,;\, ((\lambda x.\rho_2)^\flat /^2 ((\lambda x.\mu)^\flat /^1 (\lambda x.\rho_1)^\flat)) & \text{by definition} \\ = & ((\lambda x.\rho_1)^\flat /^2 (\lambda x.\mu)^\flat) \,;\, ((\lambda x.\rho_2)^\flat /^2 (\lambda x.(\mu /^1 \rho_1))^\flat) & \text{by item 1 of this lemma} \\ = & (\lambda x.(\rho_1 /^2 \mu))^\flat \,;\, (\lambda x.(\rho_2 /^2 (\mu /^1 \rho_1)))^\flat) & \text{by IH} \\ = & (\lambda x.((\rho_1 /^2 \mu) \,;\, (\rho_2 /^2 (\mu /^1 \rho_1))))^\flat & \text{by Prop. 96} \\ = & (\lambda x.((\rho_1 \,;\, \rho_2) /^2 \mu))^\flat & \text{by definition} \end{array}$$

3. We proceed by induction on σ:

3.1 If $\sigma = \mu$, then:

$$\begin{array}{rcll} (\lambda x.\rho)^\flat /^3 (\lambda x.\mu)^\flat & = & (\lambda x.\rho)^\flat /^2 (\lambda x.\mu)^\flat & \text{by definition} \\ & = & (\lambda x.(\rho /^2 \mu))^\flat & \text{by item 2 of this lemma} \\ & = & (\lambda x.(\rho /^3 \mu))^\flat & \text{by definition} \end{array}$$

3.2 If $\sigma = \sigma_1 \,;\, \sigma_2$, then:

$$\begin{array}{lll} & (\lambda x.\rho)^\flat \,/^3\, (\lambda x.(\sigma_1 \,;\, \sigma_2))^\flat & \\ = & (\lambda x.\rho)^\flat \,/^3\, ((\lambda x.\sigma_1)^\flat \,;\, (\lambda x.\sigma_2)^\flat) & \text{by Prop. 96} \\ = & ((\lambda x.\rho)^\flat \,/^3\, (\lambda x.\sigma_1)^\flat) \,/^3\, (\lambda x.\sigma_2)^\flat & \text{by definition} \\ = & (\lambda x.(\rho \,/^3\, \sigma_1))^\flat \,/^3\, (\lambda x.\sigma_2)^\flat & \text{by IH} \\ = & (\lambda x.((\rho \,/^3\, \sigma_1) \,/^3\, \sigma_2))^\flat & \text{by IH} \\ = & (\lambda x.(\rho \,/^3\, (\sigma_1 \,;\, \sigma_2)))^\flat & \text{by definition} \end{array}$$

◀

▶ **Lemma 160** (Projection of application). *The following hold:*

1. *If μ_1, μ_2 are flat multisteps and ρ_1, ρ_2 are flat rewrites such that μ_1 and ρ_1 are coinitial and μ_2 and ρ_2 are coinitial, then $(\mu_1\,\mu_2)^\flat \,/^1\, (\rho_1\,\rho_2)^\flat = ((\mu_1 \,/^1\, \rho_1)\,(\mu_2 \,/^1\, \rho_2))^\flat$.*
2. *If ρ_1, ρ_2 are flat rewrites and μ_1, μ_2 are flat multisteps such that ρ_1 and μ_1 are coinitial and ρ_2 and μ_2 are coinitial, then $(\rho_1\,\rho_2)^\flat \,/^2\, (\mu_1\,\mu_2)^\flat = ((\rho_1 \,/^2\, \mu_1)\,(\rho_2 \,/^2\, \mu_2))^\flat$.*
3. *If $\rho_1, \rho_2, \sigma_1, \sigma_2$ are flat rewrites such that ρ_1 and σ_1 are coinitial and ρ_2 and σ_2 are coinitial, then $(\rho_1\,\rho_2)^\flat \,/^3\, (\sigma_1\,\sigma_2)^\flat = ((\rho_1 \,/^3\, \sigma_1)\,(\rho_2 \,/^3\, \sigma_2))^\flat$.*

Proof. We prove each item separately:

1. Using the fact that $\overset{\flat}{\mapsto}$ is strongly normalizing (Prop. 95), we proceed by induction on the length of the longest reduction $\rho_1\,\rho_2 \overset{\flat}{\mapsto}{}^* (\rho_1\,\rho_2)^\flat$, considering four cases, depending on whether each of ρ_1 and ρ_2 is a multistep or a composition:

1.1 If both are multisteps, *i.e.* $\rho_1 = \nu_1$ and $\rho_2 = \nu_2$:

$$\begin{array}{rcll} (\mu_1\,\mu_2)^\flat \,/^1\, (\nu_1\,\nu_2)^\flat & = & (\mu_1\,\mu_2)^\flat/(\nu_1\,\nu_2)^\flat & \text{by definition} \\ & = & ((\mu_1\,\mu_2)/(\nu_1\,\nu_2))^\flat & \text{by Prop. 142} \\ & = & ((\mu_1/\nu_1)\,(\mu_2/\nu_2))^\flat & \text{by Prop. 142} \\ & = & ((\mu_1 \,/^1\, \nu_1)\,(\mu_2 \,/^1\, \nu_2))^\flat & \text{by definition} \end{array}$$

1.2 If $\rho_1 = \nu_1$ is a multistep and $\rho_2 = \rho_{21} \,;\, \rho_{22}$ is a sequence:

$$\begin{array}{lll} & (\mu_1\,\mu_2)^\flat \,/^1\, (\nu_1\,(\rho_{21} \,;\, \rho_{22}))^\flat & \\ = & (\mu_1\,\mu_2)^\flat \,/^1\, ((\nu_1\,\rho_{21})^\flat \,;\, (\nu_1^{\mathsf{tgt}}\,\rho_{22})^\flat) & \text{by Prop. 96} \\ = & ((\mu_1\,\mu_2)^\flat \,/^1\, (\nu_1\,\rho_{21})^\flat) \,/^1\, (\nu_1^{\mathsf{tgt}}\,\rho_{22})^\flat & \text{by definition} \\ = & ((\mu_1 \,/^1\, \nu_1)\,(\mu_2 \,/^1\, \rho_{21}))^\flat \,/^1\, (\nu_1^{\mathsf{tgt}}\,\rho_{22})^\flat & \text{by IH} \\ = & (((\mu_1 \,/^1\, \nu_1) \,/^1\, \nu_1^{\mathsf{tgt}})\,((\mu_2 \,/^1\, \rho_{21}) \,/^1\, \rho_{22}))^\flat & \text{by IH} \\ = & ((\mu_1 \,/^1\, (\nu_1 \,;\, \nu_1^{\mathsf{tgt}}))\,(\mu_2 \,/^1\, (\rho_{21} \,;\, \rho_{22})))^\flat & \text{by definition} \\ = & ((\mu_1 \,/^1\, (\nu_1 \,;\, \nu_1^{\blacktriangleright}))\,(\mu_2 \,/^1\, (\rho_{21} \,;\, \rho_{22})))^\flat & \text{by Prop. 96} \\ = & ((\mu_1 \,/^1\, \nu_1))\,(\mu_2 \,/^1\, (\rho_{21} \,;\, \rho_{22})))^\flat & \text{by Prop. 155, since } \nu_1 \,;\, \nu_1^{\blacktriangleright} \sim \nu_1 \end{array}$$

To justify that the IH may be applied, note that $\nu_1\,(\rho_{21} \,;\, \rho_{22}) \overset{\flat}{\mapsto} (\nu_1\,\rho_{21}) \,;\, (\nu_1^{\mathsf{tgt}}\,\rho_{22})$.

1.3 If $\rho_1 = \rho_{11} \,;\, \rho_{12}$ is a sequence and $\rho_2 = \nu_2$ is a multistep, the proof is similar as for the previous case.

1.4 If both are sequences, *i.e.* $\rho_1 = \rho_{11} \,;\, \rho_{12}$ and $\rho_2 = \rho_{21} \,;\, \rho_{22}$:

$$\begin{array}{lll} & (\mu_1\,\mu_2)^\flat \,/^1\, ((\rho_{11} \,;\, \rho_{12})\,(\rho_{21} \,;\, \rho_{22}))^\flat & \\ = & (\mu_1\,\mu_2)^\flat \,/^1\, (((\rho_{11} \,;\, \rho_{12})\,\rho_{21}^{\mathsf{src}}))^\flat \,;\, (\rho_{12}^{\mathsf{tgt}}\,(\rho_{21} \,;\, \rho_{22}))^\flat) & \text{by Prop. 96} \\ = & ((\mu_1\,\mu_2)^\flat \,/^1\, ((\rho_{11} \,;\, \rho_{12})\,\rho_{21}^{\mathsf{src}}))^\flat) \,/^1\, (\rho_{12}^{\mathsf{tgt}}\,(\rho_{21} \,;\, \rho_{22}))^\flat & \text{by definition} \\ = & ((\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12}))\,(\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}}))^\flat \,/^1\, (\rho_{12}^{\mathsf{tgt}}\,(\rho_{21} \,;\, \rho_{22}))^\flat & \text{by IH} \\ = & (((\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12})) \,/^1\, \rho_{12}^{\mathsf{tgt}})\,((\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}}) \,/^1\, (\rho_{21} \,;\, \rho_{22})))^\flat & \text{by IH} \\ = & ((\mu_1 \,/^1\, ((\rho_{11} \,;\, \rho_{12}) \,;\, \rho_{12}^{\mathsf{tgt}}))\,(\mu_2 \,/^1\, (\rho_{21}^{\mathsf{src}} \,;\, (\rho_{21} \,;\, \rho_{22}))))^\flat & \text{by definition} \\ = & ((\mu_1 \,/^1\, ((\rho_{11} \,;\, \rho_{12}) \,;\, \rho_{12}^{\blacktriangleright}))\,(\mu_2 \,/^1\, (\rho_{21}^{\blacktriangleleft} \,;\, (\rho_{21} \,;\, \rho_{22}))))^\flat & \text{by Prop. 96} \\ = & ((\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12}))\,(\mu_2 \,/^1\, (\rho_{21} \,;\, \rho_{22})))^\flat & \text{by Prop. 155} \end{array}$$

To justify that the IH may be applied, note that $(\rho_{11}\,;\,\rho_{12})\,(\rho_{21}\,;\,\rho_{22}) \overset{\flat}{\mapsto} (\rho_{11}\,;\,\rho_{12})\,\rho_{21}^{\mathsf{src}}\,;\,\rho_{12}^{\mathsf{tgt}}\,(\rho_{21}\,\rho_{22})$. To justify the last equality, note that $(\rho_{11}\,;\,\rho_{12})\,;\,\rho_{12}^{\blacktriangleright} \sim \rho_{11}\,;\,\rho_{12}$ and $\rho_{21}^{\blacktriangleleft}\,;\,(\rho_{21}\,;\,\rho_{22}) \sim \rho_{21}\,;\,\rho_{22}$.

2. Using the fact that $\overset{\flat}{\mapsto}$ is strongly normalizing (Prop. 95), we proceed by induction on the length of the longest reduction $\rho_1\,\rho_2 \overset{\flat}{\mapsto}{}^* (\rho_1\,\rho_2)^\flat$, considering four cases, depending on whether each of ρ_1 and ρ_2 is a multistep or a composition:

2.1 If both are multisteps, *i.e.* $\rho_1 = \nu_1$ and $\rho_2 = \nu_2$:

$$\begin{array}{rcll}(\nu_1\,\nu_2)^\flat\ /^2\ (\mu_1\,\mu_2)^\flat & = & (\nu_1\,\nu_2)^\flat/(\mu_1\,\mu_2)^\flat & \text{by definition}\\ & = & ((\nu_1\,\nu_2)/(\mu_1\,\mu_2))^\flat & \text{by Prop. 142}\\ & = & ((\nu_1/\mu_1)\,(\nu_2/\mu_2))^\flat & \text{by Prop. 142}\\ & = & ((\nu_1\ /^2\ \mu_1)\,(\nu_2\ /^2\ \mu_2))^\flat & \text{by definition}\end{array}$$

2.2 If $\rho_1 = \nu_1$ as a multistep and $\rho_2 = \rho_{21}\,;\,\rho_{22}$ is a sequence:

$$\begin{array}{rl} & (\nu_1\,(\rho_{21}\,;\,\rho_{22}))^\flat\ /^2\ (\mu_1\,\mu_2)^\flat\\ = & ((\nu_1\,\rho_{21})^\flat\,;\,(\nu_1^{\mathsf{tgt}}\,\rho_{22})^\flat)\ /^2\ (\mu_1\,\mu_2)^\flat\\ & \quad\text{by Prop. 96}\\ = & ((\nu_1\,\rho_{21})^\flat\ /^2\ (\mu_1\,\mu_2)^\flat)\,;\,((\nu_1^{\mathsf{tgt}}\,\rho_{22})^\flat\ /^2\ ((\mu_1\,\mu_2)^\flat\ /^1\ (\nu_1\,\rho_{21})^\flat))\\ & \quad\text{by definition}\\ = & ((\nu_1\,\rho_{21})^\flat\ /^2\ (\mu_1\,\mu_2)^\flat)\,;\,((\nu_1^{\mathsf{tgt}}\,\rho_{22})^\flat\ /^2\ ((\mu_1\ /^1\ \nu_1)\,(\mu_2\ /^1\ \rho_{21}))^\flat)\\ & \quad\text{by item 1. of this lemma}\\ = & ((\nu_1\ /^2\ \mu_1)\,(\rho_{21}\ /^2\ \mu_2))^\flat\,;\,(((\nu_1^{\mathsf{tgt}}\ /^2\ (\mu_1\ /^1\ \nu_1))\,(\rho_{22}\ /^2\ (\mu_2\ /^1\ \rho_{21})))^\flat\\ & \quad\text{by IH}\\ = & ((\nu_1\ /^2\ \mu_1)\,(\rho_{21}\ /^2\ \mu_2))^\flat\,;\,(((\nu_1\ /^2\ \mu_1)^{\mathsf{tgt}}\,(\rho_{22}\ /^2\ (\mu_2\ /^1\ \rho_{21})))^\flat\\ & \quad\text{by }(\star))\\ = & ((\nu_1\ /^2\ \mu_1)\,((\rho_{21}\ /^2\ \mu_2)\,;\,(\rho_{22}\ /^2\ (\mu_2\ /^1\ \rho_{21}))))^\flat\\ & \quad\text{by Prop. 96}\\ = & ((\nu_1\ /^2\ \mu_1)\,((\rho_{21}\,;\,\rho_{22})\ /^2\ \mu_2))^\flat\\ & \quad\text{by definition}\end{array}$$

To justify that the IH may be applied, note that $\nu_1\,(\rho_{21}\,;\,\rho_{22}) \overset{\flat}{\mapsto} (\nu_1\,\rho_{21})\,;\,(\nu_1^{\mathsf{tgt}}\,\rho_{22})$. To justify the $(\star)$ step, note that:

$$\begin{array}{rll} & (\ldots\nu_1^{\mathsf{tgt}}\ /^2\ (\mu_1\ /^1\ \nu_1)\ldots)^\flat & \\ = & (\ldots\nu_1^{\blacktriangleright}\ /^2\ (\mu_1\ /^1\ \nu_1)\ldots)^\flat & \text{by Prop. 96}\\ = & (\ldots(\mu_1\ /^1\ \nu_1)^{\blacktriangleleft}\ /^2\ (\mu_1\ /^1\ \nu_1)\ldots)^\flat & \\ = & (\ldots(\mu_1\ /^1\ \nu_1)^{\blacktriangleright}\ldots)^\flat & \text{by Lem. 149}\\ = & (\ldots(\nu_1\ /^1\ \mu_1)^{\blacktriangleright}\ldots)^\flat & \text{by Lem. 157 and Lem. 52}\\ = & (\ldots(\nu_1\ /^1\ \mu_1)^{\mathsf{tgt}}\ldots)^\flat & \text{by Prop. 96}\end{array}$$

2.3 If $\rho_1 = \rho_{11}\,;\,\rho_{12}$ as a sequence and $\rho_2 = \nu_2$ is a multistep, the proof is similar as for the previous case.

2.4 If both are sequences, *i.e.* $\rho_1 = \rho_{11}\,;\,\rho_{12}$ and $\rho_2 = \rho_{21}\,;\,\rho_{22}$, let $s = ((\rho_{21}\,;\,\rho_{22})\ /^2$

$(\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}}))^{\mathsf{src}}$ and $t = ((\rho_{11} \,;\, \rho_{12}) \,/^2\, \mu_1)^{\mathsf{tgt}}$; then:

$$\begin{array}{cl}
& ((\rho_{11} \,;\, \rho_{12})\,(\rho_{21} \,;\, \rho_{22}))^\flat \,/^2\, (\mu_1\,\mu_2)^\flat \\
= & (((\rho_{11} \,;\, \rho_{12})\,\rho_{21}^{\mathsf{src}}))^\flat \,;\, (\rho_{12}^{\mathsf{tgt}}\,(\rho_{21} \,;\, \rho_{22}))^\flat) \,/^2\, (\mu_1\,\mu_2)^\flat \\
& \quad \text{by Prop. 96} \\
= & (((\rho_{11} \,;\, \rho_{12})\,\rho_{21}^{\mathsf{src}})^\flat \,/^2\, (\mu_1\,\mu_2)^\flat) \,;\, ((\rho_{12}^{\mathsf{tgt}}\,(\rho_{21} \,;\, \rho_{22}))^\flat \,/^2\, ((\mu_1\,\mu_2)^\flat \,/^1\, ((\rho_{11} \,;\, \rho_{12})\,\rho_{21}^{\mathsf{src}})^\flat)) \\
& \quad \text{by definition} \\
= & (((\rho_{11} \,;\, \rho_{12})\,\rho_{21}^{\mathsf{src}})^\flat \,/^2\, (\mu_1\,\mu_2)^\flat) \,;\, ((\rho_{12}^{\mathsf{tgt}}\,(\rho_{21} \,;\, \rho_{22}))^\flat \,/^2\, ((\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12}))\,(\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}}))^\flat) \\
& \quad \text{by item 1. of this lemma} \\
= & (((\rho_{11} \,;\, \rho_{12}) \,/^2\, \mu_1)\,(\rho_{21}^{\mathsf{src}} \,/^2\, \mu_2))^\flat \,;\, ((\rho_{12}^{\mathsf{tgt}} \,/^2\, (\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12})))\,((\rho_{21} \,;\, \rho_{22}) \,/^2\, (\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}})))^\flat \\
& \quad \text{by IH} \\
= & (((\rho_{11} \,;\, \rho_{12}) \,/^2\, \mu_1)\,\underline{s})^\flat \,;\, ((\rho_{12}^{\mathsf{tgt}} \,/^2\, (\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12})))\,((\rho_{21} \,;\, \rho_{22}) \,/^2\, (\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}})))^\flat \\
& \quad \text{by } (\star_1) \\
= & (((\rho_{11} \,;\, \rho_{12}) \,/^2\, \mu_1)\,\underline{s})^\flat \,;\, (\underline{t}\,((\rho_{21} \,;\, \rho_{22}) \,/^2\, (\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}})))^\flat \\
& \quad \text{by } (\star_2) \\
= & (((\rho_{11} \,;\, \rho_{12}) \,/^2\, \mu_1)((\rho_{21} \,;\, \rho_{22}) \,/^2\, (\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}})))^\flat \\
& \quad \text{by Prop. 96} \\
= & (((\rho_{11} \,;\, \rho_{12}) \,/^2\, \mu_1)((\rho_{21} \,;\, \rho_{22}) \,/^2\, \mu_2))^\flat \\
& \quad \text{by } (\star_2)
\end{array}$$

To justify the $(\star_1)$ step, note that:

$$\begin{array}{rcll}
(\ldots \rho_{21}^{\mathsf{src}} \,/^2\, \mu_2 \ldots)^\flat & = & (\ldots \rho_{21}^{\blacktriangleleft} \,/^2\, \mu_2 \ldots)^\flat & \text{by Prop. 96} \\
& = & (\ldots \mu_2^{\blacktriangleleft} \,/^2\, \mu_2 \ldots)^\flat & \\
& = & (\ldots \mu_2^{\blacktriangleright} \ldots)^\flat & \text{by Prop. 142} \\
& = & (\ldots (\mu_2 \,/^1\, \rho_{21}^{\blacktriangleleft})^{\blacktriangleright} \ldots)^\flat & \text{by Prop. 142} \\
& = & (\ldots (\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}})^{\blacktriangleright} \ldots)^\flat & \text{by Prop. 96} \\
& = & (\ldots ((\rho_{21} \,;\, \rho_{22}) \,/^2\, (\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}}))^{\blacktriangleleft} \ldots)^\flat & \\
& = & (\ldots ((\rho_{21} \,;\, \rho_{22}) \,/^2\, (\mu_2 \,/^1\, \rho_{21}^{\mathsf{src}}))^{\mathsf{src}} \ldots)^\flat & \text{by Prop. 96} \\
& = & (\ldots \underline{s} \ldots)^\flat &
\end{array}$$

To justify the $(\star_2)$ step, note that:

$$\begin{array}{cll}
& (\ldots \rho_{12}^{\mathsf{tgt}} \,/^2\, (\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12}))) \ldots)^\flat & \\
= & (\ldots \rho_{12}^{\blacktriangleright} \,/^2\, (\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12}))) \ldots)^\flat & \text{by Prop. 142} \\
= & (\ldots (\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12}))^{\blacktriangleleft} \,/^2\, (\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12}))) \ldots)^\flat & \text{by Prop. 142} \\
= & (\ldots (\mu_1 \,/^1\, (\rho_{11} \,;\, \rho_{12}))^{\blacktriangleright} \ldots)^\flat & \text{by Prop. 142} \\
= & (\ldots ((\rho_{11} \,;\, \rho_{12}) \,/^2\, \mu_1)^{\blacktriangleright} \ldots)^\flat & \text{by Lem. 157 and Lem. 52} \\
= & (\ldots ((\rho_{11} \,;\, \rho_{12}) \,/^2\, \mu_1)^{\mathsf{tgt}} \ldots)^\flat & \text{by Prop. 96} \\
= & (\ldots \underline{t} \ldots)^\flat &
\end{array}$$

To justify the $(\star_3)$ step, note that:

$$\begin{array}{rcll}
(\ldots \mu_2 \,/^1\, \rho_{21}^{\mathsf{src}} \ldots)^\flat & = & (\ldots \mu_2/\rho_{21}^{\mathsf{src}} \ldots)^\flat & \text{by definition} \\
& = & (\ldots \mu_2/\mu_2^{\blacktriangleleft} \ldots)^\flat & \text{by Prop. 142, since } \mu_2 \text{ and } \rho_{21} \text{ are coinitial} \\
& = & (\ldots \mu_2^\flat \ldots)^\flat & \text{by Prop. 142} \\
& = & (\ldots \mu_2 \ldots)^\flat & \text{by Prop. 96}
\end{array}$$

To justify that the IH may be applied, note that $(\rho_{11} \,;\, \rho_{12})\,(\rho_{21} \,;\, \rho_{22}) \overset{\flat}{\mapsto} (\rho_{11} \,;\, \rho_{12})\,\rho_{21}^{\mathsf{src}} \,;\, \rho_{12}^{\mathsf{tgt}}\,(\rho_{21}\,\rho_{22})$.

3. Using the fact that $\overset{\flat}{\mapsto}$ is strongly normalizing (Prop. 95), we proceed by induction on the length of the longest reduction $\sigma_1\,\sigma_2 \overset{\flat}{\mapsto}^* (\sigma_1\,\sigma_2)^\flat$, considering four cases, depending on whether each of σ_1 and σ_2 is a multistep or a composition:

3.1 If both are multisteps, *i.e.* $\sigma_1 = \mu_1$ and $\sigma_2 = \mu_2$:

$$\begin{array}{rcll} (\rho_1\,\rho_2)^\flat \,/^3\, (\mu_1\,\mu_2)^\flat & = & (\rho_1\,\rho_2)^\flat \,/^2\, (\mu_1\,\mu_2)^\flat & \text{by definition} \\ & = & ((\rho_1 \,/^2\, \mu_1)\,(\rho_2 \,/^2\, \mu_2))^\flat & \text{by item 2. of this lemma} \\ & = & ((\rho_1 \,/^3\, \mu_1)\,(\rho_2 \,/^3\, \mu_2))^\flat & \text{by definition} \end{array}$$

3.2 If $\sigma_1 = \mu_1$ is a multistep and $\sigma_2 = \sigma_{21} \,;\, \sigma_{22}$ is a sequence:

$$\begin{array}{rll} & (\rho_1\,\rho_2)^\flat \,/^3\, (\mu_1\,(\sigma_{21} \,;\, \sigma_{22}))^\flat & \\ = & (\rho_1\,\rho_2)^\flat \,/^3\, ((\mu_1\,\sigma_{21})^\flat \,;\, (\mu_1^{\mathsf{tgt}}\,\sigma_{22})^\flat) & \text{by Prop. 96} \\ = & ((\rho_1\,\rho_2)^\flat \,/^3\, (\mu_1\,\sigma_{21})^\flat) \,/^3\, (\mu_1^{\mathsf{tgt}}\,\sigma_{22})^\flat & \text{by definition} \\ = & ((\rho_1 \,/^3\, \mu_1)\,(\rho_2 \,/^3\, \sigma_{21}))^\flat \,/^3\, (\mu_1^{\mathsf{tgt}}\,\sigma_{22})^\flat & \text{by IH} \\ = & (((\rho_1 \,/^3\, \mu_1) \,/^3\, \mu_1^{\mathsf{tgt}})\,((\rho_2 \,/^3\, \sigma_{21}) \,/^3\, \sigma_{22}))^\flat & \text{by IH} \\ = & ((\rho_1 \,/^3\, (\mu_1 \,;\, \mu_1^{\mathsf{tgt}}))\,(\rho_2 \,/^3\, (\sigma_{21} \,;\, \sigma_{22})))^\flat & \text{by definition} \\ = & ((\rho_1 \,/^3\, (\mu_1 \,;\, \mu_1^{\blacktriangleright}))\,(\rho_2 \,/^3\, (\sigma_{21} \,;\, \sigma_{22})))^\flat & \text{by Prop. 96} \\ = & ((\rho_1 \,/^3\, \mu_1)\,(\rho_2 \,/^3\, (\sigma_{21} \,;\, \sigma_{22})))^\flat & \text{by Prop. 155, since } \mu_1 \,;\, \mu_1^{\blacktriangleright} \sim \mu_1 \end{array}$$

To justify that the IH may be applied, note that $\mu_1\,(\sigma_{21} \,;\, \sigma_{22}) \overset{\flat}{\mapsto} (\mu_1, \sigma_{21}) \,;\, (\mu_1^{\mathsf{tgt}} \sigma_{22})$.

3.3 If $\sigma_1 = \sigma_{11} \,;\, \sigma_{12}$ is a sequence and $\sigma_2 = \mu_2$ is a multistep, the proof is similar as for the previous case.

3.4 If both are sequences, *i.e.* $\sigma_1 = \sigma_{11} \,;\, \sigma_{12}$ and $\sigma_2 = \sigma_{21} \,;\, \sigma_{22}$:

$$\begin{array}{rll} & (\rho_1\,\rho_2)^\flat \,/^3\, ((\sigma_{11} \,;\, \sigma_{12})\,(\sigma_{21} \,;\, \sigma_{22}))^\flat & \\ = & (\rho_1\,\rho_2)^\flat \,/^3\, (((\sigma_{11} \,;\, \sigma_{12})\,\sigma_{21}^{\mathsf{src}})^\flat \,;\, (\sigma_{12}^{\mathsf{tgt}}\,(\sigma_{21} \,;\, \sigma_{22}))^\flat) & \text{by Prop. 96} \\ = & ((\rho_1\,\rho_2)^\flat \,/^3\, ((\sigma_{11} \,;\, \sigma_{12})\,\sigma_{21}^{\mathsf{src}})^\flat) \,/^3\, (\sigma_{12}^{\mathsf{tgt}}\,(\sigma_{21} \,;\, \sigma_{22}))^\flat & \text{by definition} \\ = & ((\rho_1 \,/^3\, (\sigma_{11} \,;\, \sigma_{12}))\,(\rho_2 \,/^3\, \sigma_{21}^{\mathsf{src}}))^\flat \,/^3\, (\sigma_{12}^{\mathsf{tgt}}\,(\sigma_{21} \,;\, \sigma_{22}))^\flat & \text{by IH} \\ = & (((\rho_1 \,/^3\, (\sigma_{11} \,;\, \sigma_{12})) \,/^3\, \sigma_{12}^{\mathsf{tgt}})\,((\rho_2 \,/^3\, \sigma_{21}^{\mathsf{src}}) \,/^3\, (\sigma_{21} \,;\, \sigma_{22})))^\flat & \text{by IH} \\ = & ((\rho_1 \,/^3\, ((\sigma_{11} \,;\, \sigma_{12}) \,;\, \sigma_{12}^{\mathsf{tgt}}))\,(\rho_2 \,/^3\, (\sigma_{21}^{\mathsf{src}} \,;\, (\sigma_{21} \,;\, \sigma_{22}))))^\flat & \text{by definition} \\ = & ((\rho_1 \,/^3\, ((\sigma_{11} \,;\, \sigma_{12}) \,;\, \sigma_{12}^{\blacktriangleright}))\,(\rho_2 \,/^3\, (\sigma_{21}^{\blacktriangleleft} \,;\, (\sigma_{21} \,;\, \sigma_{22}))))^\flat & \text{by Prop. 96} \\ = & ((\rho_1 \,/^3\, (\sigma_{11} \,;\, \sigma_{12}))\,(\rho_2 \,/^3\, (\sigma_{21} \,;\, \sigma_{22})))^\flat & \text{by Prop. 155} \end{array}$$

To justify that the IH may be applied, note that $(\sigma_{11} \,;\, \sigma_{12})\,(\sigma_{21} \,;\, \sigma_{22}) \overset{\flat}{\mapsto} (\sigma_{11} \,;\, \sigma_{12})\,\sigma_{21}^{\mathsf{src}} \,;\, \sigma_{12}^{\mathsf{tgt}}\,(\sigma_{21}\,\sigma_{22})$. To justify the last equality, note that $(\sigma_{11} \,;\, \sigma_{12}) \,;\, \sigma_{12}^{\blacktriangleright} \sim \sigma_{11} \,;\, \sigma_{12}$ and $\sigma_{21}^{\blacktriangleleft} \,;\, (\sigma_{21} \,;\, \sigma_{22}) \sim \sigma_{21} \,;\, \sigma_{22}$.

◀

▶ **Proposition 161** (Properties of projection for arbitrary rewrites)**.**

1. $(\lambda x.\rho) /\!\!/ (\lambda x.\sigma) \overset{\flat}{\leftarrow}^* \lambda x.(\rho/\sigma)$
2. $(\rho_1\,\rho_2) /\!\!/ (\sigma_1\,\sigma_2) \overset{\flat}{\leftarrow}^* (\rho_1 /\!\!/ \sigma_1)\,(\rho_2 /\!\!/ \sigma_2)$*, if* ρ_1, σ_1 *are coinitial and* ρ_2, σ_2 *are coinitial.*
3. $\rho /\!\!/ (\sigma_1 \,;\, \sigma_2) = (\rho /\!\!/ \sigma_1) /\!\!/ \sigma_2$ *and* $(\rho_1 \,;\, \rho_2) /\!\!/ \sigma = (\rho_1 /\!\!/ \sigma) \,;\, (\rho_2 /\!\!/ (\sigma /\!\!/ \rho_1))$
4. $\rho /\!\!/ \rho \approx \rho^{\mathsf{tgt}}$
5. *If* $\rho \approx \sigma$ *then* $\tau /\!\!/ \rho = \tau /\!\!/ \sigma$ *and* $\rho /\!\!/ \tau \sim \sigma /\!\!/ \tau$*.*
6. $\rho \,;\, (\sigma /\!\!/ \rho) \approx \sigma \,;\, (\rho /\!\!/ \sigma)$

Proof. We prove each item separately:

1. Projection of abstraction:

$$\begin{array}{rcll}(\lambda x.\rho)/\!\!/(\lambda x.\sigma) & = & (\lambda x.\rho)^\flat/(\lambda x.\sigma)^\flat & \text{by definition}\\ & = & (\lambda x.\rho^\flat)^\flat/(\lambda x.\sigma^\flat)^\flat & \text{by Prop. 96}\\ & = & (\lambda x.(\rho^\flat/\sigma^\flat))^\flat & \text{by Lem. 159}\\ & \overset{\flat}{\hookleftarrow}{}^* & \lambda x.(\rho^\flat/\sigma^\flat) & \\ & = & \lambda x.(\rho/\!\!/\sigma) & \text{by definition}\end{array}$$

2. Projection of application:

$$\begin{array}{rcll}(\rho_1\,\rho_2)/\!\!/(\sigma_1\,\sigma_2) & \overset{\text{def}}{=} & (\rho_1\,\rho_2)^\flat/(\sigma_1\,\sigma_2)^\flat & \text{by definition}\\ & \overset{\text{def}}{=} & (\rho_1^\flat\,\rho_2^\flat)^\flat/(\sigma_1^\flat\,\sigma_2^\flat)^\flat & \text{by Prop. 96}\\ & \overset{\text{def}}{=} & ((\rho_1^\flat/\sigma_1^\flat)\,(\rho_2^\flat/\sigma_2^\flat))^\flat & \text{by Lem. 160}\\ & \overset{\flat}{\hookleftarrow}{}^* & (\rho_1^\flat/\sigma_1^\flat)\,(\rho_2^\flat/\sigma_2^\flat) & \\ & = & (\rho_1/\!\!/\sigma_1)\,(\rho_2/\!\!/\sigma_2) & \text{by definition}\end{array}$$

3. Projection of composition:
   - On one hand:

$$\begin{array}{rcll}\rho/\!\!/(\sigma_1\,;\,\sigma_2) & = & \rho^\flat/(\sigma_1^\flat\,;\,\sigma_2^\flat) & \text{by definition}\\ & = & (\rho^\flat/\sigma_1^\flat)/\sigma_2^\flat & \\ & = & (\rho/\!\!/\sigma_1)/\sigma_2^\flat & \text{by definition}\\ & = & (\rho/\!\!/\sigma_1)^\flat/\sigma_2^\flat & \text{as } \rho/\!\!/\sigma_1 \text{ is flat by construction}\\ & = & (\rho/\!\!/\sigma_1)/\!\!/\sigma_2 & \text{by definition}\end{array}$$

   - On the other hand:

$$\begin{array}{rcll}(\rho_1\,;\,\rho_2)/\!\!/\sigma & = & (\rho_1\,;\,\rho_2)^\flat/\sigma^\flat & \text{by definition}\\ & = & (\rho_1^\flat\,;\,\rho_2^\flat)/\sigma^\flat & \\ & = & (\rho_1^\flat/\sigma^\flat)\,;\,(\rho_2^\flat/(\sigma^\flat/\rho_1^\flat)) & \\ & = & (\rho_1/\!\!/\sigma)\,;\,(\rho_2^\flat/(\sigma/\!\!/\rho_1)) & \text{by definition}\\ & = & (\rho_1/\!\!/\sigma)\,;\,(\rho_2^\flat/(\sigma/\!\!/\rho_1)^\flat) & \text{as } \sigma/\!\!/\rho_1 \text{ is flat by construction}\\ & = & (\rho_1/\!\!/\sigma)\,;\,(\rho_2/\!\!/(\sigma/\!\!/\rho_1)) & \text{by definition}\end{array}$$

4. Self-erasure:

$$\begin{array}{rcll}\rho/\!\!/\rho & = & \rho^\flat/\rho^\flat & \text{by definition}\\ & \sim & (\rho^{\mathsf{tgt}})^\flat & \text{by Lem. 156}\\ & \overset{\flat}{\hookleftarrow}{}^* & \rho^{\mathsf{tgt}} & \end{array}$$

   It suffices to recall that flat permutation equivalence ($\sim$) and flattening ($\overset{\flat}{\mapsto}$) are both included in permutation equivalence ($\approx$).
5. Congruence of projection: Let $\rho \approx \sigma$. By Thm. 130 this means that $\rho^\flat \sim \sigma^\flat$. Then:
   - On one hand:

$$\begin{array}{rcll}\tau/\!\!/\rho & = & \tau^\flat/\rho^\flat & \text{by definition}\\ & = & \tau^\flat/\sigma^\flat & \text{by Prop. 155}\\ & = & \tau/\!\!/\sigma & \text{by definition}\end{array}$$

   - On the other hand:

$$\begin{array}{rcll}\rho/\!\!/\tau & = & \rho^\flat/\tau^\flat & \text{by definition}\\ & \sim & \sigma^\flat/\tau^\flat & \text{by Prop. 155}\\ & = & \sigma/\!\!/\tau & \text{by definition}\end{array}$$

6. Permutation:

$$\begin{array}{rcll} \rho\,;(\sigma/\!\!/\rho) & = & \rho\,;(\sigma^\flat/\rho^\flat) & \text{by definition} \\ & \overset{\flat}{\mapsto}^* & \rho^\flat\,;(\sigma^\flat/\rho^\flat) & \\ & \sim & \sigma^\flat\,;(\rho^\flat/\sigma^\flat) & \text{by Lem. 158} \\ & \overset{\flat}{\leftarrow\!\!\dashv}^* & \sigma\,;(\rho^\flat/\sigma^\flat) & \\ & = & \sigma\,;(\rho/\!\!/\sigma) & \text{by definition} \end{array}$$

It suffices to recall that flat permutation equivalence ($\sim$) and flattening ($\overset{\flat}{\mapsto}$) are both included in permutation equivalence ($\approx$).

◀

G.1 Characterization of empty multisteps

▶ **Lemma 162** (Characterization of empty multisteps). *Let μ be a flat multistep. Then the following are equivalent:*

1. *μ is a term,* i.e. $\mu = \underline{s}$.
2. *There exists a term s such that $\mu \sim \underline{s}$.*
3. *There exists a term s such that if $\mu \sim \rho$ then there exists a composition context* $\mathtt{K}$ *such that $\rho = \mathtt{K}\langle \underline{s}, \ldots, \underline{s}\rangle$.*
4. *There exists a term s such that if $\mu \sim \nu$ then $\nu = \underline{s}$.*

Proof.

- (1 $\implies$ 2) Let $\mu = \underline{s}$. Then it is immediate as $\mu \sim \underline{s}$.
- (2 $\implies$ 3) Let $\mu \sim \underline{s}$ and suppose that $\mu \sim \rho$. We claim that there is a composition context $\mathtt{K}$ such that $\rho = \mathtt{K}\langle \underline{s}, \ldots, \underline{s}\rangle$. First, note that we have that $\underline{s} \sim \rho$. This means that there is a sequence of applications of the axioms defining flat permutation equivalence such that $\underline{s} = \rho_0 \sim \rho_1 \sim \rho_2 \ldots \sim \rho_n = \rho$. We proceed by induction on n. If $n = 0$, it is trivial taking $\mathtt{K} := \Box$. For the inductive step, it suffices to show that if ρ_i is of the form $\rho_i = \mathtt{K}\langle \underline{s}, \ldots, \underline{s}\rangle$ then ρ_{i+1} is of the form $\rho_i = \mathtt{K}'\langle \underline{s}, \ldots, \underline{s}\rangle$. The case for the $\sim$-Assoc rule is immediate. The interesting case is the $\sim$-Perm rule. There are two subcases, depending on whether the $\sim$-Perm rule is applied forwards or backwards:
  1. **Forwards application of $\sim$-Perm.** That is, $\rho_i = \mathtt{S}\langle \underline{s}\rangle$ and $\rho_{i+1} = \mathtt{S}\langle \mu_1^\flat\,;\mu_2^\flat\rangle$ where $\mathtt{S}$ is a composition context and $\underline{s} \Leftrightarrow \mu_1\,;\mu_2$. Given that s has no rule symbols, it is easy to check by induction on s that $\mu_1 = \underline{s}$ and $\mu_2 = \underline{s}$. This concludes the proof of this case.
  2. **Backwards application of $\sim$-Perm.** That is, $\rho_i = \mathtt{S}\langle \underline{s}\,;\underline{s}\rangle$ and $\rho_{i+1} = \mathtt{S}\langle \mu_1\rangle$ where $\mu \Leftrightarrow \mu_2\,;\mu_3$ and $\mu_2^\flat = \mu_3^\flat = \underline{s}$. Note that μ_1 is a flat multistep. To finish the proof, it suffices to show that, $\mu_1 = \mu_2 = \mu_3$. This is implied by the following claim.

     Claim. Let $\mu_1 \Leftrightarrow \mu_2\,;\mu_3$ where $\mu_2^\flat = \mu_3^\flat = \underline{s}$ for some term s. Then $\mu_1 = \mu_2 = \mu_3$.

     Proof of the claim. We proceed by induction on μ_1, following the characterization of $\overset{\flat}{\mapsto}$-normal multisteps, There are three subcases, depending on the head of μ_1:

     2.1 **μ_1 headed by a variable.** Then $\mu_1 = \lambda\vec{x}.x\,\nu_{11}\ldots\nu_{1n}$. The judgment $\mu_1 \Leftrightarrow \mu_2\,;\mu_3$ must be derived by a number of applications of the SAbs rule, followed by n applications of the SApp rule, followed by an application of the SVar rule. Hence $\mu_2 = \lambda\vec{x}.x\,\nu_{21}\ldots\nu_{2n}$ and $\mu_3 = \lambda\vec{x}.x\,\nu_{31}\ldots\nu_{3n}$, where moreover $\nu_{1i} \Leftrightarrow \nu_{2i}\,;\nu_{3i}$ for all $1 \leq i \leq n$. Since $\mu_2^\flat = \mu_3^\flat$ then also $\mu_{2i}^\flat = \mu_{3i}^\flat$ for all $1 \leq i \leq n$, and moreover $\mu_{2i}^\flat$ must be a term (*i.e.* without occurrences of rule symbols), for otherwise there would be a rule symbol in $\mu_2^\flat$. Then by IH we have that $\mu_{1i}^\flat = \mu_{2i}^\flat = \mu_{3i}^\flat$ for all $1 \leq i \leq n$. This concludes the proof.

2.2 μ_1 **headed by a constant.** Similar to the previous case, when μ_1 is headed by a variable.

2.3 μ_1 **headed by a rule symbol.** We argue that this case is impossible. Indeed, if μ_1 is of the form $\lambda\vec{x}.\varrho\,\nu_{11}\ldots\nu_{1n}$ then the judgment $\mu_1 \Leftrightarrow \mu_2 \mathbin{;} \mu_3$ must be derived by a number of applications of the SAbs rule, followed by n applications of the SApp rule, followed by an application of either SRuleL or SRuleR at the head. If the judgment is derived by an application of SRuleL at the head, then we have that $\mu_2 = \lambda\vec{x}.\varrho\,\nu_{21}\ldots\nu_{2n}$ and $\mu_3 = \lambda\vec{x}.\varrho^{\mathsf{tgt}}\,\nu_{31}\ldots\nu_{3n}$ and moreover that $\mu_{1i} \Leftrightarrow \mu_{2i} \mathbin{;} \mu_{3i}$ for all $1 \leq i \leq n$. But this is impossible, given that $\mu_2^\flat$ would contain a rule symbol, contradicting the fact that $\mu_2^\flat = \underline{s}$. Using a similar argument, we note that the judgment cannot be derived by an application of the SRuleR rule at the head.

- (3 $\implies$ 4) Suppose that there exists a term s such that if $\mu \sim \rho$ then ρ is of the form $\rho = \mathtt{K}\langle\underline{s},\ldots,\underline{s}\rangle$. Moreover, suppose that $\mu \sim \nu$. Then by hypothesis ν must be of the form $\mathtt{K}\langle\underline{s},\ldots,\underline{s}\rangle$. Since ν is a multistep, containing no compositions, then $\mathtt{K}\langle=\rangle\Box$ and indeed $\nu = \underline{s}$.
- (4 $\implies$ 1) Suppose that there exists a term s such that if $\mu \sim \nu$ then $\nu = \underline{s}$. Then in particular, since $\mu \sim \mu$, we have that $\mu = \underline{s}$.

◀

▶ **Definition 163** (Empty multistep). *A multistep μ is said to be* empty *if $\mu = \underline{s}$ for some term s,* i.e. *if it does not contain rule symbols. Note that, when μ is flat, this condition is equivalent to any/all of the conditions in Lem. 162.*

▶ Remark 164. Even if $\mu^\flat$ is empty, μ is not necessarily empty. For example, $(\lambda x.\mathbf{c})\,\varrho$ is non-empty, but its $\overset{\flat}{\mapsto}$-normal form is $\mathbf{c}$, which is empty.