A Constructive Logic with Classical Proofs and Refutations (Extended Version)

Pablo Barenbaum
Departamento de Computación,
Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires, Argentina.
Universidad Nacional de Quilmes, Argentina.
Email: pbarenbaum@dc.uba.ar

Teodoro Freund
Departamento de Computación,
Facultad de Ciencias Exactas y Naturales,
Universidad de Buenos Aires, Argentina.
Email: tfreund95@gmail.com

***Abstract*—We study a conservative extension of classical propositional logic distinguishing between four modes of statement: a proposition may be affirmed or denied, and it may be strong or classical. Proofs of strong propositions must be constructive in some sense, whereas proofs of classical propositions proceed by contradiction. The system, in natural deduction style, is shown to be sound and complete with respect to a Kripke semantics. We develop the system from the perspective of the propositions-as-types correspondence by deriving a term assignment system with confluent reduction. The proof of strong normalization relies on a translation to System F with Mendler-style recursion.**

1. Introduction

Intuitionistic logic was born out of Brouwer's remark that the *law of excluded middle* ($A \vee \neg A$) allows one to prove propositions in a seemingly non-constructive way. But what constitutes a *constructive* proof, exactly? A possible answer to this question may be found in the *realizability* interpretation, also known as the Brouwer–Heyting–Kolmogorov interpretation, which establishes what kinds of mathematical constructions can be regarded as a *realizer* or *canonical proof* of a proposition. For example, a canonical proof of a conjunction $(A \wedge B)$ is given by a pair $\langle p, q \rangle$, where p and q are in turn canonical proofs of A and B respectively. From the works of Gentzen [1] and Prawitz [2] we know that, in intuitionistic natural deduction, an arbitrary proof of a proposition A can always be *normalized* to a canonical proof of A.

These ideas culminate in the *propositions-as-types* correspondence, the realization that a proposition A may be understood as a *type* that expresses the specification of a program. A proof p of A may be understood as a program fulfilling the specification A. Running the program corresponds to applying a computational procedure that normalizes the proof p to obtain a canonical proof p' of A. Under this paradigm, proofs in intuitionistic natural deduction can be identified with programs in the *simply typed λ-calculus*. This correspondence has been extended to encompass many logical systems, including first-order [3], [4] and second-order intuitionistic logic [5], [6], linear logic [7], classical logic [8], [9], [10], [11], and modal logic [12], [13]. These developments unveil the deep connection between logic and computer science, and they have practical applications in the development of programming languages and proof assistants based in type theory such as Coq and Agda.

In this paper, we define a logical system PRK, presented in natural deduction style, that distinguishes between four "modes" of stating a proposition A, which are written A^{+} (strong affirmation), A^{-} (strong denial), $A^{\oplus}$ (classical affirmation), and $A^{\ominus}$ (classical denial). As the name implies, strong affirmation is stronger than classical affirmation, *i.e.* from A^{+} one may deduce $A^{\oplus}$, and likewise from A^{-} one may deduce $A^{\ominus}$. Affirmation and denial are contradictory, *i.e.* from A^{+} and A^{-} one may derive any conclusion, and similarly for $A^{\oplus}$ and $A^{\ominus}$. This logic turns out to be a *conservative extension* of classical propositional logic, in the sense that a proposition A is classically valid if and only if $A^{\oplus}$ is valid in PRK.

System PRK is then shown to be *sound* and *complete* with respect to a Kripke-style semantics. This helps to elucidate the difference between the four modes of statement. In particular, strong affirmation and denial have a constructive "flavor"—for example, the law of excluded middle holds classically, *i.e.* $(A \vee \neg A)^{\oplus}$ is valid, but it does not hold strongly, *i.e.* $(A \vee \neg A)^{+}$ is not valid.

Furthermore, following the propositions-as-types paradigm, we derive an associated calculus λ^{PRK} and we show that it enjoys the expected meta-theoretical properties: *confluence*, *subject reduction* and *strong normalization*. Besides, we characterize the set of *normal forms*. This sheds a new light on the structure of classical proofs, and it may form the basis of the type systems for future programming languages and proof assistants.

Classical Proofs and Refutations. It is well-known that intuitionistic propositional logic enjoys the *disjunctive property*, that is, a canonical proof of a disjunction $(A \vee B)$ is given by either a canonical proof of A or a canonical proof of B. In particular, the proof *contains one bit of information*, indicating whether it encloses a proof of A or of B. In contrast, the intuitionistic notion of *refutation* (proof of a negation) is not dual to the notion of proof. For example, the set of refutations of a conjunction $(A \wedge B)$ is *not* the disjoint union of the set of refutations of A and the set of refutations of B. This is related to the fact that one of De Morgan's laws, namely $\neg(A \wedge B) \to (\neg A \vee \neg B)$, which is classically valid, does not hold intuitionistically. The reason is that the proof of a negation in intuitionistic logic proceeds by contradiction, *i.e.* the equivalence $\neg A \equiv (A \to \bot)$ holds. As a matter of fact, a refutation in intuitionistic logic *contains no information*[1].

The attempt to recover the symmetry between the notions of proof and refutation in a constructive setting lead Nelson to study logical systems with *strong negation* [15]. One way to formulate Nelson's system is to distinguish between two modes to state a proposition A, that we may call affirmation (A^+) and denial (A^-), whose witnesses are called *proofs* and *refutations* of A respectively[2]. The following (informal) equations suggest a realizability interpretation for affirmations and denials of conjunction, disjunction and negation, found in Nelson's system:

$$\begin{array}{rclrcl} (A \wedge B)^+ & \approx & A^+ \times B^+ & (A \wedge B)^- & \approx & A^- \uplus B^- \\ (A \vee B)^+ & \approx & A^+ \uplus B^+ & (A \vee B)^- & \approx & A^- \times B^- \\ (\neg A)^+ & \approx & A^- & (\neg A)^- & \approx & A^+ \end{array}$$

These equations state, for example, that the set of proofs of a conjunction $(A \wedge B)$ is the cartesian product of the set of proofs of A and the set of proofs of B, while the set of refutations of a conjunction $(A \wedge B)$ is the disjoint union of the set of refutations of A and the set of refutations of B.

This paper was conceived with the goal in mind of providing a **realizability interpretation for classical logic** based on this strong notion of negation. Long-established embeddings of classical logic into intuitionistic logic, such as Gödel's, are based on *double-negation translations*. These translations rely on the equivalence $A \equiv \neg\neg A$, which is classically, but not intuitionistically, valid.

Our starting point is a different equivalence, namely $A \equiv (\neg A \to A)$, which is again classically, but not intuitionistically, valid. To formulate the interpretation, we introduce a further distinction, according to which a proposition A may be qualified as *strong* or *classical*, resulting in four possible modes:

	affirmation	denial
strong	A^+	A^-
classical	$A^\oplus$	$A^\ominus$

As before, the witness of an affirmation (resp. denial) is called a proof (resp. refutation). Our first intuition is that a classical proof of a proposition A should be given by a construction that transforms a *strong* refutation of A into a strong proof of A. Hence, informally speaking, the realizability interpretation should include an equation like $A^\oplus \approx (A^- \to A^+)$, where $(X \to Y)$ is expected to denote the set of "transformations", from X to Y in some suitable sense.

In a preliminary version of this work, we explored a realizability interpretation based on such an equation, and its dual equation, $A^\ominus \approx (A^+ \to A^-)$. But, unfortunately, we were not able to formulate a well-behaved system from the computational point of view[3]. The study of proof normalization suggests that the "right" equations should instead be $A^\oplus \approx (A^\ominus \to A^+)$ and its dual, $A^\ominus \approx (A^\oplus \to A^-)$. This means that a classical proof of a proposition A should be given by a transformation that takes a *classical* refutation of A as an input and produces a strong proof of A as an output. This is indeed the path that we follow.

The complete set of equations that suggest the realizability interpretation that we study in this paper is:

$$\begin{array}{rclrcl} (A \wedge B)^+ & \approx & A^\oplus \times B^\oplus & (A \wedge B)^- & \approx & A^\ominus \uplus B^\ominus \\ (A \vee B)^+ & \approx & A^\oplus \uplus B^\oplus & (A \vee B)^- & \approx & A^\ominus \times B^\ominus \\ (\neg A)^+ & \approx & A^\ominus & (\neg A)^- & \approx & A^\oplus \\ A^\oplus & \approx & A^\ominus \to A^+ & A^\ominus & \approx & A^\oplus \to A^- \end{array}$$

Observe that a strong proof of a conjunction is given by a pair of *classical* (and not strong) proofs. Similarly for the other connectives, *e.g.* a strong refutation of $\neg A$ is given by a *classical* (and not a strong) proof of A. For the sake of brevity, in this paper we will only consider three logical connectives: conjunction, disjunction, and negation. Extending our results and techniques to incorporate other propositional connectives, such as implication, and truth and falsity constants, should not present major obstacles.

One technical difficulty that we confront is the fact that the last two equations are mutually recursive. This means, in particular, that these equations cannot be understood as a translation from formulae of PRK to formulae of other systems (such as the simply typed λ-calculus), at least not in the naive sense. However, as we shall see, these recursive equations do fulfill Mendler's *positivity requirement* [16], which allows us to give a translation from λ^{PRK} to System F extended with (non-strictly positive) recursive type constraints.

Structure of This Paper. The remainder of this paper is organized as follows. In Section 2 (**Natural Deduction**) we present the proof system PRK in natural deduction style, and we study some basic facts, such as weakening and substitution. In Section 3 (**Kripke Semantics**) we define an ad hoc notion of Kripke model, and we show that PRK is sound and complete with respect to this Kripke semantics, *i.e.*, a sequent $\Gamma \vdash A$ is provable in PRK if and only if it holds in every Kripke model. In Section 4 (**The λ^{PRK}-calculus**) we derive a term assignment for PRK, and we endow it with a small-step reduction semantics. We show that the system is confluent and that it enjoys subject reduction. To show that λ^{PRK} is strongly normalizing, we rely on the aforementioned translation to System F extended with recursive type constraints. We also provide an inductive characterization of the

1. As attested for example by the fact that, in *homotopy type theory*, a type of the form $\neg A$ can always be shown to be a mere proposition, *i.e.* if it is inhabited, it is equivalent to the unit type; see for instance [14, Section 3.6].

2. These are called "P-realizers" and "N-realizers" by Nelson.

3. The difficulty is that it is not obvious how to normalize a proof of falsity derived from $A^\oplus$ and $A^\ominus$, *i.e.* a contradiction obtained from combining a classical proof and a classical refutation of a proposition A.

set of normal forms, and we show that an extensionality rule akin to η-reduction may be incorporated to the system. In Section 5 (**Relation with Classical Logic**) we show that PRK is a conservative extension of classical logic. We show how this provides a new computational interpretation for classical logic. In Section 6 (**Conclusion**) we conclude, and we discuss related and future work.

2. The Natural Deduction System PRK

In this section, we define the logical system PRK, formulated in natural deduction style (Def. 1). We then prove that some typical reasoning principles, namely weakening, cut, and substitution, as well as some principles specific to this system, are admissible in PRK (Lem. 2). An important result in this section is the projection lemma (Lem. 4). We also formulate an explicit duality principle (Lem. 5).

We suppose given a denumerable set of *propositional variables* $\alpha, \beta, \gamma, \ldots$. The set of *pure propositions* is given by the abstract syntax:

$$
\begin{array}{rcll}
A, B, C, \ldots & ::= & \alpha & \text{propositional variable} \\
& \mid & A \wedge B & \text{conjunction} \\
& \mid & A \vee B & \text{disjunction} \\
& \mid & \neg A & \text{negation}
\end{array}
$$

The set of *moded propositions* (or just *propositions*) is given by the abstract syntax:

$$
\begin{array}{rcll}
P, Q, R, \ldots ::= & ::= & A^{+} & \text{strong affirmation} \\
& \mid & A^{-} & \text{strong denial} \\
& \mid & A^{\oplus} & \text{classical afirmation} \\
& \mid & A^{\ominus} & \text{classical denial}
\end{array}
$$

As mentioned in the introduction, propositions are classified into four modes, which arise from discriminating two dimensions. The first dimension (called *sign*) distinguishes between *affirmations* (A^{+} and $A^{\oplus}$) and *denials* (A^{-} and $A^{\ominus}$), sometimes also called *positive* and *negative* propositions. The second dimension (called *strength*) distinguishes between *strong propositions* (A^{+} and A^{-}) and *classical propositions* ($A^{\oplus}$ and $A^{\ominus}$). Note that modes cannot be nested, *e.g.* $(A^{+} \wedge B^{+})^{-}$ is not a well-formed proposition.

The *opposite proposition* $P^{\sim}$ of a given proposition P is defined by flipping the sign, but preserving the strength:

$$
\begin{array}{lcllcl}
(A^{+})^{\sim} & \stackrel{\text{def}}{=} & A^{-} & (A^{-})^{\sim} & \stackrel{\text{def}}{=} & A^{+} \\
(A^{\oplus})^{\sim} & \stackrel{\text{def}}{=} & A^{\ominus} & (A^{\ominus})^{\sim} & \stackrel{\text{def}}{=} & A^{\oplus}
\end{array}
$$

The *classical projection* of a given proposition P is written $\bigcirc P$ and defined by preserving the sign and making the strength classical:

$$
\begin{array}{lcllcl}
\bigcirc(A^{+}) & \stackrel{\text{def}}{=} & A^{\oplus} & \bigcirc(A^{-}) & \stackrel{\text{def}}{=} & A^{\ominus} \\
\bigcirc(A^{\oplus}) & \stackrel{\text{def}}{=} & A^{\oplus} & \bigcirc(A^{\ominus}) & \stackrel{\text{def}}{=} & A^{\ominus}
\end{array}
$$

Note that $P^{\sim\sim} = P$, $\bigcirc\bigcirc P = \bigcirc P$, and $\bigcirc(P^{\sim}) = (\bigcirc P)^{\sim}$.

Definition 1 (System PRK)**.** Judgments in PRK are of the form $\Gamma \vdash P$, where Γ is a finite *set* of moded propositions, *i.e.* we work implicitly up to structural rules of contraction and exchange. Derivability of judgments is defined inductively by the following inference schemes.

Except for the first two rules, the system is defined following the realizability interpretation of propositions discussed in the introduction. For instance, rules $\mathrm{I}\wedge^{+}$ and $\mathrm{E}\wedge_i^{+}$ embody the equation for the strong affirmation of a conjunction, $(A \wedge B)^{+} \approx (A^{\oplus} \times B^{\oplus})$.

$$
\frac{}{\Gamma, P \vdash P}\,\textsc{Ax}
\qquad
\frac{\Gamma \vdash P \quad \Gamma \vdash P^{\sim} \quad P \text{ strong}}{\Gamma \vdash Q}\,\textsc{Abs}
$$

$$
\frac{\Gamma \vdash A^{\oplus} \quad \Gamma \vdash B^{\oplus}}{\Gamma \vdash (A \wedge B)^{+}}\,\mathrm{I}\wedge^{+}
\qquad
\frac{\Gamma \vdash A^{\ominus} \quad \Gamma \vdash B^{\ominus}}{\Gamma \vdash (A \vee B)^{-}}\,\mathrm{I}\vee^{-}
$$

$$
\frac{\Gamma \vdash (A_1 \wedge A_2)^{+} \quad i \in \{1,2\}}{\Gamma \vdash A_i{}^{\oplus}}\,\mathrm{E}\wedge_i^{+}
$$

$$
\frac{\Gamma \vdash (A_1 \vee A_2)^{-} \quad i \in \{1,2\}}{\Gamma \vdash A_i{}^{\ominus}}\,\mathrm{E}\vee_i^{-}
$$

$$
\frac{\Gamma \vdash A_i{}^{\oplus} \quad i \in \{1,2\}}{\Gamma \vdash (A_1 \vee A_2)^{+}}\,\mathrm{I}\vee_i^{+}
\qquad
\frac{\Gamma \vdash A_i{}^{\ominus} \quad i \in \{1,2\}}{\Gamma \vdash (A_1 \wedge A_2)^{-}}\,\mathrm{I}\wedge_i^{-}
$$

$$
\frac{\Gamma \vdash (A \vee B)^{+} \quad \Gamma, A^{\oplus} \vdash P \quad \Gamma, B^{\oplus} \vdash P}{\Gamma \vdash P}\,\mathrm{E}\vee^{+}
$$

$$
\frac{\Gamma \vdash (A \wedge B)^{-} \quad \Gamma, A^{\ominus} \vdash P \quad \Gamma, B^{\ominus} \vdash P}{\Gamma \vdash P}\,\mathrm{E}\wedge^{-}
$$

$$
\frac{\Gamma \vdash A^{\ominus}}{\Gamma \vdash (\neg A)^{+}}\,\mathrm{I}\neg^{+}
\qquad
\frac{\Gamma \vdash A^{\oplus}}{\Gamma \vdash (\neg A)^{-}}\,\mathrm{I}\neg^{-}
$$

$$
\frac{\Gamma \vdash (\neg A)^{+}}{\Gamma \vdash A^{\ominus}}\,\mathrm{E}\neg^{+}
\qquad
\frac{\Gamma \vdash (\neg A)^{-}}{\Gamma \vdash A^{\oplus}}\,\mathrm{E}\neg^{-}
$$

$$
\frac{\Gamma, A^{\ominus} \vdash A^{+}}{\Gamma \vdash A^{\oplus}}\,\mathrm{IC}^{+}
\qquad
\frac{\Gamma, A^{\oplus} \vdash A^{-}}{\Gamma \vdash A^{\ominus}}\,\mathrm{IC}^{-}
$$

$$
\frac{\Gamma \vdash A^{\oplus} \quad \Gamma \vdash A^{\ominus}}{\Gamma \vdash A^{+}}\,\mathrm{EC}^{+}
\qquad
\frac{\Gamma \vdash A^{\ominus} \quad \Gamma \vdash A^{\oplus}}{\Gamma \vdash A^{-}}\,\mathrm{EC}^{-}
$$

Rule Ax is the standard axiom rule. Rule Abs is the *absurdity* rule, which allows one to derive any proposition Q from a strong proposition P and its opposite $P^{\sim}$. Rules $\mathrm{I}\wedge^{+}$ and $\mathrm{E}\wedge_i^{+}$ are introduction and elimination rules for the strong affirmation of a conjunction. Rules $\mathrm{I}\vee_i^{+}$ and $\mathrm{E}\vee^{+}$ are introduction and elimination rules for the strong affirmation of a disjunction; note that $\mathrm{E}\vee^{+}$ allows one to conclude *any* proposition. Rules $\mathrm{I}\neg^{+}$ and $\mathrm{E}\neg^{+}$ are introduction and elimination rules for the strong affirmation of a negation. Rules IC^{+} and EC^{+} are introduction and elimination rules for classical affirmation (resembling introduction and elimination rules for implication: IC^{+} resembles the deduction theorem, while EC^{+} resembles *modus ponens*). The negative rules are dual to the positive ones, *i.e.* the rules for an affirmation of a given connective have the same structure as the rules for the denial of the dual connective. Note that conjunction is dual to disjunction and negation and classical proposition are dual to themselves.

In the rest of this paper, we frequently use the following lemma without explicit mention. It establishes a number of basic reasoning principles that are valid in PRK.

Lemma 2. *The following inference schemes are admissible in* PRK*:*

1. **Weakening** (W)*: if* $\Gamma \vdash P$ *then* $\Gamma, Q \vdash P$.
2. **Cut** (CUT)*: if* $\Gamma, P \vdash Q$ *and* $\Gamma \vdash P$ *then* $\Gamma \vdash Q$.
3. **Substitution***: if* $\Gamma \vdash Q$ *then* $\Gamma[\alpha := A] \vdash Q[\alpha := A]$*, where* $-[\alpha := A]$ *denotes the substitution of the propositional variable* α *for the pure proposition* A.
4. **Generalized absurdity** (ABS$'$)*: if* $\Gamma \vdash P$ *and* $\Gamma \vdash P^{\sim}$*, where* P *is not necessarily strong, then* $\Gamma \vdash Q$.
5. **Projection of conclusions** (PC)*: if* $\Gamma \vdash P$ *then* $\Gamma \vdash \bigcirc P$.
6. **Contraposition** (CONTRA)*: if* P *is classical and* $\Gamma, P \vdash Q$ *then* $\Gamma, Q^{\sim} \vdash P^{\sim}$.
7. **Classical strengthening** (CS)*: if* P *is classical and* $\Gamma, P^{\sim} \vdash P$ *then* $\Gamma \vdash P$.

Proof. **Weakening**, **cut**, and **substitution** are routine proofs by induction on the derivation of the first judgment.

For **generalized absurdity**, suppose that $\Gamma \vdash P$ and $\Gamma \vdash P^{\sim}$. If P is strong, applying the ABS rule we may conclude $\Gamma \vdash Q$. If P is classical, there are two cases, depending on whether P is positive or negative. If P is positive, *i.e.* $P = A^{\oplus}$ then:

$$\dfrac{\dfrac{\Gamma \vdash A^{\oplus} \qquad \Gamma \vdash A^{\ominus}}{\Gamma \vdash A^{+}}\,\text{EC}^{+} \qquad \dfrac{\Gamma \vdash A^{\ominus} \qquad \Gamma \vdash A^{\oplus}}{\Gamma \vdash A^{-}}\,\text{EC}^{-}}{\Gamma \vdash Q}\,\text{ABS}$$

If P is negative, *i.e.* $P = A^{\ominus}$, the proof is symmetric.

For **projection of conclusions**, if P is classical, *i.e.* of the form $A^{\oplus}$ or $A^{\ominus}$, we are done. If P is strong, *i.e.* of the form A^{+} or A^{-}, we conclude by applying the IC^{+} or the IC^{-} rule respectively. For example, if $P = A^{+}$:

$$\dfrac{\dfrac{\Gamma \vdash A^{+}}{\Gamma, A^{\ominus} \vdash A^{+}}\,\text{W}}{\Gamma \vdash A^{\oplus}}\,\text{IC}^{+}$$

For **contraposition** we only study the case when P is positive, *i.e.* $P = A^{\oplus}$; the negative case is symmetric. So let $\Gamma, A^{\oplus} \vdash Q$. Then:

$$\dfrac{\dfrac{\dfrac{\Gamma, A^{\oplus} \vdash Q}{\Gamma, Q^{\sim}, A^{\oplus} \vdash Q}\,\text{W} \qquad \dfrac{}{\Gamma, Q^{\sim}, A^{\oplus} \vdash Q^{\sim}}\,\text{Ax}}{\Gamma, Q^{\sim}, A^{\oplus} \vdash A^{-}}\,\text{ABS}'}{\Gamma, Q^{\sim} \vdash A^{\ominus}}\,\text{IC}^{-}$$

For **classical strengthening** we only study the case when P is positive, *i.e.* $P = A^{\oplus}$; the negative case is symmetric. So let $\Gamma, A^{\ominus} \vdash A^{\oplus}$. Then:

$$\dfrac{\dfrac{\Gamma, A^{\ominus} \vdash A^{\oplus} \qquad \dfrac{}{\Gamma, A^{\ominus} \vdash A^{\ominus}}\,\text{Ax}}{\Gamma, A^{\ominus} \vdash A^{+}}\,\text{EC}^{+}}{\Gamma \vdash A^{\oplus}}\,\text{IC}^{+}$$

□

Example 3 (Law of excluded middle)**.** *The law of excluded middle holds classically in* PRK*, that is,* $\vdash (A \vee \neg A)^{\oplus}$*. Indeed, let* $\Gamma = \{(A \vee \neg A)^{\ominus}, (\neg A)^{\ominus}\}$*, and let* π *be the following derivation:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A^{\ominus} \vdash (\neg A)^{\ominus}}\,\text{Ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, A^{\ominus}, (\neg A)^{\ominus} \vdash A^{\ominus}}\,\text{Ax}}{\Gamma, A^{\ominus}, (\neg A)^{\ominus} \vdash (\neg A)^{+}}\,\text{I}\neg^{+}}{\Gamma, A^{\ominus} \vdash (\neg A)^{\oplus}}\,\text{IC}^{+}}{\Gamma, A^{\ominus} \vdash A^{+}}\,\text{ABS}'}{\Gamma \vdash A^{\oplus}}\,\text{IC}^{+}}{\Gamma \vdash (A \vee \neg A)^{+}}\,\text{I}\vee_1^{+}}{\Gamma \vdash (A \vee \neg A)^{\oplus}}\,\text{IC}^{+}$$

Then we have that:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(A \vee \neg A)^{\ominus}, (\neg A)^{\ominus} \vdash (A \vee \neg A)^{\ominus}}\,\text{Ax} \qquad \begin{array}{c}\vdots \\ \pi\end{array}}{(A \vee \neg A)^{\ominus}, (\neg A)^{\ominus} \vdash (A \vee \neg A)^{-}}\,\text{EC}^{-}}{(A \vee \neg A)^{\ominus}, (\neg A)^{\ominus} \vdash A^{\ominus}}\,\text{E}\vee_i^{-}}{(A \vee \neg A)^{\ominus}, (\neg A)^{\ominus} \vdash (\neg A)^{+}}\,\text{I}\neg^{+}}{(A \vee \neg A)^{\ominus} \vdash (\neg A)^{\oplus}}\,\text{IC}^{+}}{(A \vee \neg A)^{\ominus} \vdash (A \vee \neg A)^{+}}\,\text{I}\vee_2^{+}}{\vdash (A \vee \neg A)^{\oplus}}\,\text{IC}^{+}$$

Dually, the law of non-contradiction holds classically in PRK*, that is,* $\vdash (A \wedge \neg A)^{\ominus}$ *holds. Results from the following section will entail that the strong law of excluded middle,* $\vdash (A \vee \neg A)^{+}$*, does not hold in* PRK *(see Ex. 8). The reader may attempt to derive this judgment to convince herself that it does not hold.*

Projection Lemma. The proof of the following lemma is subtle. It will be a key tool in order to prove completeness of PRK with respect to the Kripke semantics:

Lemma 4. *If* $\Gamma, P \vdash Q$ *then* $\Gamma, \bigcirc P \vdash \bigcirc Q$.

Proof. By induction on the derivation of $\Gamma, P \vdash Q$. The difficult cases are conjunction and disjunction elimination. [See Section A.1 in the appendix for the proof.] □

A corollary obtained from iterating the projection lemma is that if $P_1, \ldots, P_n \vdash Q$ then $\bigcirc P_1, \ldots, \bigcirc P_n \vdash \bigcirc Q$.

Duality Principle. The *dual* of a pure proposition A is written $A^{\perp}$ and defined as:

$$\begin{array}{rclcrcl} \alpha^{\perp} & \overset{\text{def}}{=} & \alpha & \qquad & (A \wedge B)^{\perp} & \overset{\text{def}}{=} & A^{\perp} \vee B^{\perp} \\ (A \vee B)^{\perp} & \overset{\text{def}}{=} & A^{\perp} \wedge B^{\perp} & & (\neg A)^{\perp} & \overset{\text{def}}{=} & \neg(A^{\perp}) \end{array}$$

The dual of a proposition P is written $P^\perp$ and defined as:

$$\begin{array}{llll} (A^+)^\perp \stackrel{\text{def}}{=} (A^\perp)^- & & (A^-)^\perp \stackrel{\text{def}}{=} (A^\perp)^+ \\ (A^\oplus)^\perp \stackrel{\text{def}}{=} (A^\perp)^\ominus & & (A^\ominus)^\perp \stackrel{\text{def}}{=} (A^\perp)^\oplus \end{array}$$

The following duality principle is then straightforward to prove by induction on the derivation of the judgment:

Lemma 5. *If $P_1, \ldots, P_n \vdash Q$ then $P_1^\perp, \ldots, P_n^\perp \vdash Q^\perp$.*

3. Kripke Semantics for PRK

In this section, we define a Kripke semantics (Def. 6, Def. 7), for which system PRK turns out to be sound (Prop. 11) and complete (Thm. 17). Recall that a Kripke model $\mathcal{M}$ in intuitionistic logic[4] is given by a set $\mathcal{W}$ of elements called *worlds*, a partial order $\leq$ on $\mathcal{W}$ called the *accessibility relation*, and for each world $w \in \mathcal{W}$ a set $\mathcal{V}_w$ of propositional variables verifying a *monotonicity* property, namely, $w \leq w'$ implies $\mathcal{V}_w \subseteq \mathcal{V}_{w'}$. A relation of forcing $\mathcal{M}, w \Vdash A$ is defined for each proposition A by structural recursion on A. In the base case, $\mathcal{M}, w \Vdash \alpha$ is declared to hold for a propositional variable α whenever $\alpha \in \mathcal{V}_w$.

This standard notion of Kripke model is adapted for PRK by replacing the set $\mathcal{V}_w$ with two sets $\mathcal{V}_w^+$ and $\mathcal{V}_w^-$ and by imposing an additional condition we call *stabilization*, stating that a propositional variable must eventually belong to the union $\mathcal{V}_w^+ \cup \mathcal{V}_w^-$, but never to the intersection $\mathcal{V}_w^+ \cap \mathcal{V}_w^-$. The relation of forcing $\mathcal{M}, w \Vdash P$ is then defined in such a way that $\mathcal{M}, w \Vdash \alpha^+$ is declared to hold if $\alpha \in \mathcal{V}_w^+$. Similarly, α^- is declared to hold if $\alpha \in \mathcal{V}_w^-$. One difficulty that we find is how to define the forcing relation for a classical proposition like $A^\oplus$. The forcing relation for $A^\oplus$ should behave, informally speaking, like an intuitionistic implication "$A^\ominus \to A^+$". However this does not provide a *bona fide* definition, because the interpretation of $A^\oplus$ depends on $A^\ominus$, and the interpretation of $A^\ominus$ depends in turn on $A^\oplus$. What we do is define the interpretations of $A^\oplus$ and $A^\ominus$ without referring to each other. A key lemma (Lem. 10) then ensures that $A^\oplus$ is given the same semantics as an intuitionistic implication of the form "$A^\ominus \to A^+$".

Definition 6. A *Kripke model* (for PRK) is a structure $\mathcal{M} = (\mathcal{W}, \leq, \mathcal{V}^+, \mathcal{V}^-)$ where $\mathcal{W} = \{w, w', \ldots\}$ is a set of worlds, $\leq$ is a partial order on $\mathcal{W}$, and for each world $w \in \mathcal{W}$ there are sets $\mathcal{V}_w^+$ and $\mathcal{V}_w^-$ of propositional variables, such that the following conditions hold:

1. **Monotonicity.** If $w \leq w'$ then $\mathcal{V}_w^+ \subseteq \mathcal{V}_{w'}^+$ and $\mathcal{V}_w^- \subseteq \mathcal{V}_{w'}^-$.
2. **Stabilization.** For all $w \in \mathcal{W}$ and all α, there exists $w' \geq w$ such that $\alpha \in \mathcal{V}_{w'}^+ \triangle \mathcal{V}_{w'}^-$.

Note that we write $w' \geq w$ for $w \leq w'$, and $\triangle$ denotes the symmetric difference on sets, that is, $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.

4. See for instance [17, Section 5.3].

The definition of the forcing relation is given by induction on the following notion of *measure* $\#(-)$ of a proposition P:

$$\begin{array}{lll} \#(A^+) \stackrel{\text{def}}{=} 2|A| & & \#(A^-) \stackrel{\text{def}}{=} 2|A| \\ \#(A^\oplus) \stackrel{\text{def}}{=} 2|A|+1 & & \#(A^\ominus) \stackrel{\text{def}}{=} 2|A|+1 \end{array}$$

where $|A|$ denotes the *size*, *i.e.* the number of symbols, in the formula A. Note in particular that $\#(A^\oplus) = \#(A^\ominus) > \#(A^+) = \#(A^-)$, that $\#((A \star B)^+) = \#((A \star B)^-) > \#(A^\oplus) = \#(A^\ominus)$ for $\star \in \{\wedge, \vee\}$, and that $\#((\neg A)^+) = \#((\neg A)^-) > \#(A^\oplus) = \#(A^\ominus)$.

Definition 7 (Forcing)**.** Given a Kripke model, we define the *forcing* relation, written $\mathcal{M}, w \Vdash P$ for each world $w \in \mathcal{W}$ and each proposition P, as follows, by induction on the *measure* $\#(P)$:

$$\begin{array}{lcl} \mathcal{M}, w \Vdash \alpha^+ & \iff & \alpha \in \mathcal{V}_w^+ \\ \mathcal{M}, w \Vdash \alpha^- & \iff & \alpha \in \mathcal{V}_w^- \\ \mathcal{M}, w \Vdash (A \wedge B)^+ & \iff & \mathcal{M}, w \Vdash A^\oplus \text{ and } \mathcal{M}, w \Vdash B^\oplus \\ \mathcal{M}, w \Vdash (A \wedge B)^- & \iff & \mathcal{M}, w \Vdash A^\ominus \text{ or } \mathcal{M}, w \Vdash B^\ominus \\ \mathcal{M}, w \Vdash (A \vee B)^+ & \iff & \mathcal{M}, w \Vdash A^\oplus \text{ or } \mathcal{M}, w \Vdash B^\oplus \\ \mathcal{M}, w \Vdash (A \vee B)^- & \iff & \mathcal{M}, w \Vdash A^\ominus \text{ and } \mathcal{M}, w \Vdash B^\ominus \\ \mathcal{M}, w \Vdash (\neg A)^+ & \iff & \mathcal{M}, w \Vdash A^\ominus \\ \mathcal{M}, w \Vdash (\neg A)^- & \iff & \mathcal{M}, w \Vdash A^\oplus \\ \mathcal{M}, w \Vdash A^\oplus & \iff & \mathcal{M}, w' \nVdash A^- \text{ for all } w' \geq w \\ \mathcal{M}, w \Vdash A^\ominus & \iff & \mathcal{M}, w' \nVdash A^+ \text{ for all } w' \geq w \end{array}$$

Furthermore, if Γ is a (possibly infinite) set of propositions, we write:

$$\begin{array}{lcl} \mathcal{M}, w \Vvdash \Gamma & \iff & \mathcal{M}, w \Vdash P \text{ for every } P \in \Gamma \\ \mathcal{M}, \Gamma \Vvdash P & \iff & \mathcal{M}, w \Vvdash \Gamma \text{ implies } \mathcal{M}, w \Vdash P \text{ for every } w \\ \Gamma \Vvdash P & \iff & \mathcal{M}, \Gamma \Vvdash P \text{ for every Kripke model } \mathcal{M} \end{array}$$

Note that most cases in the definition of forcing do not mention the accessibility relation, other than for classical propositions.

Example 8 (Counter-model for the strong excluded middle)**.** *There is a Kripke model $\mathcal{M}$ with a world w_0 such that $\mathcal{M}, w_0 \nVdash (\alpha \vee \neg\alpha)^+$. Indeed, let $\mathcal{P}$ be the set of all propositional variables, and let $\mathcal{M}$ be the Kripke model such that $\mathcal{W} = \{w_0, w_1, w_2\}$ with $w_0 \leq w_1$ and $w_0 \leq w_2$, where $\mathcal{V}^+$ and $\mathcal{V}^-$ are defined as follows:*

	$\mathcal{V}^+$	$\mathcal{V}^-$
w_0	$\varnothing$	$\varnothing$
w_1	$\mathcal{P}$	$\varnothing$
w_2	$\varnothing$	$\mathcal{P}$

It is easy to verify that $\mathcal{M}$ is a Kripke model and that $\mathcal{M}, w_0 \nVdash (\alpha \vee \neg\alpha)^+$. Note, on the other hand, that the classical excluded middle holds, i.e. $\mathcal{M}, w_0 \Vdash (\alpha \vee \neg\alpha)^\oplus$.

Before going on, we introduce typical nomenclature. If Γ is a possibly infinite set of propositions, we say that $\Gamma \vdash Q$ holds whenever the judgment $\Delta \vdash Q$ is derivable in PRK for some finite subset $\Delta \subseteq \Gamma$. A set Γ of propositions is *consistent* if there is a proposition P such that $\Gamma \nvdash P$. Otherwise, Γ is *inconsistent*.

In the remainder of this section we shall prove that PRK is sound and complete with respect to this notion of Kripke model. *i.e.* that $\Gamma \vdash P$ holds if and only if $\Gamma \Vdash P$ holds. We begin by establishing some basic properties of the forcing relation. [See Section A.2 in the appendix for the proofs.]

Lemma 9 (Properties of Forcing)**.**

1. **Monotonicity.** *If $\mathcal{M}, w \Vdash P$ and $w \leq w'$ then $\mathcal{M}, w' \Vdash P$.*
2. **Stabilization.** *For every world w and every proposition P, there is a world $w' \geq w$ such that either $\mathcal{M}, w' \Vdash P$ or $\mathcal{M}, w' \Vdash P^{\sim}$ hold, but not both.*
3. **Non-contradiction.** *If $\mathcal{M}, w \Vdash P$ then $\mathcal{M}, w \nVdash P^{\sim}$.*

To prove **soundness**, we first need an auxiliary lemma that gives necessary and sufficient conditions for a classical proposition to hold. [See Section A.3 in the appendix for the full proof of soundness.]

Lemma 10 (Rule of Classical Forcing)**.**

1. $(\mathcal{M}, w \Vdash A^{\oplus})$ *if and only if* $(\forall w' \geq w)((\mathcal{M}, w' \Vdash A^{\ominus}) \implies (\mathcal{M}, w' \Vdash A^{+}))$.
2. $(\mathcal{M}, w \Vdash A^{\ominus})$ *if and only if* $(\forall w' \geq w)((\mathcal{M}, w' \Vdash A^{\oplus}) \implies (\mathcal{M}, w' \Vdash A^{-}))$.

With these tools at our disposal, it is immediate to prove soundness:

Proposition 11 (Soundness)**.** *If $\Gamma \vdash P$ then $\Gamma \Vdash P$.*

Proof. By induction on the derivation of $\Gamma \vdash P$. The interesting cases are the IC^{+} and IC^{-} rules, which follow from the rule of classical forcing (Lem. 10), and the ABS, EC^{+}, and EC^{-} rules, which follow from the property of non-contradiction (Lem. 9). □

To prove **completeness**, the methodology that we follow is the standard one, which proceeds by contraposition assuming that $\Gamma \nvdash P$ and building a counter-model. The counter-model is given by a Kripke model $\mathcal{M}_0$ and a world w such that $\mathcal{M}_0, w \Vdash \Gamma$ but $\mathcal{M}_0, w \nVdash P$. In fact, the choice of the Kripke model $\mathcal{M}_0$ does not depend on Γ nor P. Rather, $\mathcal{M}_0$ is always chosen to be the *canonical* Kripke model whose worlds are "saturated" sets of propositions (*prime theories*, sometimes called *disjunctive theories*). Completeness is obtained by taking Γ and *saturating* it it to a prime theory Γ' which then verifies $\mathcal{M}_0, \Gamma' \Vdash \Gamma$ but $\mathcal{M}_0, \Gamma' \nVdash P$. [In the remainder of this section, the proofs of the technical lemmas are only sketched; see Section A.4 in the appendix for the full proofs.]

Definition 12 (Prime theory)**.** A *prime theory* is a set of propositions Γ such that the following hold:

1. **Closure by deduction.** If $\Gamma \vdash P$ then $P \in \Gamma$.
2. **Consistency.** Γ is consistent.
3. **Disjunctive property.**
   - If $(A \vee B)^{+} \in \Gamma$ then either $A^{\oplus} \in \Gamma$ or $B^{\oplus} \in \Gamma$.
   - If $(A \wedge B)^{-} \in \Gamma$ then either $A^{\ominus} \in \Gamma$ or $B^{\ominus} \in \Gamma$.

Lemma 13 (Saturation)**.** *Let Γ be a consistent set of propositions, and let Q be a proposition such that $\Gamma \nvdash Q$. Then there exists a prime theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash Q$.*

Proof. [Lem. 48 in the appendix.] □

Definition 14 (Canonical model)**.** The *canonical model* is the structure $\mathcal{M}_0 = (\mathcal{W}_0, \subseteq, \mathcal{V}^{+}, \mathcal{V}^{-})$, where:

1. $\mathcal{W}_0$ is the set of all prime theories.
2. $\subseteq$ denotes the set-theoretic inclusion between prime theories.
3. $\mathcal{V}^{+}_{\Gamma} = \{\alpha \mid \alpha^{+} \in \Gamma\}$ and $\mathcal{V}^{-}_{\Gamma} = \{\alpha \mid \alpha^{-} \in \Gamma\}$.

Lemma 15. *The canonical model is a Kripke model.*

Proof. [Lem. 50 in the appendix.] The difficult part is proving the stabilization property, which relies on the fact that if Γ is a consistent set and P is a proposition, then $\Gamma \cup \{P\}$ and $\Gamma \cup \{P^{\sim}\}$ are not both inconsistent. □

Lemma 16 (Main Semantic Lemma)**.** *Let Γ be a prime theory. Then $\mathcal{M}_0, \Gamma \Vdash P$ holds in the canonical model if and only if $P \in \Gamma$.*

Proof. [Lem. 51 in the appendix.] By induction on the measure $\#(P)$. The difficult case is when P is a classical proposition, which requires resorting to the Saturation lemma (Lem. 13). □

Theorem 17 (Completeness)**.** *If $\Gamma \Vdash P$ then $\Gamma \vdash P$.*

Proof. The proof is by contraposition, *i.e.* let $\Gamma \nvdash P$ and let us show that there is a Kripke model $\mathcal{M}$ and a world w such that $\mathcal{M}, w \Vdash \Gamma$ but $\mathcal{M}, w \nVdash P$. Note that Γ is consistent, so by Saturation (Lem. 13) there exists a prime theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash P$. Note that $\mathcal{M}_0, \Gamma' \Vdash \Gamma$ because, by the Main Semantic Lemma (Lem. 16), we have that $\mathcal{M}_0, \Gamma' \Vdash Q$ for every $Q \in \Gamma \subseteq \Gamma'$. Moreover, also by the Main Semantic Lemma (Lem. 16), we have that $\mathcal{M}_0, \Gamma' \nVdash P$ because $P \notin \Gamma'$. □

4. Propositions as types: the λ^{PRK}-Calculus

In this section, we formulate a typed λ-calculus, called λ^{PRK}, by deriving a system of term assignment for derivations in PRK, and furnishing it with reduction rules. Besides the basic results of confluence (Prop. 25) and subject reduction (Prop. 24) the central result in this section is a translation (Def. 28, Def. 29) from PRK to System F extended with recursive type constraints, following Mendler [16]. The translation maps each type P of PRK to a type $[\![P]\!]$, and each term t of type P to a term $[\![t]\!]$ of type $[\![P]\!]$. Recursion is needed in order to be able to translate classical propositions, which are characterized by the recursive equations discussed in the introduction, $A^{\oplus} \approx (A^{\ominus} \to A^{+})$ and its dual $A^{\ominus} \approx (A^{\oplus} \to A^{-})$. Moreover the translation is such that each reduction step $t \to s$ in λ^{PRK} is simulated in one or more steps $[\![t]\!] \to^{+} [\![s]\!]$ in the extended System F. This translation provides a *syntactical model* for λ^{PRK} in the sense of [18], and one of its consequences is that λ^{PRK} is strongly normalizing (Thm. 32). This allows us to prove *canonicity* (Thm. 35), for which we study an inductive characterization of the set of normal forms. Finally, we consider an extension $\lambda^{\text{PRK}}_{\eta}$ of the system that incorporates an extensionality rule for classical proofs (Def. 36, Thm. 37).

Propositions $P, Q, \dots$ are sometimes also called *types* in this section. We assume given a denumerable set of variables $x, y, z, \dots$. The set of *typing contexts* is defined by the grammar $\Gamma ::= \varnothing \mid \Gamma, x : P$, where each variable is assumed to occur at most once in a typing context. Typing contexts are considered implicitly up to reordering[5]

The set of terms is given by the following abstract syntax. The letter i ranges over the set $\{1, 2\}$. Some terms are decorated with a plus or a minus sign. In the grammar we write "$^{\pm}$" to stand for either "$^{+}$" or "$^{-}$".

$t, s, u, \dots$	$::=$	x	variable
	$\mid$	$t \blacktriangleright\!\!\blacktriangleleft_P s$	absurdity
	$\mid$	$\langle t, s\rangle^{\pm}$	$\wedge^+$ / $\vee^-$ introduction
	$\mid$	$\pi^{\pm}_i(t)$	$\wedge^+$ / $\vee^-$ elimination
	$\mid$	$\mathsf{in}^{\pm}_i(t)$	$\vee^+$ / $\wedge^-$ introduction
	$\mid$	$\delta^{\pm} t\,[_{x:P}.s][_{y:Q}.u]$	$\vee^+$ / $\wedge^-$ elimination
	$\mid$	$\nu^{\pm} t$	$\neg^+$ / $\neg^-$ introduction
	$\mid$	$\mu^{\pm} t$	$\neg^+$ / $\neg^-$ elimination
	$\mid$	$\mathsf{IC}^{\pm}_{(x:P)}.\,t$	classical introduction
	$\mid$	$t \bullet^{\pm} s$	classical elimination

The notions of free and bound occurrences of variables are defined as expected considering that $\delta^{\pm} t\,[_{x:P}.s][_{y:Q}.u]$ binds occurrences of x in s and occurrences of y in u, whereas $\mathsf{IC}^{\pm}_{x:P}.\,t$ binds occurrences of x in t. We implicitly work modulo α-renaming of bound variables. We write $\mathsf{fv}(t)$ for the set of free variables of t, and $t[x := s]$ for the capture-avoiding substitution of x by s in t. Sometimes we omit type decorations if they are irrelevant or clear from the context, for example, we may write $\mathsf{IC}^{+}_{x}.\,t$ rather than $\mathsf{IC}^{+}_{(x:A^{\ominus})}.\,t$, and $t \blacktriangleright\!\!\blacktriangleleft s$ rather than $t \blacktriangleright\!\!\blacktriangleleft_P s$. Sometimes we also omit the name of unused bound variables, writing "_" instead; *e.g.* if $x \notin \mathsf{fv}(t)$ we may write $\mathsf{IC}^{+}_{\_}.\,t$ rather than $\mathsf{IC}^{+}_{(x:A^{\ominus})}.\,t$.

Definition 18 (The λ^{PRK} type system)**.** Typing judgments are of the form $\Gamma \vdash t : P$. Derivability of judgments is defined inductively by the following typing rules:

$$\frac{}{\Gamma, x : P \vdash x : P}\,\textsc{Ax} \qquad \frac{\Gamma \vdash t : P \quad \Gamma \vdash s : P^{\sim} \quad P \text{ strong}}{\Gamma \vdash t \blacktriangleright\!\!\blacktriangleleft_Q s : Q}\,\textsc{Abs}$$

$$\frac{\Gamma \vdash t : A^{\oplus} \quad \Gamma \vdash s : B^{\oplus}}{\Gamma \vdash \langle t, s\rangle^{+} : (A \wedge B)^{+}}\,\mathrm{I}\wedge^{+} \qquad \frac{\Gamma \vdash t : A^{\ominus} \quad \Gamma \vdash s : B^{\ominus}}{\Gamma \vdash \langle t, s\rangle^{-} : (A \vee B)^{-}}\,\mathrm{I}\vee^{-}$$

$$\frac{\Gamma \vdash t : (A_1 \wedge A_2)^{+} \quad i \in \{1,2\}}{\Gamma \vdash \pi^{+}_i(t) : {A_i}^{\oplus}}\,\mathrm{E}\wedge^{+}_i$$

$$\frac{\Gamma \vdash t : (A_1 \vee A_2)^{-} \quad i \in \{1,2\}}{\Gamma \vdash \pi^{-}_i(t) : {A_i}^{\ominus}}\,\mathrm{E}\vee^{-}_i$$

$$\frac{\Gamma \vdash t : {A_i}^{\oplus} \quad i \in \{1,2\}}{\Gamma \vdash \mathsf{in}^{+}_i(t) : (A_1 \vee A_2)^{+}}\,\mathrm{I}\vee^{+}_i \qquad \frac{\Gamma \vdash t : {A_i}^{\ominus} \quad i \in \{1,2\}}{\Gamma \vdash \mathsf{in}^{-}_i(t) : (A_1 \wedge A_2)^{-}}\,\mathrm{I}\wedge^{-}_i$$

$$\frac{\Gamma \vdash t : (A \vee B)^{+} \quad \Gamma, x : A^{\oplus} \vdash s : P \quad \Gamma, y : B^{\oplus} \vdash u : P}{\Gamma \vdash \delta^{+} t\,[_{x:A^{\oplus}}.s][_{y:B^{\oplus}}.u] : P}\,\mathrm{E}\vee^{+}$$

$$\frac{\Gamma \vdash t : (A \wedge B)^{-} \quad \Gamma, x : A^{\ominus} \vdash s : P \quad \Gamma, y : B^{\ominus} \vdash u : P}{\Gamma \vdash \delta^{-} t\,[_{x:A^{\ominus}}.s][_{y:B^{\ominus}}.u] : P}\,\mathrm{E}\wedge^{-}$$

$$\frac{\Gamma \vdash t : A^{\ominus}}{\Gamma \vdash \nu^{+} t : (\neg A)^{+}}\,\mathrm{I}\neg^{+} \qquad \frac{\Gamma \vdash t : A^{\oplus}}{\Gamma \vdash \nu^{-} t : (\neg A)^{-}}\,\mathrm{I}\neg^{-}$$

$$\frac{\Gamma \vdash t : (\neg A)^{+}}{\Gamma \vdash \mu^{+} t : A^{\ominus}}\,\mathrm{E}\neg^{+} \qquad \frac{\Gamma \vdash t : (\neg A)^{-}}{\Gamma \vdash \mu^{-} t : A^{\oplus}}\,\mathrm{E}\neg^{-}$$

$$\frac{\Gamma, x : A^{\ominus} \vdash t : A^{+}}{\Gamma \vdash \mathsf{IC}^{+}_{(x:A^{\ominus})}.\,t : A^{\oplus}}\,\mathrm{IC}^{+} \qquad \frac{\Gamma, x : A^{\oplus} \vdash t : A^{-}}{\Gamma \vdash \mathsf{IC}^{+}_{(x:A^{\oplus})}.\,t : A^{\ominus}}\,\mathrm{IC}^{-}$$

$$\frac{\Gamma \vdash t : A^{\oplus} \quad \Gamma \vdash s : A^{\ominus}}{\Gamma \vdash t \bullet^{+} s : A^{+}}\,\mathrm{EC}^{+} \qquad \frac{\Gamma \vdash t : A^{\ominus} \quad \Gamma \vdash s : A^{\oplus}}{\Gamma \vdash t \bullet^{-} s : A^{-}}\,\mathrm{EC}^{-}$$

Remark 19. Each typing rule in λ^{PRK} (Def. 18) corresponds exactly to the rule of the same name in PRK (Def. 1). It is immediate to show that $P_1, \dots, P_n \vdash Q$ is derivable in PRK if and only if $x_1 : P_1, \dots, x_n : P_n \vdash t : Q$ is derivable in λ^{PRK} for some term t.

We begin by studying properties of λ^{PRK} from the *logical* point of view, as a type system. In particular, the following lemma adapts some of the results in Lem. 2 and Ex. 3 to λ^{PRK}, providing explicit proof terms for derivations.

Lemma 20. *The following rules are admissible in λ^{PRK}:*

1. **Weakening** *(W): If $\Gamma \vdash t : P$ and $x \notin \mathsf{fv}(t)$ then $\Gamma, x : Q \vdash t : P$.*
2. **Cut** *(Cut): if $\Gamma, x : P \vdash t : Q$ and $\Gamma \vdash s : P$ then $\Gamma \vdash t[x := s] : Q$.*
3. **Generalized absurdity** *(Abs′): if $\Gamma \vdash t : P$ and $\Gamma \vdash s : P^{\sim}$, where P is not necessarily strong, there is a term $t \bowtie_Q s$ such that $\Gamma \vdash t \bowtie_Q s : Q$.*
4. **Contraposition** *(Contra): if P is classical and $\Gamma, x : P \vdash t : Q$, there is a term $\updownarrow^{y}_{x}(t)$ such that $\Gamma, y : Q^{\sim} \vdash \updownarrow^{y}_{x}(t) : P^{\sim}$.*
5. **Excluded middle***: there is a term $\pitchfork^{+}_{A}$ such that $\Gamma \vdash \pitchfork^{+}_{A} : (A \vee \neg A)^{\oplus}$.*
6. **Non-contradiction***: there is a term $\pitchfork^{-}_{A}$ such that $\Gamma \vdash \pitchfork^{-}_{A} : (A \wedge \neg A)^{\ominus}$.*

Proof. **Weakening** and **cut** are routine by induction on the derivation of the first premise of the rule. For **generalized absurdity**, it suffices to take:

$$t \bowtie_Q s \stackrel{\text{def}}{=} \begin{cases} t \blacktriangleright\!\!\blacktriangleleft_Q s & \text{if } P \text{ is strong} \\ (t \bullet^{+} s) \blacktriangleright\!\!\blacktriangleleft_Q (s \bullet^{-} t) & \text{if } P = A^{\oplus} \\ (t \bullet^{-} s) \blacktriangleright\!\!\blacktriangleleft_Q (s \bullet^{+} t) & \text{if } P = A^{\ominus} \end{cases}$$

5. Remark that the type system λ^{PRK} is a *refinement* of the logical system PRK because contexts are *multisets*, rather than *sets*, of assumptions: there is no structural rule of contraction in λ^{PRK}. This means, for example, that in λ^{PRK} there are two different derivations of the sequent $P, P \vdash P$ using the Ax rule, depending on which one of the two assumptions is used, whereas in PRK there is only one such proof. This is a typical situation in a propositions-as-types setting.

For **contraposition**, it suffices to take:

$$\updownarrow_x^y(t) \stackrel{\text{def}}{=} \begin{cases} \mathsf{IC}^-_{(x:A^\oplus)}.\,(t \bowtie_{A^-} y) & \text{if } P = A^\oplus \\ \mathsf{IC}^-_{(x:A^\ominus)}.\,(t \bowtie_{A^+} y) & \text{if } P = A^\ominus \end{cases}$$

For **excluded middle**, it suffices to take:

$$\pitchfork^+_A \stackrel{\text{def}}{=} \mathsf{IC}^+_{(x:(A\vee\neg A)^\ominus)}.\,\mathsf{in}^+_2(\mathsf{IC}^+_{(y:\neg A^\ominus)}.\,\nu^+\pi^-_1(x \bullet^- \Delta^+_{y,A}))$$
$$\Delta^+_{y,A} \stackrel{\text{def}}{=} \mathsf{IC}^+_{(\_:(A\vee\neg A)^\ominus)}.\,\mathsf{in}^+_1(\mathsf{IC}^+_{(z:A^\ominus)}.\,(y \bowtie_{A^+} \mathsf{IC}^+_{(\_:\neg A^\ominus)}.\,\nu^+ z))$$

Dually, for **non-contradiction**:

$$\pitchfork^-_A \stackrel{\text{def}}{=} \mathsf{IC}^-_{(x:(A\wedge\neg A)^\oplus)}.\,\mathsf{in}^-_2(\mathsf{IC}^-_{(y:\neg A^\oplus)}.\,\nu^-\pi^+_1(x \bullet^+ \Delta^-_{y,A}))$$
$$\Delta^-_{y,A} \stackrel{\text{def}}{=} \mathsf{IC}^-_{(\_:(A\wedge\neg A)^\oplus)}.\,\mathsf{in}^-_1(\mathsf{IC}^-_{(z:A^\oplus)}.\,(y \bowtie_{A^-} \mathsf{IC}^-_{(\_:\neg A^\oplus)}.\,\nu^- z))$$

□

We now turn to studying the *computational* properties of λ^{PRK}, provided with the following notion of reduction:

Definition 21 (The λ^{PRK}-calculus)**.** Typable terms of λ^{PRK} are endowed with the following rewriting rules, closed under arbitrary contexts.

$$\begin{array}{rcll}
\pi^\pm_i(\langle t_1, t_2\rangle^\pm) & \xrightarrow{\mathsf{proj}} & t_i & \text{if } i \in \{1,2\} \\
\delta^\pm(\mathsf{in}^\pm_i(t))\,[_x.s_1][_x.s_2] & \xrightarrow{\mathsf{case}} & s_i[x:=t] & \text{if } i \in \{1,2\} \\
\mu^\pm(\nu^\pm t) & \xrightarrow{\mathsf{neg}} & t & \\
(\mathsf{IC}^\pm_x.\,t) \bullet^\pm s & \xrightarrow{\mathsf{beta}} & t[x:=s] & \\
\langle t_1, t_2\rangle^\pm \bowtie \mathsf{in}^\mp_i(s) & \xrightarrow{\mathsf{absPairInj}} & t_i \bowtie s & \text{if } i \in \{1,2\} \\
\mathsf{in}^\pm_i(t) \bowtie \langle s_1, s_2\rangle^\mp & \xrightarrow{\mathsf{absInjPair}} & t \bowtie s_i & \text{if } i \in \{1,2\} \\
(\nu^\pm t) \bowtie (\nu^\mp s) & \xrightarrow{\mathsf{absNeg}} & t \bowtie s &
\end{array}$$

If many occurrences of "$\pm$" appear in the same expression, they are all supposed to stand for the same sign (either $+$ or $-$), and $\mp$ is supposed to stand for the opposite sign.

Example 22. *If $x : A^\ominus \vdash t : A^+$ and $y : A^\oplus \vdash s : A^-$ then:*

$$\begin{array}{rl}
& (\nu^-(\mathsf{IC}^+_x.\,t)) \bowtie (\nu^+(\mathsf{IC}^-_y.\,s)) \\
\longrightarrow & (\mathsf{IC}^+_x.\,t) \bowtie (\mathsf{IC}^-_y.\,s) \\
= & ((\mathsf{IC}^+_x.\,t) \bullet^+ (\mathsf{IC}^-_y.\,s)) \bowtie ((\mathsf{IC}^-_y.\,s) \bullet^- (\mathsf{IC}^+_x.\,t)) \\
\longrightarrow & t[x:=(\mathsf{IC}^-_y.\,s)] \bowtie ((\mathsf{IC}^-_y.\,s) \bullet^- (\mathsf{IC}^+_x.\,t)) \\
\longrightarrow & t[x:=(\mathsf{IC}^-_y.\,s)] \bowtie s[y:=(\mathsf{IC}^+_x.\,t)]
\end{array}$$

A first observation is that PRK's duality principle (Lem. 5) can be strengthened to obtain a **computational duality principle** for λ^{PRK}. The proof is immediate given that all typing and reduction rules are symmetric:

Lemma 23. *If $t^\perp$ is the term that results from flipping all the signs in t, then $\Gamma \vdash t : P$ if and only if $\Gamma^\perp \vdash t^\perp : P^\perp$, and $t \to s$ if and only if $t^\perp \to s^\perp$.*

The second computational property that we study is **subject reduction**, also known as *type preservation*. This fundamental property ensures that reduction is well-defined over the set of typable terms. More precisely:

Proposition 24. *If $\Gamma \vdash t : P$ and $t \to s$, then $\Gamma \vdash s : P$.*

Proof. The core of the proof consists in checking that each rewriting rule preserves the type of the term. [See Section A.5 in the appendix for the proof.] □

Third, the λ^{PRK}-calculus enjoys **confluence**, the basic property of a rewriting system stating that given reduction sequences $t_0 \to^* t_1$ and $t_0 \to^* t_2$ there must exist a term t_3 such that $t_1 \to^* t_3$ and $t_2 \to^* t_3$.

Proposition 25. *The λ^{PRK}-calculus is confluent.*

Proof. The rewriting system λ^{PRK} can be modeled as a higher-order rewriting system (HRS) in the sense of Nipkow[6]. This HRS is *orthogonal*, *i.e.* left-linear without critical pairs, which entails that it is confluent [19]. □

Our next goal is to prove that λ^{PRK} enjoys **strong normalization**, that is, that there are no infinite reduction sequences $t_1 \to t_2 \to t_3 \to \ldots$. To do so, we give a translation to System F extended with *recursive type constraints*.

Type constraints are a way to define types as solutions to recursive equations. For instance, the type T of binary trees is given by $\mathsf{T} \equiv 1 + (\mathsf{T} \times \mathsf{T})$. In our case, the idea is to define $A^\oplus$ and $A^\ominus$ as solutions to the mutually recursive equations $A^\oplus \equiv (A^\ominus \to A^+)$ and $A^\ominus \equiv (A^\oplus \to A^-)$.

We begin by recalling the extended System F and its relevant properties.

System F Extended with Recursive Type Constraints. In this subsection we recall the definition of System $\mathrm{F}\langle\mathcal{C}\rangle$, *i.e.* System F parameterized by an *arbitrary* set of recursive type constraints $\mathcal{C}$, as formulated by Mendler [16].

The set of *types* in System $\mathrm{F}\langle\mathcal{C}\rangle$ is given by $A ::= \alpha \mid A \to A \mid \forall\alpha.A$ where $\alpha, \beta, \ldots$ are called *base types*. The set of *terms* is given by $t ::= x^A \mid \lambda x^A.\,t \mid t\,t \mid \lambda\alpha.\,t \mid t\,A$, where $\lambda\alpha.\,t$ is type abstraction and $t\,A$ is type application. A *type constraint* is an equation of the form $\alpha \equiv A$. System $\mathrm{F}\langle\mathcal{C}\rangle$ is parameterized by a set $\mathcal{C}$ of type constraints. Each set $\mathcal{C}$ of type constraints induces a notion of equivalence between types, written $A \equiv B$ and defined as the congruence generated by $\mathcal{C}$. Typing rules are those of the usual System F [20, Section 11.3] extended with a *conversion* rule:

$$\frac{\Gamma \vdash t : A \qquad A \equiv B}{\Gamma \vdash t : B}\,\textsc{Conv}$$

Variables occurring *positively* (resp. *negatively*) in a type A are written $\mathsf{p}(A)$ (resp. $\mathsf{n}(A)$) and defined as usual:

$$\begin{array}{rclrcl}
\mathsf{p}(\alpha) & \stackrel{\text{def}}{=} & \{\alpha\} & \mathsf{n}(\alpha) & \stackrel{\text{def}}{=} & \varnothing \\
\mathsf{p}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{n}(A) \cup \mathsf{p}(B) & \mathsf{n}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{p}(A) \cup \mathsf{n}(B) \\
\mathsf{p}(\forall\alpha.A) & \stackrel{\text{def}}{=} & \mathsf{p}(A) \setminus \{\alpha\} & \mathsf{n}(\forall\alpha.A) & \stackrel{\text{def}}{=} & \mathsf{n}(A) \setminus \{\alpha\}
\end{array}$$

A set of type constraints $\mathcal{C}$ verifies the *positivity condition* if for every type constraint $(\alpha \equiv A) \in \mathcal{C}$ and every type B such that $\alpha \equiv B$ one has that $\alpha \notin \mathsf{n}(B)$. Mendler's main result [16, Theorem 13] is:

Theorem 26 (Mendler, 1991)**.** *If $\mathcal{C}$ verifies the positivity condition, then System $\mathrm{F}\langle\mathcal{C}\rangle$ is strongly normalizing.*

6. It suffices to model it with a single sort ι, with constants such as $\pi^+_i : \iota \to \iota$, $\mathsf{IC}^- : (\iota \to \iota) \to \iota$, etc., and rules such as $\delta^+(\mathsf{in}^+_1 x)\,f\,g \to f\,x$. Strictly speaking, two constants for $\bowtie$ are needed, depending on the signs.

We define the empty ($\mathbf{0}$), unit ($\mathbf{1}$), product ($A \times B$), and sum types ($A + B$) via their usual encodings in System F (see for instance [20, Section 11.3]). For example, the product type is defined as $(A \times B) \stackrel{\text{def}}{=} \forall\alpha.((A \to B \to \alpha) \to \alpha)$. with a constructor $\langle t, s\rangle$ and an eliminator $\pi_i(t)$. [See Section B.1 for a more detailed description of the Extended System F.]

System F Extended with $\mathcal{C}_{\mathbf{pn}}$ Constraints. In this subsection, we describe System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$, an extension of System F with a *specific* set of recursive type constraints called $\mathcal{C}_{\mathbf{pn}}$. Given that the set of base types is countably infinite, we may assume without loss of generality that, for any two types A, B in System F there are two type variables, called $\mathbf{p}_{A,B}$ and $\mathbf{n}_{A,B}$. More precisely, the set of type variables can be partitioned as $\mathbf{V} \uplus \mathbf{P} \uplus \mathbf{N}$ in such a way that the propositional variables of λ^{PRK} are identified with type variables of $\mathbf{V}$, and there are bijective mappings $(A, B) \mapsto \mathbf{p}_{A,B} \in \mathbf{P}$ and $(A, B) \mapsto \mathbf{n}_{A,B} \in \mathbf{N}$. Note that we do not forbid A and B to have occurrences of type variables in $\mathbf{P}$ and $\mathbf{N}$.[7]

System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$ is given by extending System F with the set of recursive type constraints $\mathcal{C}_{\mathbf{pn}}$, including the following equations for all types A, B:

$$\mathbf{p}_{A,B} \equiv (\mathbf{n}_{A,B} \to A) \qquad \mathbf{n}_{A,B} \equiv (\mathbf{p}_{A,B} \to B)$$

This extension is in fact **strongly normalizing**:

Corollary 27. *System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$ is strongly normalizing.*

Proof. A corollary of the previous theorem. It suffices to show that the recursive type constraints $\mathcal{C}_{\mathbf{pn}}$ verify Mendler's positivity condition. The proof of this fact is slightly technical. [See Section A.6 in the appendix for the proof.] ☐

Translating λ^{PRK} to System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$. We are now in a position to define the translation from λ^{PRK} to System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$.

Definition 28 (Translation of Propositions)**.** A proposition P of λ^{PRK} is translated into a type $[\![P]\!]$ of System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$, according to the following definition, given by induction on the *measure* $\#(P)$ (defined in Section 3):

$$\begin{array}{rclrcl}
[\![\alpha^+]\!] & \stackrel{\text{def}}{=} & \alpha & [\![\alpha^-]\!] & \stackrel{\text{def}}{=} & \alpha \to \mathbf{0} \\
[\![(A \wedge B)^+]\!] & \stackrel{\text{def}}{=} & [\![A^\oplus]\!] \times [\![B^\oplus]\!] & [\![(A \wedge B)^-]\!] & \stackrel{\text{def}}{=} & [\![A^\ominus]\!] + [\![B^\ominus]\!] \\
[\![(A \vee B)^+]\!] & \stackrel{\text{def}}{=} & [\![A^\oplus]\!] + [\![B^\oplus]\!] & [\![(A \vee B)^-]\!] & \stackrel{\text{def}}{=} & [\![A^\ominus]\!] \times [\![B^\ominus]\!] \\
[\![(\neg A)^+]\!] & \stackrel{\text{def}}{=} & \mathbf{1} \to [\![A^\ominus]\!] & [\![(\neg A)^-]\!] & \stackrel{\text{def}}{=} & \mathbf{1} \to [\![A^\oplus]\!] \\
[\![A^\oplus]\!] & \stackrel{\text{def}}{=} & \mathbf{p}_{[\![A^+]\!],[\![A^-]\!]} & [\![A^\ominus]\!] & \stackrel{\text{def}}{=} & \mathbf{n}_{[\![A^+]\!],[\![A^-]\!]}
\end{array}$$

Moreover, a typing context $\Gamma = (x_1 : P_1, \dots, x_n : P_n)$ is translated as $[\![\Gamma]\!] \stackrel{\text{def}}{=} (x_1 : [\![P_1]\!], \dots, x_n : [\![P_n]\!])$.

Note that the translation of propositions mimicks the equations for the realizability interpretation discussed in the introduction. In fact, the translation of $[\![A^\oplus]\!]$ is $\mathbf{p}_{[\![A^+]\!],[\![A^-]\!]}$, which is equivalent to $\mathbf{n}_{[\![A^+]\!],[\![A^-]\!]} \to [\![A^+]\!]$ according to the recursive type constraints in $\mathcal{C}_{\mathbf{pn}}$, and this in turn equals $[\![A^\ominus]\!] \to [\![A^+]\!]$, just as required. Similarly for the translation of $A^\ominus$. The translation of $(\neg A)^+$ is $(\mathbf{1} \to [\![A^\ominus]\!])$ rather than just $[\![A^\ominus]\!]$ for a technical reason, in order to ensure that each reduction step in λ^{PRK} is simulated by *at least one* step in System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$.

Definition 29 (Translation of Terms)**.** First, we define a family of terms $\vdash \mathsf{abs}^P_Q : [\![P]\!] \to [\![P^\sim]\!] \to [\![Q]\!]$ in System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$ as follows, by induction on the measure $\#(P)$:

$$\begin{array}{rcl}
\mathsf{abs}^{\alpha^+}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathcal{E}_{[\![Q]\!]}(y\, x) \\
\mathsf{abs}^{\alpha^-}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathcal{E}_{[\![Q]\!]}(x\, y) \\
\mathsf{abs}^{(A\wedge B)^+}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \delta y\, [_z.\mathsf{abs}^{A^\oplus}_Q\ \pi_1(x)\, z][_z.\mathsf{abs}^{B^\oplus}_Q\ \pi_2(x)\, z] \\
\mathsf{abs}^{(A\wedge B)^-}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \delta x\, [_z.\mathsf{abs}^{A^\ominus}_Q\ z\, \pi_1(y)][_z.\mathsf{abs}^{B^\ominus}_Q\ z\, \pi_2(x)] \\
\mathsf{abs}^{(A\vee B)^+}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \delta x\, [_z.\mathsf{abs}^{A^\oplus}_Q\ x\, \pi_1(y)][_z.\mathsf{abs}^{B^\oplus}_Q\ x\, \pi_2(y)] \\
\mathsf{abs}^{(A\vee B)^-}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \delta y\, [_z.\mathsf{abs}^{A^\ominus}_Q\ \pi_1(x)\, z][_z.\mathsf{abs}^{B^\oplus}_Q\ \pi_2(x)\, z] \\
\mathsf{abs}^{(\neg A)^+}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathsf{abs}^{A^\ominus}_Q\ (x\, \star)\, (y\, \star) \\
\mathsf{abs}^{(\neg A)^-}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathsf{abs}^{A^\oplus}_Q\ (x\, \star)\, (y\, \star) \\
\mathsf{abs}^{A^\oplus}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathsf{abs}^{A^+}_Q\ (x\, y)\, (y\, x) \\
\mathsf{abs}^{A^\ominus}_Q & \stackrel{\text{def}}{=} & \lambda x\, y.\, \mathsf{abs}^{A^-}_Q\ (x\, y)\, (y\, x)
\end{array}$$

where: $\mathcal{E}_A(t)$ denotes an inhabitant of A whenever t is an inhabitant of the empty type; $\delta t\, [_x.s][_x.u]$ is the eliminator of the sum type; $\pi_i(t)$ is the eliminator of the product type; and $\star$ denotes the trivial inhabitant of the unit type. Now each typable term $\Gamma \vdash t : P$ in λ^{PRK} can be translated into a term $[\![\Gamma]\!] \vdash [\![t]\!] : [\![P]\!]$ of System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$ as follows:

$$\begin{array}{rcl}
[\![x]\!] & \stackrel{\text{def}}{=} & x \\
[\![t \bowtie_Q s]\!] & \stackrel{\text{def}}{=} & \mathsf{abs}^P_Q\ [\![t]\!]\ [\![s]\!] \\
 & & \quad \text{if } \Gamma \vdash t : P \text{ and } \Gamma \vdash s : P^\sim \\
[\![\langle t, s\rangle^\pm]\!] & \stackrel{\text{def}}{=} & \langle [\![t]\!], [\![s]\!]\rangle \\
[\![\pi^\pm_i(t)]\!] & \stackrel{\text{def}}{=} & \pi_i([\![t]\!]) \\
[\![\mathsf{in}^\pm_i(t)]\!] & \stackrel{\text{def}}{=} & \mathsf{in}_i([\![t]\!]) \\
[\![\delta^\pm t\, [_{(x:P)}.s][_{(y:Q)}.u]]\!] & \stackrel{\text{def}}{=} & \delta[\![t]\!]\, [_{(x:[\![P]\!])}.[\![s]\!]][_{(y:[\![Q]\!])}.[\![u]\!]] \\
[\![\nu^\pm t]\!] & \stackrel{\text{def}}{=} & \lambda x^{\mathbf{1}}.\, [\![t]\!] \quad \text{ where } x \notin \mathsf{fv}(t) \\
[\![\mu^\pm t]\!] & \stackrel{\text{def}}{=} & [\![t]\!]\, \star \\
[\![\mathsf{IC}^\pm_{(x:P)}.\, t]\!] & \stackrel{\text{def}}{=} & \lambda x^{[\![P]\!]}.\, [\![t]\!] \\
[\![t \bullet^\pm s]\!] & \stackrel{\text{def}}{=} & [\![t]\!]\, [\![s]\!]
\end{array}$$

It is easy to check that $[\![\Gamma]\!] \vdash [\![t]\!] : [\![P]\!]$ holds in System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$ by induction on the derivation of the judgment $\Gamma \vdash t : P$ in λ^{PRK}. Two straightforward properties of the translation are:

Lemma 30. 1. $\mathsf{fv}([\![t]\!]) = \mathsf{fv}(t)$; 2. $[\![t[x:=s]]\!] = [\![t]\!][x:=[\![s]\!]]$.

The key result is the following **simulation** lemma from which strong normalization follows:

Lemma 31. *If $t \to s$ in λ^{PRK} then $[\![t]\!] \to^+ [\![s]\!]$ in System F$\langle\mathcal{C}_{\mathbf{pn}}\rangle$.*

Proof. [Lem. 55 in the appendix.] By case analysis on the rewriting rule used to derive the step $t \to s$. The interesting case is when the rule is applied at the root of the term. As an

7. An alternative, perhaps cleaner, presentation would be to define types inductively as $A, B, \ldots ::= \alpha \mid \mathbf{p}_{A,B} \mid \mathbf{n}_{A,B} \mid A \to B \mid \forall\alpha.A$.

illustrative example, consider an instance of the $\mathsf{absPairInj}$ rule, with $\Gamma \vdash t_1 : A_1{}^{\oplus}$, and $\Gamma \vdash t_2 : A_2{}^{\oplus}$, and $\Gamma \vdash s : A_i{}^{\ominus}$ for some $i \in \{1, 2\}$. Then:

$$
\begin{array}{ll}
 & [\![\langle t_1, t_2\rangle^{+} \blacktriangleright\!\!\blacktriangleleft_P \mathsf{in}_i^{-}(s)]\!] \\
= & \mathsf{abs}_P^{(A_1 \wedge A_2)^{+}} \langle [\![t_1]\!], [\![t_2]\!]\rangle\, \mathsf{in}_i([\![s]\!]) \\
\to^{+} & \delta\mathsf{in}_i([\![s]\!])[_{(z:[\![A_1{}^{\ominus}]\!])}.\mathsf{abs}_P^{A_1{}^{\oplus}}\ \pi_1(\langle [\![t_1]\!], [\![t_2]\!]\rangle)\, z] \\
 & \qquad\qquad [_{(z:[\![A_2{}^{\ominus}]\!])}.\mathsf{abs}_P^{A_2{}^{\oplus}}\ \pi_2(\langle [\![t_1]\!], [\![t_2]\!]\rangle)\, z] \\
\to & \mathsf{abs}_P^{A_i{}^{\oplus}}\ \pi_i(\langle [\![t_1]\!], [\![t_2]\!]\rangle)\, [\![s]\!] \\
\to & \mathsf{abs}_P^{A_i{}^{\oplus}}\ [\![t_i]\!]\, [\![s]\!] \\
\to^{+} & \mathsf{abs}_P^{A_i{}^{+}}\ ([\![t_i]\!]\, [\![s]\!])([\![s]\!]\, [\![t_i]\!]) \\
= & [\![(t_i \bullet^{+} s) \blacktriangleright\!\!\blacktriangleleft_P (t_i \bullet^{-} s)]\!] \\
= & [\![t_i \bowtie_P s]\!]
\end{array}
$$
□

Theorem 32. *The λ^{PRK}-calculus is strongly normalizing.*

Proof. An easy consequence of Lem. 31 given that the extended System F is strongly normalizing (Coro. 27). □

Canonicity. In the previous subsections we have shown that the λ^{PRK}-calculus enjoys subject reduction and strong normalization. This implies that each typable term t reduces to a normal form t' of the same type. In this subsection, these results are refined to prove a *canonicity* theorem, stating that each closed, typable term t reduces to a *canonical* term t' of the same type. For example, canonical terms of type $(A \vee B)^{+}$ are of the form $\mathsf{in}_i^{+}(t)$. From the logical point of view, this means that given a strong proof of $(A \vee B)$, in a context without assumptions, one can always recover either a classical proof of A or a classical proof of B. This shows that PRK has a form of disjunctive property.

First we provide an inductive characterization of the set of **normal forms** of λ^{PRK}.

Definition 33 (Normal terms)**.** The sets of *normal terms* $(N, \ldots)$ and *neutral terms* $(S, \ldots)$ are defined mutually inductively by:

$$
\begin{array}{lllllll}
N & ::= & S & \mid & \langle N, N\rangle^{\pm} & \mid & \mathsf{in}_i^{\pm}(N) \\
 & \mid & \nu^{\pm} N & \mid & \mathsf{IC}^{\pm}_{x:P}.\,N & & \\
 & & & & & & \\
S & ::= & x & \mid & \pi_i^{\pm}(S) & \mid & \delta^{\pm} S\,[_x.N][_x.N] \\
 & \mid & \mu^{\pm} S & \mid & S \bullet^{\pm} N & & \\
 & \mid & S \blacktriangleright\!\!\blacktriangleleft_P N & \mid & N \blacktriangleright\!\!\blacktriangleleft_P S & &
\end{array}
$$

Proposition 34. *A term is in the grammar of normal terms if and only if it is a normal form,* i.e. *it does not reduce in λ^{PRK}.*

Proof. Straightforward by induction. [See Section A.8 in the appendix for a detailed proof.] □

In order to state a canonicity theorem succintly, we introduce some nomenclature. A term is *canonical* if it has any of the following shapes:

$$\langle t_1, t_2\rangle^{\pm} \qquad \mathsf{in}_i^{\pm}(t) \qquad \nu^{\pm} t \qquad \mathsf{IC}^{\pm}_{x}.\,t$$

A typing context is *classical* if all the assumptions are classical, *i.e.* of the form $A^{\oplus}$ or $A^{\ominus}$. Recall that a context is a term $\mathtt{C}$ with a single free occurrence of a hole $\Box$, and that $\mathtt{C}\langle t\rangle$ denotes the capturing substitution of the term t into the hole of $\mathtt{C}$. A *case-context* is a context of the form $\mathtt{K} ::= \Box \mid \delta^{\pm}\mathtt{K}\,[_x.t][_y.s]$. An *eliminative context* is a context of the form $\mathtt{E} ::= \Box \mid \pi_i^{\pm}(\mathtt{E}) \mid \delta^{\pm}\mathtt{E}\,[_x.t][_y.s] \mid \mu^{\pm}\mathtt{E}$. Note that $\Box \bullet^{\pm} t$ is not eliminative and that all case-contexts are eliminative. An *explosion* is a term of the form $t \blacktriangleright\!\!\blacktriangleleft_P s$ or of the form $t \bullet^{\pm} s$. A term is *closed* if it has no free variables. A term is *open* if it not closed, *i.e.* it has at least one free variable.

The following theorem has three parts; the first one provides guarantees for *closed* terms, whereas the two other ones provide weaker guarantees for terms typable under an arbitrary classical context.

Theorem 35 (Canonicity)**.**

1. *Let $\vdash t : P$. Then t reduces to a canonical term.*
2. *Let $\Gamma \vdash t : P$ where Γ is classical and P is strong. Then either $t \to^{*} t'$ where t' is canonical or $t \to^{*} \mathtt{K}\langle t'\rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion.*
3. *Let $\Gamma \vdash t : P$ where Γ and P are classical. Then either $t \to^{*} \mathsf{IC}^{\pm}_{x}.\,t'$ or $t \to^{*} \mathtt{E}\langle t'\rangle$, where $\mathtt{E}$ is an eliminative context and t' is a variable or an open explosion.*

Proof. By subject reduction (Prop. 24) and strong normalization (Thm. 32) the term t reduces to a normal form t'. Moreover, by Prop. 34, we have that t' is generated by the grammar of Def. 33. The proof then proceeds by induction on the derivation of t' in the grammar of normal terms. [See Section A.9 in the appendix for the proof.] □

Extensionality for Classical Proofs. To conclude the syntactic study of λ^{PRK}, we discuss that an extensionality rule, akin to η-reduction in the λ-calculus, may be incorporated to λ^{PRK}, obtaining a calculus $\lambda^{\text{PRK}}_{\eta}$.

Definition 36. The $\lambda^{\text{PRK}}_{\eta}$-calculus is defined by extending the λ^{PRK} calculus with the following reduction rule:

$$\mathsf{IC}^{\pm}_{x}.\,(t \bullet^{\pm} x) \xrightarrow{\mathsf{eta}} t \qquad \text{if } x \notin \mathsf{fv}(t)$$

Theorem 37. *The $\lambda^{\text{PRK}}_{\eta}$-calculus enjoys subject reduction, and it is strongly normalizing and confluent.*

Proof. Subject reduction is straightforward, extending Prop. 24 with an easy case for the eta rule. Local confluence is also straightforward by examining the critical pairs. The key lemma to prove strong normalization is that eta reduction steps can be postponed after steps of other kinds. [See Section A.10 for the details.] □

5. Embedding Classical Logic into PRK

Intuitionistic logic *refines* classical logic: each intuitionistically valid formula A is also classically valid, but there may be many classically equivalent "readings" of a classical formula which are not intuitionistically equivalent, such as $\neg(A \wedge \neg B)$ and $\neg A \vee B$. System PRK refines classical logic in a similar sense. For example, the classical sequent $\alpha \vdash \alpha$

may be "read" in PRK in various different ways, such as $\alpha^+ \vdash \alpha^\oplus$ and $\alpha^\oplus \vdash \alpha^+$, of which the former holds but the latter does not. In this section we show that PRK is *conservative* (Prop. 38) with respect to classical logic, and that classical logic may be *embedded* (Thm. 39) in PRK. We also describe the computational behavior of the terms resulting from this embedding (Lem. 40).

First, we claim that PRK is a **conservative extension** of classical logic, *i.e.* if $A_1^\oplus, \dots, A_n^\oplus \vdash B^\oplus$ holds in PRK then the sequent $A_1, \dots, A_n \vdash B$ holds in classical logic. In general:

Proposition 38. *Define $c(P)$ as follows:*

$$\begin{array}{ll} c(A^\oplus) \stackrel{\text{def}}{=} A & c(A^\ominus) \stackrel{\text{def}}{=} \neg A \\ c(A^+) \stackrel{\text{def}}{=} A & c(A^-) \stackrel{\text{def}}{=} \neg A \end{array}$$

If the sequent $P_1, \dots, P_n \vdash Q$ holds in PRK *then the sequent $c(P_1), \dots, c(P_n) \vdash c(Q)$ holds in classical propositional logic.*

Proof. By induction on the derivation of the judgment, observing that all the inference rules in PRK are mapped to classically valid inferences. For example, for the $\mathrm{E}\wedge^-$ rule, note that if $\Gamma \vdash \neg(A \wedge B)$ and $\Gamma, \neg A \vdash C$ and $\Gamma, \neg B \vdash C$ hold in classical propositional logic then $\Gamma \vdash C$. □

Second, we claim that classical logic may be **embedded** in PRK, that is:

Theorem 39. *If $A_1, \dots, A_n \vdash B$ holds in classical logic then $A_1^\oplus, \dots, A_n^\oplus \vdash B^\oplus$ holds in* PRK.

Proof. The proof is by induction on the proof of the sequent $A_1, \dots, A_n \vdash B$ in Gentzen's system of natural deduction for classical logic NK, including introduction and elimination rules for conjunction, disjunction, and negation (encoding falsity as the pure proposition $\bot \stackrel{\text{def}}{=} (\alpha_0 \wedge \neg\alpha_0)$ for some fixed propositional variable α_0), the explosion principle, and the law of excluded middle. We build the corresponding proof terms in λ^{PRK}:

1. **Conjunction introduction.** Let $\Gamma \vdash t : A^\oplus$ and $\Gamma \vdash s : B^\oplus$. Then $\Gamma \vdash \langle t, s\rangle^{\mathcal{C}} : (A \wedge B)^\oplus$ where:

$$\langle t, s\rangle^{\mathcal{C}} \stackrel{\text{def}}{=} \mathsf{IC}^+_{(\_:(A\wedge B)^\ominus)}.\, \langle t, s\rangle^+$$

2. **Conjunction elimination.** Let $\Gamma \vdash t : (A_1 \wedge A_2)^\oplus$. Then $\Gamma \vdash \pi_i^{\mathcal{C}}(t) : A_i{}^\oplus$ where:

$$\pi_i^{\mathcal{C}}(t) \stackrel{\text{def}}{=} \mathsf{IC}^+_{(x:A_i{}^\ominus)}.\, \pi_i^+(t \bullet^+ \mathsf{IC}^-_{(\_:(A_1\wedge A_2)^\oplus)}.\, \mathsf{in}_i^-(x)) \bullet^+ x$$

3. **Disjunction introduction.** Let $\Gamma \vdash t : A_i{}^\oplus$. Then $\Gamma \vdash \mathsf{in}_i^{\mathcal{C}}(t) : (A_1 \vee A_2)^\oplus$ where:

$$\mathsf{in}_i^{\mathcal{C}}(t) \stackrel{\text{def}}{=} \mathsf{IC}^+_{(\_:(A_1\vee A_2)^\ominus)}.\, \mathsf{in}_i^+(t)$$

4. **Disjunction elimination.** Let $\Gamma \vdash t : (A \vee B)^\oplus$ and $\Gamma, x : A^\oplus \vdash s : C^\oplus$ and $\Gamma, x : B^\oplus \vdash u : C^\oplus$. Then $\Gamma \vdash \delta^{\mathcal{C}} t\, [_{(x:A^\oplus)}.s][_{(x:B^\oplus)}.u] : C^\oplus$, where:

$$\begin{array}{ll} \mathsf{IC}^+_{(y:C^\ominus)}. & \delta^+ \; (t \bullet^+ \mathsf{IC}^-_{(\_:(A\vee B)^\oplus)}.\, \langle \updownarrow^y_x (s), \updownarrow^y_x (u)\rangle^-) \\ & \quad [_{(x:A^\oplus)}.s \bullet^+ y] \\ & \quad [_{(x:B^\oplus)}.u \bullet^+ y] \end{array}$$

Recall that $\updownarrow^y_x (t)$ stands for the witness of contraposition (Lem. 20).

5. **Negation introduction.** By Lem. 20 we have that $\Gamma \vdash \pitchfork^-_{\alpha_0} : (\alpha_0 \wedge \neg\alpha_0)^\ominus$, that is $\Gamma \vdash \pitchfork^-_{\alpha_0} : \bot^\ominus$. Moreover, suppose that $\Gamma, x : A^\oplus \vdash t : \bot^\oplus$. Then $\Gamma \vdash \Lambda^{\mathcal{C}}_{(x:A^\oplus)}.\, t : (\neg A)^\oplus$, where:

$$\Lambda^{\mathcal{C}}_{(x:A^\oplus)}.\, t \stackrel{\text{def}}{=} \mathsf{IC}^+_{(\_:(\neg A)^\ominus)}.\, \nu^+ \mathsf{IC}^-_{(x:A^\oplus)}.\, (t \bowtie_{A^-} \pitchfork^-_{\alpha_0})$$

6. **Negation elimination.** Let $\Gamma \vdash t : (\neg A)^\oplus$ and $\Gamma \vdash s : A^\oplus$. Then $\Gamma \vdash t \,\#^{\mathcal{C}} s : \bot^\oplus$, where:

$$t \,\#^{\mathcal{C}} s \stackrel{\text{def}}{=} t \bowtie_{\bot^\oplus} \mathsf{IC}^-_{(\_:(\neg A)^\oplus)}.\, \nu^- s$$

7. **Explosion.** Let $\Gamma \vdash t : \bot^\oplus$. Then $\Gamma \vdash (t \bowtie_Q \pitchfork^-_{\alpha_0}) : Q$.

8. **Excluded middle.** It suffices to take $\pitchfork^{\mathcal{C}}_A \stackrel{\text{def}}{=} \pitchfork^+_A$. Then by Lem. 20, $\Gamma \vdash \pitchfork^{\mathcal{C}}_A : (A \vee \neg A)^\oplus$. □

Finally, this embedding may be understood as providing a **computational interpretation** for classical logic. In fact, besides the introduction and elimination rules that have been proved above, implication may be defined as an abbreviation, $(A \Rightarrow B) \stackrel{\text{def}}{=} (\neg A \vee B)$, and witnesses for its introduction rule $\lambda^{\mathcal{C}}_{x:A}.\, t$ and its elimination rule $t \,@^{\mathcal{C}}\, s$ may be defined as follows. If $\Gamma, x : A^\oplus \vdash t : B^\oplus$ then $\Gamma \vdash \lambda^{\mathcal{C}}_{(x:A)}.\, t : (A \Rightarrow B)^\oplus$ where:

$$\begin{array}{rcl} \lambda^{\mathcal{C}}_x.\, t & \stackrel{\text{def}}{=} & \mathsf{IC}^+_{(y:(A\Rightarrow B)^\ominus)}.\, \mathsf{in}_2^+(t[x:=\mathbf{X}_y]) \\ \mathbf{X}_y & \stackrel{\text{def}}{=} & \mathsf{IC}^+_{(z:A^\ominus)}.\, (\mu^-(\mathbf{X}'_{y,z} \bullet^- \mathsf{IC}^+_{(\_:(\neg A)^\ominus)}.\, \nu^+ z)) \bullet^+ z \\ \mathbf{X}'_{y,z} & \stackrel{\text{def}}{=} & \pi_1^+(y \bullet^- \mathsf{IC}^+_{(\_:(A\Rightarrow B)^\ominus)}.\, \mathsf{in}_1^+(\mathsf{IC}^+_{(\_:(\neg A)^\ominus)}.\, \nu^+ z)) \end{array}$$

If $\Gamma \vdash t : (A \Rightarrow B)^\oplus$ and $\Gamma \vdash s : A^\oplus$, then $\Gamma \vdash t \,@^{\mathcal{C}}\, s : B^\oplus$, where:

$$\begin{array}{ll} t \,@^{\mathcal{C}}\, s \stackrel{\text{def}}{=} & \mathsf{IC}^+_{(x:B^\ominus)}. \\ & \quad \delta^+ \; (t \bullet^+ \mathsf{IC}^-_{(\_:(A\Rightarrow B)^\oplus)}.\, \langle (\mathsf{IC}^-_{(\_:(\neg A)^\oplus)}.\, \nu^- s), x\rangle^-) \\ & \quad\quad [_{(y:(\neg A)^\oplus)}.s \bowtie_{B^+} \mu^-(y \bullet^+ \mathsf{IC}^-_{(\_:(\neg A)^\oplus)}.\, \nu^- x)] \\ & \quad\quad [_{(z:B^\oplus)}.z \bullet^+ x] \end{array}$$

Lemma 40. *The following hold in $\lambda^{\text{PRK}}_\eta$ (with* eta *reduction):*

$$\begin{array}{rcl} \pi_i^{\mathcal{C}}(\langle t_1, t_2\rangle^{\mathcal{C}}) & \to^* & t_i \\ \delta^{\mathcal{C}} \mathsf{in}_i^{\mathcal{C}}(t)\, [_x.s_1][_x.s_2] & \to^* & s_i[x:=t] \\ (\lambda^{\mathcal{C}}_x.\, t) \,@^{\mathcal{C}}\, s & \to^* & t[x:=s] \\ \delta^{\mathcal{C}} \; \pitchfork^{\mathcal{C}}_A \; [_x.s_1][_x.s_2] & \to^* & \mathsf{IC}^+_y.\, (s_2[x:=s_1^*] \bullet^+ y) \end{array}$$

where $s_1^* := \mathsf{IC}^+_{\_}.\, \nu^+(\mathsf{IC}^-_x.\, s_1 \bowtie y)$.

Proof. By calculation. The last rule describes the behaviour of the law of excluded middle. [See Section A.11 in the appendix.] □

6. Conclusion

This work explores a logical system PRK, formulated in natural deduction style (Def. 1), based on a, to the best of our knowledge, new realizability interpretation for classical logic. The key idea is that a classical proof of a proposition can be understood as a transformation from a classical refutation to a strong proof of the proposition. We summarize our contributions: system PRK has been shown to be **sound** (Prop. 11) and **complete** (Thm. 17) with respect to a Kripke semantics. A calculus λ^{PRK} (Def. 18, Def. 21) based on PRK via the propositions-as-types correspondence, has been defined. The calculus enjoys good properties, the most relevant ones being **confluence** (Prop. 25), **subject reduction** (Prop. 24), **strong normalization** (Thm. 32), and **canonicity** (Thm. 35). Finally, we have shown that PRK is a **conservative extension** (Prop. 38) of classical logic, and classical logic may be **embedded** (Thm. 39) in PRK. This provides a **computational interpretation** (Lem. 40) for classical logic.

Future Work. It is a natural question whether λ^{PRK} can be extended to second-order logic. In fact, formulating such a system is straightforward by extending the realizability interpretation described in the introduction with equations:

$$\begin{array}{lcllcl} (\forall\alpha.A)^+ & \approx & \forall\alpha.(A^\oplus) & (\forall\alpha.A)^- & \approx & \exists\alpha.(A^\ominus) \\ (\exists\alpha.A)^+ & \approx & \exists\alpha.(A^\oplus) & (\exists\alpha.A)^- & \approx & \forall\alpha.(A^\ominus) \end{array}$$

For instance, introduction and elimination rules for positive universal quantification in second-order λ^{PRK} would be:

$$\frac{\Gamma \vdash t : A^\oplus \qquad \alpha \notin \mathsf{fv}(\Gamma)}{\Gamma \vdash \lambda^+\alpha.\, t : (\forall\alpha.\, A)^+}\,\mathrm{I}\forall^+ \qquad \frac{\Gamma \vdash t : (\forall\alpha.\, B)^+}{\Gamma \vdash t \bullet^+ A : B[\alpha := A]^\oplus}\,\mathrm{E}\forall^+$$

From the logical point of view, the system turns out to be a conservative extension of second-order classical logic, and from the computational point of view it still enjoys confluence and subject reduction. However, the techniques described in this paper do not suffice to prove strong normalization. A different normalization proof, possibly based on Tait–Girard's technique of *reducibility candidates*, should be explored.

We have not addressed decision problems, such as determining the validity of a formula in PRK, corresponding to the type inhabitation problem for λ^{PRK}. Unfortunately, λ^{PRK} does not enjoy the *subformula property*. For a counterexample consider $(x \bowtie_{\beta^+} y) \bowtie_{\alpha^+} (x \bowtie_{\beta^-} y)$, which is a normal term of type α^+ under the context $x : \alpha^+, y : \alpha^-$ such that the unrelated type β^+ appears in the derivation.

We have not stated explicitly a computational rule for negation, *i.e.* for $(\Lambda^{\mathcal{C}}_x.\, t) \#^{\mathcal{C}} s$ in Lem. 40 [but see Section A.11.3 in the appendix]. Intriguingly, it does not reduce to $t[x := s]$ in general, *i.e.* the inference schemes for classical negation that we have constructed are not consistent with a definition of negation as $\neg A \equiv (A \to \bot)$ in intuitionistic logic. We believe this to be not just an artifact of a faulty construction, but due to a deeper reason.

Related Work. That classical logic may be embedded in intuitionistic logic has been known as early as Glivenko's proof of his theorem in the late 1920s. For a long time, however, the generalized belief seemed to be that classical proofs had no computational content. In the late 1980s, Griffin [8] remarked that the type of Felleisen's $\mathcal{C}$ operator (similar to `call/cc`) corresponds to Peirce's law $(((A \to B) \to A) \to A)$. This sparked research on calculi for classical logic. Many of these works are based on classical axioms that behave as *control operators*, *i.e.* operators that can manipulate their computational context. The literature is abundant on this topic—we limit ourselves to pointing out some influential works.

Parigot [11] proposes a calculus $\lambda\mu$ based on cut elimination in natural deduction with multiple conclusions for second-order classical logic. Its control operator μ is related with the rule we call contraposition. The study of $\lambda\mu$ is mature: topics such as separability [21], [22], [23], abstract machines [24], call-by-need [25], intersection types [26], and encodings into linear logic [27], [28] have been developed.

Barbanera and Berardi [10] propose a symmetric λ-calculus based on a system of natural deduction including a "symmetric application" operator $(t \bigstar s)$, closely related to our witness of absurdity $(t \bowtie s)$. The system of [10] is sound and complete with respect to second-order classical logic, and it is strongly normalizing, but not confluent.

Curien and Herbelin [9] derive a calculus $\bar{\lambda}\mu\tilde{\mu}$ from Gentzen's classical sequent calculus. This exposes the symmetry between a program yielding an output and a continuation consuming an input. The interaction between a program and a continuation, written $\langle t \mid s \rangle$ is also reminiscent to our witness of absurdity $(t \bowtie s)$. Many variants of this system have been studied; for example, recently, Miquey [29] has extended $\bar{\lambda}\mu\tilde{\mu}$ to incorporate dependent types.

Classical calculi such as $\lambda\mu$ and $\bar{\lambda}\mu\tilde{\mu}$ are typically translated into the λ-calculus by means of continuation-passing style (CPS) translations, whereas our translation from λ^{PRK} to the extended System F is simpler. Part of the complexity appears to be factored into the proof that classical inference schemes hold in λ^{PRK} (*i.e.* Lem. 40).

The works of Andreoli [30] and Girard [31] in linear logic introduced the notions of *focusing* and *polarity*, which allow to formulate linear, intuitionistic, and classical logic as fragments of a single system (Unified Logic). Our notions of positive and negative formulae, which express affirmation and denial, should not be confused with the subtler notions of positive and negative formulae in the sense of polarity.

Krivine [32] defines a realizability interpretation for classical logic using an abstract machine λ_c that extends Krivine's abstract machine with further instructions. This approach is based on the idea that adding logical axioms corresponds to adding instructions to the machine, and it has been adapted to provide computational meaning to reasoning principles such as the axiom of dependent choice [33], [34].

Ilik, Lee, and Herbelin [35] study a Kripke semantics for classical logic. Note that our work in Section 3 provides a different Kripke semantics for PRK, and hence for classical logic. The semantics given in [35] and our own have some similarities, but the relation between them is not obvious. For example, in [35] a Kripke model involves a relation of "exploding" world, which has no counterpart in our system.

Acknowledgments

This work was partially supported by project grant ECOS Sud A17C01. The authors would like to thank

Eduardo Bonelli and the anonymous reviewers for feedback on an early draft.

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A. Technical Appendix

A.1. Proof of the Projection Lemma (Lem. 4)

Lemma 41 (Projection). *If* $\Gamma, P \vdash Q$ *then* $\Gamma, \bigcirc P \vdash \bigcirc Q$.

Proof. We call P the *target assumption*. The proof proceeds by induction on the derivation of $\Gamma, P \vdash Q$. We only study the cases with positive signs, the negative cases are symmetric.

• AX: let $\Gamma, Q \vdash Q$. There are two cases, depending on whether the target assumption is in Γ or not.

1. *If the target assumption is in* Γ, i.e. $\Gamma = \Gamma', P$. Note that we have $\Gamma', \bigcirc P, Q \vdash Q$ by the AX rule. By truncating the conclusion (Lem. 2) we conclude that $\Gamma', \bigcirc P, Q \vdash \bigcirc Q$, as required.
2. *If the target assumption is* Q. Then we have that $\Gamma, \bigcirc Q \vdash \bigcirc Q$ by the AX rule.

• ABS: let $\Gamma, P \vdash Q$ be derived from $\Gamma, P \vdash R$ and $\Gamma, P \vdash R^{\sim}$ for some strong proposition R. By IH we have that $\Gamma, \bigcirc P \vdash \bigcirc R$ and $\Gamma, \bigcirc P \vdash \bigcirc R^{\sim}$ so by the generalized absurdity rule (ABS$'$) we have that $\Gamma, \bigcirc P \vdash \bigcirc Q$.

• I$\wedge^+$: let $\Gamma, P \vdash (A \wedge B)^+$ be derived from $\Gamma, P \vdash A^{\oplus}$ and $\Gamma, P \vdash A^{\oplus}$. By IH, $\Gamma, \bigcirc P \vdash A^{\oplus}$ and $\Gamma, \bigcirc P \vdash A^{\oplus}$. By the I$\wedge^+$ rule, $\Gamma, \bigcirc P \vdash (A \wedge B)^+$. Projecting the conclusion (Lem. 2), $\Gamma, \bigcirc P \vdash (A \wedge B)^{\oplus}$ as required.

• E$\wedge_i^+$: let $\Gamma, P \vdash A_i{}^{\oplus}$ be derived from $\Gamma, P \vdash (A_1 \wedge A_2)^+$. Then the proof is of the form:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots & \vdots \\ \pi & \xi\end{matrix}}{\Gamma, \bigcirc P, A_i{}^{\ominus} \vdash (A_1 \wedge A_2)^+}\,\text{EC}^+}{\Gamma, \bigcirc P, A_i{}^{\ominus} \vdash A_i{}^{\oplus}}\,\text{E}\wedge_i^+}{\Gamma, \bigcirc P \vdash A_i{}^{\oplus}}\,\text{CS}$$

where:

$$\pi \stackrel{\text{def}}{=} \left(\dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P \vdash (A_1 \wedge A_2)^{\oplus}}}{\Gamma, \bigcirc P, A_i{}^{\ominus} \vdash (A_1 \wedge A_2)^{\oplus}}\,\text{W}\right)$$

$$\xi \stackrel{\text{def}}{=} \left(\dfrac{\dfrac{\dfrac{}{\Gamma, \bigcirc P, A_i{}^{\ominus} \vdash A_i{}^{\ominus}}\,\text{Ax}}{\Gamma, \bigcirc P, A_i{}^{\ominus} \vdash (A_1 \wedge A_2)^-}\,\text{I}\wedge_i^-}{\Gamma, \bigcirc P, A_i{}^{\ominus} \vdash (A_1 \wedge A_2)^{\ominus}}\,\text{PC}\right)$$

• I$\vee_i^+$: let $\Gamma, P \vdash (A_1 \vee A_2)^+$ be derived from $\Gamma, P \vdash A_i{}^{\oplus}$. By IH, $\Gamma, \bigcirc P \vdash A_i{}^{\oplus}$. By the I$\vee_i^+$ rule, $\Gamma, \bigcirc P \vdash (A_1 \vee A_2)^+$. Projecting the conclusion (Lem. 2), $\Gamma, \bigcirc P \vdash (A_1 \vee A_2)^{\oplus}$.

• E$\vee^+$: let $\Gamma, P \vdash Q$ be derived from $\Gamma, P \vdash (A_1 \vee A_2)^+$ and $\Gamma, P, A_i{}^{\oplus} \vdash Q$ for each $i \in \{1, 2\}$. By IH, $\Gamma, \bigcirc P \vdash (A_1 \vee A_2)^{\oplus}$ and $\Gamma, \bigcirc P, A_i{}^{\oplus} \vdash \bigcirc Q$ for each $i \in \{1, 2\}$. Then the proof is of the form:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots & \vdots \\ \rho & \xi\end{matrix}}{\Gamma, \bigcirc P, \bigcirc Q^{\sim} \vdash (A_1 \vee B_2)^+}\,\text{EC}^+ \quad \begin{matrix}\vdots \\ \pi_1\end{matrix} \quad \begin{matrix}\vdots \\ \pi_2\end{matrix}}{\Gamma, \bigcirc P, \bigcirc Q^{\sim} \vdash \bigcirc Q}\,\text{E}\vee^+}{\Gamma, \bigcirc P \vdash \bigcirc Q}\,\text{CS}$$

where:

$$\rho \stackrel{\text{def}}{=} \left(\dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P \vdash (A_1 \vee A_2)^{\oplus}}}{\Gamma, \bigcirc P, \bigcirc Q^{\sim} \vdash (A_1 \vee A_2)^{\oplus}}\,\text{W}\right)$$

$$\xi \stackrel{\text{def}}{=} \left(\dfrac{\begin{matrix}\vdots & \vdots \\ \xi_1 & \xi_2\end{matrix}}{\Gamma, \bigcirc P, \bigcirc Q^{\sim} \vdash (A_1 \vee A_2)^{\ominus}}\,\text{I}\vee^-\right)$$

and for each $i \in \{1, 2\}$ the derivations π_i and ξ_i are given by:

$$\pi_i \stackrel{\text{def}}{=} \left(\dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P, A_i{}^{\oplus} \vdash \bigcirc Q}}{\Gamma, \bigcirc P, \bigcirc Q^{\sim}, A_i{}^{\oplus} \vdash \bigcirc Q}\,\text{W}\right)$$

$$\xi_i \stackrel{\text{def}}{=} \left(\dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P, A_i{}^{\oplus} \vdash \bigcirc Q}}{\Gamma, \bigcirc P, \bigcirc Q^{\sim} \vdash A_i{}^{\ominus}}\,\text{CONTRA}\right)$$

• I$\neg^+$: let $\Gamma, P \vdash (\neg A)^+$ be derived from $\Gamma, P \vdash A^{\ominus}$. By IH we have that $\Gamma, \bigcirc P \vdash A^{\ominus}$. By the I$\neg^+$ rule, $\Gamma, \bigcirc P \vdash (\neg A)^+$. Projecting the conclusion (Lem. 2), $\Gamma, \bigcirc P \vdash (\neg A)^{\oplus}$.

• E$\neg^+$: let $\Gamma, P \vdash A^{\ominus}$ be derived from $\Gamma, P \vdash (\neg A)^+$. Then the proof is of the form:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots & \vdots \\ \pi & \xi\end{matrix}}{\Gamma, \bigcirc P, A^{\oplus} \vdash (\neg A)^+}\,\text{EC}^+}{\Gamma, \bigcirc P, A^{\oplus} \vdash A^{\ominus}}\,\text{E}\neg^+}{\Gamma, \bigcirc P \vdash A^{\ominus}}\,\text{CS}$$

where:

$$\pi \stackrel{\text{def}}{=} \left(\dfrac{\dfrac{\text{IH}}{\Gamma, \bigcirc P \vdash (\neg A)^{\oplus}}}{\Gamma, \bigcirc P, A^{\oplus} \vdash (\neg A)^{\oplus}}\,\text{W}\right)$$

$$\xi \stackrel{\text{def}}{=} \left(\dfrac{\dfrac{\dfrac{}{\Gamma, \bigcirc P, A^{\oplus}, (\neg A)^{\oplus} \vdash A^{\oplus}}\,\text{Ax}}{\Gamma, \bigcirc P, A^{\oplus}, (\neg A)^{\oplus} \vdash (\neg A)^-}\,\text{I}\neg^-}{\Gamma, \bigcirc P, A^{\oplus} \vdash (\neg A)^{\ominus}}\,\text{IC}^-\right)$$

• IC$^+$: let $\Gamma, P \vdash A^{\oplus}$ be derived from $\Gamma, P, A^{\ominus} \vdash A^+$. By IH, $\Gamma, \bigcirc P, A^{\ominus} \vdash A^{\oplus}$, so by Lem. 2 we have that $\Gamma, \bigcirc P \vdash A^{\oplus}$.

• EC$^+$: let $\Gamma, P \vdash A^+$ be derived from $\Gamma, P \vdash A^\oplus$ and $\Gamma, P \vdash A^\ominus$. Then, in particular, by IH on the first premise, we have $\Gamma, \bigcirc P \vdash A^\oplus$, as required. ☐

A.2. Proof of Properties of Forcing (Lem. 9)

Lemma 42 (Monotonicity of forcing)**.** *If $\mathcal{M}, w \Vdash P$ and $w \leq w'$ then $\mathcal{M}, w' \Vdash P$.*

Proof. By induction on the measure $\#(P)$. We only check the positive propositions; the negative cases are dual—*e.g.* the proof for $(A \wedge B)^-$ is symmetric to the proof for $(A \vee B)^+$:

1. *Propositional variable, $P = \alpha^+$.* Let $\mathcal{M}, w \Vdash \alpha^+$, that is $\alpha \in \mathcal{V}^+_w$. Then by the monotonicity property we have that $\alpha \in \mathcal{V}^+_{w'}$, so $\mathcal{M}, w' \Vdash \alpha^+$.
2. *Conjunction, $P = (A \wedge B)^+$.* Let $\mathcal{M}, w \Vdash (A \wedge B)^+$, that is $\mathcal{M}, w \Vdash A^\oplus$ and $\mathcal{M}, w \Vdash B^\oplus$. Then by IH $\mathcal{M}, w' \Vdash A^\oplus$ and $\mathcal{M}, w' \Vdash B^\oplus$ so $\mathcal{M}, w' \Vdash (A \wedge B)^+$.
3. *Disjunction, $P = (A \vee B)^+$.* Let $\mathcal{M}, w \Vdash (A \vee B)^+$, that is $\mathcal{M}, w \Vdash A^\oplus$ or $\mathcal{M}, w \Vdash B^\oplus$. We consider the two possibilities. On one hand, if $\mathcal{M}, w \Vdash A^\oplus$ then by IH $\mathcal{M}, w' \Vdash A^\oplus$ so $\mathcal{M}, w' \Vdash (A \vee B)^+$. On the other hand, if $\mathcal{M}, w \Vdash B^\oplus$ then by IH $\mathcal{M}, w' \Vdash B^\oplus$ so $\mathcal{M}, w' \Vdash (A \vee B)^+$.
4. *Negation, $P = (\neg A)^+$.* Let $\mathcal{M}, w \Vdash (\neg A)^+$, that is $\mathcal{M}, w \Vdash A^\ominus$. Then by IH $\mathcal{M}, w' \Vdash A^\ominus$ so $\mathcal{M}, w' \Vdash (\neg A)^+$.
5. *Classical proposition, $P = A^\oplus$.* Let $\mathcal{M}, w \Vdash A^\oplus$, that is, for every $w'' \geq w$ we have that $\mathcal{M}, w'' \nVdash A^-$. Our goal is to prove that $\mathcal{M}, w' \Vdash A^\oplus$, so let $w'' \geq w'$ and let us check that $\mathcal{M}, w'' \nVdash A^-$. Indeed, given that $w'' \geq w' \geq w$ we have that $\mathcal{M}, w'' \nVdash A^-$.

☐

Lemma 43 (Stabilization of forcing)**.** *For every world w and every proposition P, there is a world $w' \geq w$ such that either $\mathcal{M}, w' \Vdash P$ or $\mathcal{M}, w' \Vdash P^\sim$, but not both.*

Proof. By induction on the measure $\#(P)$. We only check the positive propositions; the negative cases are dual.

1. *Propositional variable, $P = \alpha^+$ and $P^\sim = \alpha^-$.* By the stabilization property, there exists $w' \geq w$ such that $\alpha \in \mathcal{V}^+_{w'} \triangle \mathcal{V}^-_{w'}$, *i.e.* $\alpha \in \mathcal{V}^+_{w'}$ or $\alpha \in \mathcal{V}^-_{w'}$ but not both, so we consider two cases:
   - 1.1 If $\alpha \in \mathcal{V}^+_{w'} \setminus \mathcal{V}^-_{w'}$ then $\mathcal{M}, w' \Vdash \alpha^+$ and $\mathcal{M}, w' \nVdash \alpha^-$.
   - 1.2 If $\alpha \in \mathcal{V}^-_{w'} \setminus \mathcal{V}^+_{w'}$ then $\mathcal{M}, w' \Vdash \alpha^-$ and $\mathcal{M}, w' \nVdash \alpha^+$.
2. *Conjunction, $P = (A \wedge B)^+$ and $P^\sim = (A \wedge B)^-$.* By IH there is a world $w_1 \geq w$ such that either $\mathcal{M}, w_1 \Vdash A^\oplus$ or $\mathcal{M}, w_1 \Vdash A^\ominus$ but not both, so we consider two subcases:
   - 2.1 If $\mathcal{M}, w_1 \Vdash A^\oplus$ and $\mathcal{M}, w_1 \nVdash A^\ominus$, then by IH there is a world $w_2 \geq w_1$ such that either $\mathcal{M}, w_2 \Vdash B^\oplus$ or $\mathcal{M}, w_2 \Vdash B^\ominus$ but not both, so we consider two further subcases:
     - 2.1.1 If $\mathcal{M}, w_2 \Vdash B^\oplus$ and $\mathcal{M}, w_2 \nVdash B^\ominus$, then we take $w' := w_2$. By monotonicity (Lem. 42) we have that $\mathcal{M}, w_2 \Vdash A^\oplus$ so indeed $\mathcal{M}, w_2 \Vdash (A \wedge B)^+$. We are left to show that $\mathcal{M}, w_2 \nVdash (A \wedge B)^-$. We already know that $\mathcal{M}, w_2 \nVdash B^\ominus$, so to conclude it suffices to show that $\mathcal{M}, w_2 \nVdash A^\ominus$. Indeed, suppose that $\mathcal{M}, w_2 \Vdash A^\ominus$ holds. By IH there exists $w_3 \geq w_2$ such that either $\mathcal{M}, w_3 \Vdash A^\oplus$ or $\mathcal{M}, w_3 \Vdash A^\ominus$ but *not both*. However, by monotonicity (Lem. 42) — given that both $\mathcal{M}, w_2 \Vdash A^\oplus$ and $\mathcal{M}, w_2 \Vdash A^\ominus$ hold— we know that both $\mathcal{M}, w_3 \Vdash A^\oplus$ and $\mathcal{M}, w_3 \Vdash A^\ominus$ hold, a contradiction.
     - 2.1.2 If $\mathcal{M}, w_2 \Vdash B^\ominus$ and $\mathcal{M}, w_2 \nVdash B^\oplus$, then we take $w' := w_2$, and we have that $\mathcal{M}, w_2 \Vdash (A \wedge B)^-$ and $\mathcal{M}, w_2 \nVdash (A \wedge B)^+$.
   - 2.2 If $\mathcal{M}, w_1 \Vdash A^\ominus$ and $\mathcal{M}, w_1 \nVdash A^\oplus$, then we take $w' := w_1$, and we have that $\mathcal{M}, w_1 \Vdash (A \wedge B)^-$ and $\mathcal{M}, w_1 \nVdash (A \wedge B)^+$.
3. *Disjunction, $P = (A \vee B)^+$ and $P^\sim = (A \vee B)^-$.* By IH there is a world $w_1 \geq w$ such that either $\mathcal{M}, w_1 \Vdash A^\oplus$ or $\mathcal{M}, w_1 \Vdash A^\ominus$ but not both, so we consider two subcases:
   - 3.1 If $\mathcal{M}, w_1 \Vdash A^\oplus$ and $\mathcal{M}, w_1 \nVdash A^\ominus$, then we take $w' := w_1$, and we have that $\mathcal{M}, w_1 \Vdash (A \vee B)^+$ and $\mathcal{M}, w_1 \nVdash (A \vee B)^-$.
   - 3.2 If $\mathcal{M}, w_1 \Vdash A^\ominus$ and $\mathcal{M}, w_1 \nVdash A^\oplus$, then by IH there is a world $w_2 \geq w_1$ such that either $\mathcal{M}, w_2 \Vdash B^\oplus$ or $\mathcal{M}, w_2 \Vdash B^\ominus$ but not both, so we consider two further subcases:
     - 3.2.1 If $\mathcal{M}, w_2 \Vdash B^\oplus$ and $\mathcal{M}, w_2 \nVdash B^\ominus$, then we take $w' := w_2$, and we have that $\mathcal{M}, w_2 \Vdash (A \vee B)^+$ and $\mathcal{M}, w_2 \nVdash (A \vee B)^-$.
     - 3.2.2 If $\mathcal{M}, w_2 \Vdash B^\ominus$ and $\mathcal{M}, w_2 \nVdash B^\oplus$, then we take $w' := w_2$. By monotonicity (Lem. 42) we have that $\mathcal{M}, w_2 \Vdash A^\ominus$ so indeed $\mathcal{M}, w_2 \Vdash (A \vee B)^-$. We are left to show that $\mathcal{M}, w_2 \nVdash (A \vee B)^+$. We already know that $\mathcal{M}, w_2 \nVdash B^\oplus$, so we are left to show that $\mathcal{M}, w_2 \nVdash A^\oplus$. Indeed, suppose that $\mathcal{M}, w_2 \Vdash A^\oplus$ holds. By IH there exists $w_3 \geq w_2$ such that either $\mathcal{M}, w_3 \Vdash A^\oplus$ or $\mathcal{M}, w_3 \Vdash A^\ominus$ holds but *not both*. However, by monotonicity (Lem. 42) —given that both

$\mathcal{M}, w_2 \Vdash A^{\oplus}$ and $\mathcal{M}, w_2 \Vdash A^{\ominus}$ hold— we know that both $\mathcal{M}, w_3 \Vdash A^{\oplus}$ and $\mathcal{M}, w_3 \Vdash A^{\ominus}$ hold, a contradiction.

4. *Negation, $P = (\neg A)^+$ and $P^\sim = (\neg A)^-$.* By IH there is a world $w' \geq w$ such that either $\mathcal{M}, w' \Vdash A^{\oplus}$ or $\mathcal{M}, w' \Vdash A^{\ominus}$ hold but not both, so we consider two cases:
   - 4.1 If $\mathcal{M}, w' \Vdash A^{\oplus}$ and $\mathcal{M}, w' \nVdash A^{\ominus}$, then $\mathcal{M}, w' \Vdash (\neg A)^-$ and $\mathcal{M}, w' \nVdash (\neg A)^+$.
   - 4.2 If $\mathcal{M}, w' \Vdash A^{\ominus}$ and $\mathcal{M}, w' \nVdash A^{\oplus}$, then $\mathcal{M}, w' \Vdash (\neg A)^+$ and $\mathcal{M}, w' \nVdash (\neg A)^-$.
5. *Classical proposition, $P = A^{\oplus}$ and $P^\sim = A^{\ominus}$.* By IH there is a world $w' \geq w$ such that either $\mathcal{M}, w' \Vdash A^+$ or $\mathcal{M}, w' \Vdash A^-$ but not both. We consider two subcases:
   - 5.1 If $\mathcal{M}, w' \Vdash A^+$ and $\mathcal{M}, w' \nVdash A^-$, then we claim that $\mathcal{M}, w' \Vdash A^{\oplus}$ and $\mathcal{M}, w' \nVdash A^{\ominus}$. Indeed, let us prove each condition:
     - 5.1.1 In order to show that $\mathcal{M}, w' \Vdash A^{\oplus}$, it suffices to check that given $w'' \geq w'$ we have that $\mathcal{M}, w'' \nVdash A^-$. Indeed, suppose that $\mathcal{M}, w'' \Vdash A^-$. Then by IH there exists $w''' \geq w''$ such that either $\mathcal{M}, w''' \Vdash A^+$ or $\mathcal{M}, w''' \Vdash A^-$ but *not both*. However, by monotonicity (Lem. 42) —given that both $\mathcal{M}, w' \Vdash A^+$ and $\mathcal{M}, w'' \Vdash A^-$ hold, and $w' \leq w'' \leq w'''$— we know that both $\mathcal{M}, w''' \Vdash A^+$ and $\mathcal{M}, w''' \Vdash A^-$ hold, a contradiction.
     - 5.1.2 In order to show that $\mathcal{M}, w' \nVdash A^{\ominus}$, it suffices to note that $\mathcal{M}, w' \Vdash A^+$, which contradicts the definition of $\mathcal{M}, w' \Vdash A^{\ominus}$, given that accessibility is reflexive, *i.e.* $w' \leq w'$.
   - 5.2 If $\mathcal{M}, w' \Vdash A^-$ and $\mathcal{M}, w' \nVdash A^+$, then we claim that $\mathcal{M}, w' \Vdash A^{\ominus}$ and $\mathcal{M}, w' \nVdash A^{\oplus}$. Indeed, let us prove each condition:
     - 5.2.1 In order to show that $\mathcal{M}, w' \Vdash A^{\ominus}$, it suffices to check that given $w'' \geq w'$ we have that $\mathcal{M}, w'' \nVdash A^+$. Indeed, suppose that $\mathcal{M}, w'' \Vdash A^+$. Then by IH there exists $w''' \geq w''$ such that either $\mathcal{M}, w''' \Vdash A^+$ and $\mathcal{M}, w''' \Vdash A^-$ but *not both*. However, by monotonicity (Lem. 42) —given that both $\mathcal{M}, w'' \Vdash A^+$ and $\mathcal{M}, w' \Vdash A^-$ hold, and $w' \leq w'' \leq w'''$— we know that both $\mathcal{M}, w''' \Vdash A^+$ and $\mathcal{M}, w''' \Vdash A^-$ hold, a contradiction.
     - 5.2.2 In order to show that $\mathcal{M}, w' \nVdash A^{\oplus}$ it suffices to note that $\mathcal{M}, w' \Vdash A^-$, which contradicts the definition of $\mathcal{M}, w' \Vdash A^{\oplus}$, given that accessibility is reflexive, *i.e.* $w' \leq w'$.

□

Lemma 44 (Non-contradiction of forcing)**.** *If $\mathcal{M}, w \Vdash P$ then $\mathcal{M}, w \nVdash P^\sim$.*

Proof. Suppose that both $\mathcal{M}, w \Vdash P$ and $\mathcal{M}, w \Vdash P^\sim$ hold. By stabilization (Lem. 43) there is a world $w' \geq w$ such that either $\mathcal{M}, w' \Vdash P$ or $\mathcal{M}, w' \Vdash P^\sim$ but *not both*. However, by monotonicity (Lem. 42) we know that both $\mathcal{M}, w' \Vdash P$ and $\mathcal{M}, w' \Vdash P^\sim$ must hold, a contradiction. □

A.3. Proof of Soundness of PRK with respect to the Kripke semantics

Lemma 45 (Rule of classical forcing)**.**

1. $(\mathcal{M}, w \Vdash A^{\oplus})$ *if and only if, for all* $w' \geq w$, $(\mathcal{M}, w' \Vdash A^{\ominus})$ *implies* $(\mathcal{M}, w' \Vdash A^+)$.
2. $(\mathcal{M}, w \Vdash A^{\ominus})$ *if and only if, for all* $w' \geq w$, $(\mathcal{M}, w' \Vdash A^{\oplus})$ *implies* $(\mathcal{M}, w' \Vdash A^-)$.

Proof. We only prove the first item. The second one is symmetric, flipping all the signs.

($\Rightarrow$) Suppose that $\mathcal{M}, w \Vdash A^{\oplus}$, let $w' \geq w$, and let us show that the implication $(\mathcal{M}, w' \Vdash A^{\ominus}) \implies (\mathcal{M}, w' \Vdash A^+)$ holds. In fact, the implication holds vacuously, given that $\mathcal{M}, w' \Vdash A^{\oplus}$ by monotonicity (Lem. 9), and therefore $\mathcal{M}, w' \nVdash A^{\ominus}$ by non-contradiction (Lem. 9).

($\Leftarrow$) Suppose that for every $w' \geq w$ the implication $(\mathcal{M}, w' \Vdash A^{\ominus}) \implies (\mathcal{M}, w' \Vdash A^+)$ holds. Let us show that $\mathcal{M}, w \Vdash A^{\oplus}$ holds, *i.e.* that for every $w' \geq w$ we have that $\mathcal{M}, w' \nVdash A^-$. Let w' be a world such that $w' \geq w$ and, by contradiction, suppose that $\mathcal{M}, w' \Vdash A^-$. Then by non-contradiction (Lem. 9) we have that $\mathcal{M}, w' \nVdash A^+$. Hence, to obtain a contradiction, using the implication of the hypothesis, it suffices to show that $\mathcal{M}, w' \Vdash A^{\ominus}$, that is, that for every $w'' \geq w'$ we have that $\mathcal{M}, w'' \nVdash A^+$. Indeed, let $w'' \geq w'$. By monotonicity (Lem. 9) $\mathcal{M}, w'' \Vdash A^-$, so by non-contradiction (Lem. 9) $\mathcal{M}, w'' \nVdash A^+$, as required.

□

Proposition 46 (Soundness)**.** *If $\Gamma \vdash P$ is provable in* PRK, *then $\Gamma \Vdash\!\!\vdash P$.*

Proof. By induction on the derivation of $\Gamma \vdash P$. The axiom rule, and the introduction and elimination rules for conjunction, disjunction, and negation are straightforward using the definition of Kripke model. The interesting cases are the following rules:

1. ABS: let $\Gamma \vdash Q$ be derived from $\Gamma \vdash P$ and $\Gamma \vdash P^\sim$ for some strong proposition P. Suppose that $\mathcal{M}, w \Vdash\!\!\vdash \Gamma$ holds in an arbitrary world w under an arbitrary Kripke model $\mathcal{M}$, and let us show that

$\mathcal{M}, w \Vdash Q$. Note that by IH we have that $\mathcal{M}, w \Vdash P$ and $\mathcal{M}, w \Vdash P^{\sim}$. But this is impossible by non-contradiction (Lem. 9). Hence $\mathcal{M}, w \Vdash Q$.

2. IC^{+}: let $\Gamma \vdash A^{\oplus}$ be derived from $\Gamma, A^{\ominus} \vdash A^{+}$. Suppose that $\mathcal{M}, w \Vvdash \Gamma$ holds in an arbitrary world w under an arbitrary Kripke model $\mathcal{M}$, and let us show that $\mathcal{M}, w \Vdash A^{\oplus}$. We claim that for every $w' \geq w$ the implication $(\mathcal{M}, w' \Vdash A^{\ominus}) \implies (\mathcal{M}, w' \Vdash A^{+})$ holds. Indeed, suppose that $\mathcal{M}, w' \Vdash A^{\ominus}$. Moreover, by monotonicity (Lem. 9), we have that $\mathcal{M}, w' \Vvdash \Gamma$. So $\mathcal{M}, w' \Vvdash \Gamma, A^{\ominus}$ holds. Hence by IH we have that $\mathcal{M}, w' \Vdash A^{+}$. Given that the implication $(\mathcal{M}, w' \Vdash A^{\ominus}) \implies (\mathcal{M}, w' \Vdash A^{+})$ holds for all $w' \geq w$, using the rule of classical forcing (Lem. 10) we conclude that $\mathcal{M}, w \Vdash A^{\oplus}$, as required.
3. IC^{-}: similar to the IC^{+} case.
4. EC^{+}, EC^{-}: similar to the ABS case.

□

A.4. Auxiliary lemmas to prove Completeness of PRK with respect to the Kripke semantics

In the following proof we use an encoding of falsity with the pure proposition $\bot \stackrel{\text{def}}{=} (\alpha_0 \wedge \neg\alpha_0)$ for some fixed propositional variable α_0. Remark that $\Gamma \vdash \bot^{\ominus}$ is provable, being an instance of the law of non-contradiction (Ex. 3).

Lemma 47 (Consistent extension). *Let Γ be a consistent set, and let P be a proposition. Then $\Gamma \cup \{P\}$ and $\Gamma \cup \{P^{\sim}\}$ are not both inconsistent.*

Proof. Suppose that $\Gamma \cup \{P\}$ and $\Gamma \cup \{P^{\sim}\}$ are both inconsistent. In particular we have that $\Gamma, P \vdash \bot^{\oplus}$ and $\Gamma, P^{\sim} \vdash \bot^{\oplus}$. By the projection lemma (Lem. 4) we have that $\Gamma, \bigcirc P \vdash \bot^{\oplus}$ and $\Gamma, \bigcirc P^{\sim} \vdash \bot^{\oplus}$. Moreover, by contraposition (Lem. 2) we have that $\Gamma, \bot^{\ominus} \vdash \bigcirc P^{\sim}$ and $\Gamma, \bot^{\ominus} \vdash \bigcirc P$. Since $\bot^{\ominus}$ is provable (Ex. 3), applying the cut rule (Lem. 2) we have that $\Gamma \vdash \bigcirc P^{\sim}$ and $\Gamma \vdash \bigcirc P$. The generalized absurdity rule allows us to derive $\Gamma \vdash Q$ for any Q from these two sequents, so Γ is inconsistent. This contradicts the hypothesis that Γ is consistent. □

Lemma 48 (Saturation). *Let Γ be a consistent set of propositions, and let Q be a proposition such that $\Gamma \nvdash Q$. Then there exists a prime theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash Q$.*

Proof. Consider an enumeration of all propositions $(P_1, P_2, \ldots)$. We build a sequence of sets $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$, with the invariant that $\Gamma_n \nvdash Q$ for all $n \geq 0$, according to the following construction.

In the n-th step, suppose that $\Gamma_1, \ldots, \Gamma_n$ have already been constructed, and consider the first proposition P in the enumeration such that $\Gamma_n \vdash P$ but the disjunctive property fails for P, that is, either P is of the form $(A \vee B)^{+}$ with $A^{\oplus}, B^{\oplus} \notin \Gamma_n$ or P is of the form $(A \wedge B)^{-}$ with $A^{\ominus}, B^{\ominus} \notin \Gamma_n$. There are two subcases:

1. If $P = (A \vee B)^{+}$ with $A^{\oplus}, B^{\oplus} \notin \Gamma_n$, note that $\Gamma_n, A^{\oplus} \vdash Q$ and $\Gamma_n, B^{\oplus} \vdash Q$ cannot both hold simultaneously. Indeed, if both $\Gamma_n, A^{\oplus} \vdash Q$ and $\Gamma_n, B^{\oplus} \vdash Q$ hold, given that also $\Gamma_n \vdash (A \vee B)^{+}$, applying $\mathrm{E}\vee^{+}$ we would have $\Gamma_n \vdash Q$, contradicting the hypothesis. Hence we may define Γ_{n+1} as follows:
$$\Gamma_{n+1} \stackrel{\text{def}}{=} \begin{cases} \Gamma_n \cup \{A^{\oplus}\} & \text{if } \Gamma_n, A^{\oplus} \nvdash Q \\ \Gamma_n \cup \{B^{\oplus}\} & \text{otherwise} \end{cases}$$
Note that, in the second case, $\Gamma_n, B^{\oplus} \nvdash Q$ holds.
2. If $P = (A \wedge B)^{-}$ with $A^{\ominus}, B^{\ominus} \notin \Gamma_n$, the construction is similar, defining Γ_{n+1} as either $\Gamma_n \cup \{A^{\ominus}\}$ or $\Gamma_n \cup \{B^{\ominus}\}$.

Now we define Γ_ω and Γ' as follows:
$$\begin{array}{rcl} \Gamma_\omega & \stackrel{\text{def}}{=} & \bigcup_{n \in \mathbb{N}} \Gamma_n \\ \Gamma' & \stackrel{\text{def}}{=} & \Gamma_\omega \cup \{A^{\pm} \mid \Gamma_\omega \vdash A^{\pm}\} \end{array}$$
Note that $\Gamma \subseteq \Gamma_\omega \subseteq \Gamma'$. Moreover, we claim that Γ' is a prime theory:

Closure by deduction. Let $\Gamma' \vdash P$, and let us show that $P \in \Gamma'$. Since all assumptions in Γ' of the form $A^{\pm}$ are provable from Γ_ω, this means that $\Gamma_\omega \vdash P$ by the cut rule (Lem. 20). We consider four subcases, depending on whether P is a strong/classical proof/refutation. We only study the positive cases; the negative cases are symmetric:

1. *Strong proof,* i.e. $P = A^{+}$. Then $\Gamma_\omega \vdash A^{+}$ so $A^{+} \in \Gamma'$ by definition of Γ'.
2. *Classical proof,* i.e. $P = A^{\oplus}$. Then $\Gamma_\omega \vdash A^{\oplus}$ so in particular $\Gamma_\omega \vdash (A \vee A)^{+}$ applying the $\mathrm{I}\vee_1^{+}$ rule. Then there is an n_0 such that $\Gamma_n \vdash (A \vee A)^{+}$ for all $n \geq n_0$. Then it cannot be the case that $A^{\oplus} \notin \Gamma_n$ for all $n \geq n_0$, because the proposition $(A \vee A)^{+}$ must be eventually treated by the construction of $(\Gamma_n)_{n \in \mathbb{N}}$ above. This means that there is an $n \geq n_0$ such that $A^{\oplus} \in \Gamma_n$, and therefore $A^{\oplus} \in \Gamma_\omega \subseteq \Gamma'$, as required.

Consistency. It suffices to note that $\Gamma' \nvdash Q$. Indeed, suppose that $\Gamma' \vdash Q$. Then $\Gamma_\omega \vdash Q$ by the cut rule (Lem. 20), so there exists an n_0 such that $\Gamma_n \vdash Q$ for all $n \geq n_0$. This contradicts the invariant of the construction of $(\Gamma_n)_{n \in \mathbb{N}}$ above.

Disjunctive property. We consider only the positive case. The negative case is symmetric. Suppose that $\Gamma' \vdash (A \vee B)^{+}$. Then $\Gamma_\omega \vdash (A \vee B)^{+}$ by the cut rule (Lem. 20), so there exists an n_0 such that $\Gamma_n \vdash (A \vee B)^{+}$ for all $n \geq n_0$. Then it cannot be the case that $A^{\oplus}, B^{\oplus} \notin \Gamma_n$ for all $n \geq n_0$, because the proposition $(A \vee B)^{+}$ must be eventually treated by the construction of $(\Gamma_n)_{n \in \mathbb{N}}$ above. This means that there is an $n \geq n_0$ such that either $A^{\oplus} \in \Gamma_n$ or $B^{\oplus} \in \Gamma_n$, and therefore we have that either $A^{\oplus} \in \Gamma_\omega \subseteq \Gamma'$, or $B^{\oplus} \in \Gamma_\omega \subseteq \Gamma'$, as required.

Finally, note that $\Gamma' \nvdash Q$, as has already been shown in the proof of consistency above. □

Definition 49 (Canonical model)**.** The *canonical model* is the structure $\mathcal{M}_0 = (\mathcal{W}_0, \subseteq, \mathcal{V}^+, \mathcal{V}^-)$:

1. $\mathcal{W}_0$ is the set of all prime theories, *i.e.* $\mathcal{W}_0 \stackrel{\text{def}}{=} \{\Gamma \mid \Gamma \text{ is prime}\}$.
2. $\subseteq$ is the set-theoretic inclusion between prime theories.
3. $\mathcal{V}^+_\Gamma = \{\alpha \mid \alpha^+ \in \Gamma\}$ and $\mathcal{V}^-_\Gamma = \{\alpha \mid \alpha^- \in \Gamma\}$.

Lemma 50. *The canonical model is a Kripke model.*

Proof. Let us check the two required properties. **Monotonicity** is immediate, since if $\Gamma \subseteq \Gamma'$ then $\alpha^\pm \in \Gamma$ implies $\alpha^\pm \in \Gamma'$. For **stabilization**, let Γ be a prime theory and let α be a propositional variable. First note that $\Gamma \cup \{\alpha^+\}$ and $\Gamma \cup \{\alpha^-\}$ cannot both be inconsistent, by the consistent extension lemma (Lem. 47). We consider two subcases, depending on whether $\Gamma \cup \{\alpha^+\}$ is consistent:

1. *If* $\Gamma \cup \{\alpha^+\}$ *is consistent.* Then $\Gamma, \alpha^+ \nvdash \alpha^-$ because $\Gamma, \alpha^+ \vdash \alpha^-$ would make the set $\Gamma \cup \{\alpha^+\}$ inconsistent. Then by saturation (Lem. 48) there is a prime theory $\Gamma' \supseteq \Gamma \cup \{\alpha^+\}$ such that $\Gamma' \nvdash \alpha^-$. Hence we have that $\Gamma' \supseteq \Gamma$ with $\alpha \in \mathcal{V}^+_{\Gamma'} \setminus \mathcal{V}^-_{\Gamma'}$.
2. *Otherwise, so* $\Gamma \cup \{\alpha^-\}$ *is consistent.* Similarly as in the previous case, we have that $\Gamma, \alpha^- \nvdash \alpha^+$, so by saturation (Lem. 48) there is a prime theory $\Gamma' \supseteq \Gamma \cup \{\alpha^-\}$ such that $\Gamma' \nvdash \alpha^+$, and this implies that $\alpha \in \mathcal{V}^-_{\Gamma'} \setminus \mathcal{V}^+_{\Gamma'}$.

□

Lemma 51 (Main Semantic Lemma)**.** *Let* Γ *be a prime theory. Then* $\mathcal{M}_0, \Gamma \Vdash P$ *holds in the canonical model if and only if* $P \in \Gamma$.

Proof. We proceed by induction on the measure $\#(P)$. We only study the positive cases, the negative cases are symmetric.

Propositional variable, $P = \alpha^+$**.**

$$\mathcal{M}_0, \Gamma \Vdash \alpha^+ \iff \alpha \in \mathcal{V}^+_\Gamma \iff \alpha^+ \in \Gamma$$

Strong conjunction, $P = (A \wedge B)^+$**.**

$$\begin{array}{lll} & \mathcal{M}_0, \Gamma \Vdash (A \wedge B)^+ & \\ \iff & \mathcal{M}_0, \Gamma \Vdash A^\oplus \text{ and } \mathcal{M}_0, \Gamma \Vdash B^\oplus & \\ \iff & A^\oplus \in \Gamma \text{ and } B^\oplus \in \Gamma & \text{by IH} \\ \iff & (A \wedge B)^+ \in \Gamma & \end{array}$$

The last equivalence uses the fact that Γ is closed by deduction, using rule $\mathrm{I}\wedge^+$ for the "only if" direction and rules $\mathrm{E}\wedge^+_1, \mathrm{E}\wedge^+_2$ for the "if" direction.

Strong disjunction, $P = (A \vee B)^+$**.**

$$\begin{array}{lll} & \mathcal{M}_0, \Gamma \Vdash (A \vee B)^+ & \\ \iff & \mathcal{M}_0, \Gamma \Vdash A^\oplus \text{ or } \mathcal{M}_0, \Gamma \Vdash B^\oplus & \\ \iff & A^\oplus \in \Gamma \text{ or } B^\oplus \in \Gamma & \text{by IH} \\ \iff & (A \vee B)^+ \in \Gamma & \end{array}$$

The last equivalence uses the fact that Γ is a prime theory, using rules $\mathrm{I}\vee^+_1$ and $\mathrm{I}\vee^+_2$ for the "only if" direction, and the fact that Γ is disjunctive for the "if" direction.

Strong negation, $P = (\neg A)^+$**.**

$$\begin{array}{llll} \mathcal{M}_0, \Gamma \Vdash (\neg A)^+ & \iff & \mathcal{M}_0, \Gamma \Vdash A^\ominus & \\ & \iff & A^\ominus \in \Gamma & \text{by IH} \\ & \iff & (\neg A)^+ \in \Gamma & \end{array}$$

The last equivalence uses the fact that Γ is closed by deduction, using rule $\mathrm{I}\neg^+$ for the "only if" direction and rule $\mathrm{E}\neg^+$ for the "if" direction.

Classical proposition, $P = A^\oplus$**.**

$$\begin{array}{llll} \mathcal{M}_0, \Gamma \Vdash A^\oplus & \iff & \forall \Gamma' \supseteq \Gamma,\ \mathcal{M}_0, \Gamma' \nVdash A^- & \\ & \iff & \forall \Gamma' \supseteq \Gamma,\ A^- \notin \Gamma' & \text{by IH} \\ & \iff & A^\oplus \in \Gamma & \end{array}$$

Note that Γ' does not vary over arbitrary sets of propositions, but only over prime theories. To justify the last equivalence, we prove each implication separately:

($\Rightarrow$) We show the contrapositive. Let $A^\oplus \notin \Gamma$ and let us show that there is a prime theory $\Gamma' \supseteq \Gamma$ such that $A^- \in \Gamma'$. First we claim that $\Gamma \cup \{A^-\}$ is consistent.

> *Proof of the claim.* Suppose by contradiction that $\Gamma \cup \{A^-\}$ is inconsistent. Then in particular $\Gamma, A^- \vdash \bot^\oplus$. (Recall that we encode falsity as $\bot \stackrel{\text{def}}{=} (\alpha_0 \wedge \neg\alpha_0)$). By the projection lemma (Lem. 4) we have that $\Gamma, A^\ominus \vdash \bot^\oplus$. By contraposition (Lem. 2) $\Gamma, \bot^\ominus \vdash A^\oplus$. Since $\bot^\ominus$ is provable (Ex. 3), by the cut rule (Lem. 2) we have that $\Gamma \vdash A^\oplus$. But Γ is closed by deduction, so $A^\oplus \in \Gamma$. This contradicts the fact that $A^\oplus \notin \Gamma$ and concludes the proof of the claim.

Now since $\Gamma \cup \{A^-\}$ is consistent, by saturation (Lem. 48), we may extend it to a prime theory $\Gamma' \supseteq \Gamma \cup \{A^-\}$. This concludes this case.

($\Leftarrow$) Suppose that $A^\oplus \in \Gamma$, and let $\Gamma' \supseteq \Gamma$ such that $A^- \in \Gamma'$. Then since Γ' is closed by deduction, using the IC^+ rule we have that $A^\ominus \in \Gamma'$. Since Γ' contains both $A^\oplus$ and $A^\ominus$, using the generalized absurdity rule we may derive an arbitrary proposition from Γ', which means that Γ' is inconsistent, contradicting the fact that Γ' is a prime theory.

□

A.5. Proof of Subject Reduction of λ^{PRK} (Prop. 24)

Proposition 52 (Subject reduction)**.** *If* $\Gamma \vdash t : P$ *and* $t \to s$*, then* $\Gamma \vdash s : P$.

Proof. Since reduction is closed under arbitrary contexts, the term on the left hand side is of the form $\mathtt{C}\langle t_0 \rangle$ and it reduces to $\mathtt{C}\langle t_1 \rangle$ contracting the redex t_0. We proceed by induction on the context $\mathtt{C}$ under which the rewriting step takes place. The interesting case is when the context is empty. All other cases are easy by resorting to the IH. We proceed by case analysis on each of the reduction rules. Note that each rule actually stands for two rules, depending on the instantiations of the signs. We write only one of these cases; if the signs are flipped the proof is symmetric. We use the admissible typing rules Cut and Abs′ (Lem. 20).

- proj: let $i \in \{1,2\}$. We have:

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t_1 : {A_1}^{\oplus}} \qquad \dfrac{\pi_2}{\Gamma \vdash t_2 : {A_2}^{\oplus}}}{\Gamma \vdash \langle t_1, t_2\rangle^+ : (A_1 \wedge A_2)^+}\,\mathrm{I}\wedge^+}{\Gamma \vdash t_i : {A_i}^{\oplus}}\,\mathrm{E}\wedge_i^+$$

Then:

$$\dfrac{\pi_i}{\Gamma \vdash t_i : A^{\oplus}}$$

- case: let $i \in \{1,2\}$. We have:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : {A_i}^{\oplus}}}{\Gamma \vdash \mathsf{in}_i^+(t) : (A_1 \vee A_2)^+}\,\mathrm{I}\vee_i^+ \qquad \pi_1 \qquad \pi_2}{\Gamma \vdash \delta^+(\mathsf{in}_i^+(t))\,[_x.s_1][_x.s_2] : P}\,\mathrm{E}\vee^+$$

where for each $j \in \{1,2\}$, the derivation π_j is:

$$\dfrac{\pi'_j}{\Gamma, x : {A_j}^{\oplus} \vdash s_j : P}$$

Then:

$$\dfrac{\dfrac{\pi'_i}{\Gamma, x : {A_i}^{\oplus} \vdash s_i : P} \qquad \dfrac{\pi}{\Gamma \vdash t : {A_i}^{\oplus}}}{\Gamma \vdash s_i[x:=t] : P}\,\textsc{Cut}$$

- neg: We have:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : A^{\ominus}}}{\Gamma \vdash \nu^+ t : (\neg A)^+}\,\mathrm{I}\neg^+}{\Gamma \vdash \mu^+(\nu^+ t) : A^{\ominus}}\,\mathrm{E}\neg^+$$

Then:

$$\dfrac{\pi}{\Gamma \vdash t : A^{\ominus}}$$

- beta: we have:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma, x : A^{\ominus} \vdash t : A^+}}{\Gamma \vdash \mathsf{IC}_x^+.\,t : A^{\oplus}}\,\mathrm{IC}^+ \qquad \dfrac{\pi'}{\Gamma \vdash s : A^{\ominus}}}{\Gamma \vdash (\mathsf{IC}_x^+.\,t) \bullet^+ s : A^+}\,\mathrm{EC}^+$$

Then:

$$\dfrac{\dfrac{\pi}{\Gamma, x : A^{\ominus} \vdash t : A^+} \qquad \dfrac{\pi'}{\Gamma \vdash s : A^{\ominus}}}{\Gamma \vdash t[x:=s] : A^+}\,\textsc{Cut}$$

- absPairInj: we have:

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash t_1 : {A_1}^{\oplus}} \qquad \dfrac{\pi_2}{\Gamma \vdash t_2 : {A_2}^{\oplus}}}{\Gamma \vdash \langle t_1, t_2\rangle^+ : (A_1 \wedge A_2)^+}\,\mathrm{I}\wedge^+ \qquad \dfrac{\dfrac{\pi'}{\Gamma \vdash s : {A_i}^{\ominus}}}{\Gamma \vdash \mathsf{in}_i^-(s) : (A_1 \wedge A_2)^-}\,\mathrm{I}\wedge_i^-}{\Gamma \vdash \langle t_1, t_2\rangle^+ \blacktriangleright\!\!\blacktriangleleft_P \mathsf{in}_i^-(s) : P}\,\textsc{Abs}$$

Then:

$$\dfrac{\dfrac{\pi_i}{\Gamma \vdash t_i : {A_i}^{\oplus}} \qquad \dfrac{\pi'}{\Gamma \vdash s : {A_i}^{\ominus}}}{\Gamma \vdash t_i \bowtie_P s : P}\,\textsc{Abs}'$$

- absInjPair: similar to the previous case.
- absNeg:

$$\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash t : A^{\ominus}}}{\Gamma \vdash \nu^+ t : (\neg A)^+}\,\mathrm{I}\neg^+ \qquad \dfrac{\dfrac{\pi'}{\Gamma \vdash s : A^{\oplus}}}{\Gamma \vdash \nu^- s : (\neg A)^-}\,\mathrm{I}\neg^-}{\Gamma \vdash (\nu^+ t) \blacktriangleright\!\!\blacktriangleleft_P (\nu^- s) : P}\,\textsc{Abs}$$

Then:

$$\dfrac{\dfrac{\pi}{\Gamma \vdash t : A^{\ominus}} \qquad \dfrac{\pi'}{\Gamma \vdash s : A^{\oplus}}}{\Gamma \vdash t \bowtie_P s : P}\,\textsc{Abs}'$$

□

A.6. Proof of the Positivity Condition for Coro. 27

Definition 53. Recall that the set of type constraints $\mathcal{C}_{\mathbf{pn}}$ is given by all equations of the following form, for all types A, B of System F:

$$\mathbf{p}_{A,B} \equiv (\mathbf{n}_{A,B} \to A) \qquad \mathbf{n}_{A,B} \equiv (\mathbf{p}_{A,B} \to B)$$

Proposition 54. *The set of type constraints $\mathcal{C}_{\mathbf{pn}}$ verifies Mendler's positivity condition (stated in the body of the paper, and also in the appendix in Def. 77).*

Proof. Define the *complexity* of a type as follows:

$$\begin{array}{rcl} ||\alpha|| & \stackrel{\text{def}}{=} & 1 \quad \text{if } \alpha \in \mathbf{V} \\ ||\mathbf{p}_{A,B}|| = ||\mathbf{n}_{A,B}|| = ||A \to B|| & \stackrel{\text{def}}{=} & 1 + ||A|| + ||B|| \\ ||\forall\alpha.A|| & \stackrel{\text{def}}{=} & 1 + ||A|| \end{array}$$

Recall that $\mathsf{p}(A)$ (resp. $\mathsf{n}(A)$) stand for the set of type variables occurring positively (resp. negatively) in a given type A. Moreover, the set of type variables occurring *weakly positively* (resp. *weakly negatively*) in A are written $\mathsf{p}^{\mathtt{w}}(A)$ (resp. $\mathsf{n}^{\mathtt{w}}(A)$) and defined as follows:

$$\begin{array}{rcl} \mathsf{p}^{\mathtt{w}}(\alpha) & \stackrel{\text{def}}{=} & \{\alpha\} \quad \text{if } \alpha \in \mathbf{V} \\ \mathsf{p}^{\mathtt{w}}(\mathbf{p}_{A,B}) & \stackrel{\text{def}}{=} & \{\mathbf{p}_{A,B}\} \cup \mathsf{p}^{\mathtt{w}}(A) \cup \mathsf{n}^{\mathtt{w}}(B) \\ \mathsf{p}^{\mathtt{w}}(\mathbf{n}_{A,B}) & \stackrel{\text{def}}{=} & \{\mathbf{n}_{A,B}\} \cup \mathsf{n}^{\mathtt{w}}(A) \cup \mathsf{p}^{\mathtt{w}}(B) \\ \mathsf{p}^{\mathtt{w}}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{n}^{\mathtt{w}}(A) \cup \mathsf{p}^{\mathtt{w}}(B) \\ \mathsf{p}^{\mathtt{w}}(\forall\alpha.A) & \stackrel{\text{def}}{=} & \mathsf{p}^{\mathtt{w}}(A) \setminus \{\alpha\} \end{array}$$

$$\begin{array}{rcl}
\mathsf{n}^{\mathtt{w}}(\alpha) & \stackrel{\text{def}}{=} & \varnothing \quad \text{if } \alpha \in \mathbf{V} \\
\mathsf{n}^{\mathtt{w}}(\mathbf{p}_{A,B}) & \stackrel{\text{def}}{=} & \mathsf{n}^{\mathtt{w}}(A) \cup \mathsf{p}^{\mathtt{w}}(B) \\
\mathsf{n}^{\mathtt{w}}(\mathbf{n}_{A,B}) & \stackrel{\text{def}}{=} & \mathsf{p}^{\mathtt{w}}(A) \cup \mathsf{n}^{\mathtt{w}}(B) \\
\mathsf{n}^{\mathtt{w}}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{p}^{\mathtt{w}}(A) \cup \mathsf{n}^{\mathtt{w}}(B) \\
\mathsf{n}^{\mathtt{w}}(\forall\alpha.A) & \stackrel{\text{def}}{=} & \mathsf{n}^{\mathtt{w}}(A) \setminus \{\alpha\}
\end{array}$$

It is easy to check that $\mathsf{p}(A) \subseteq \mathsf{p}^{\mathtt{w}}(A)$ and $\mathsf{n}(A) \subseteq \mathsf{n}^{\mathtt{w}}(A)$ by simultaneous induction on A. It is also easy to check that if $\alpha \in \mathsf{p}^{\mathtt{w}}(A) \cup \mathsf{n}^{\mathtt{w}}(A)$ then $||\alpha|| \leq ||A||$, by induction on A. Moreover, let X, Y be types. A type A is said to be (X,Y)-*positive* if $\mathbf{p}_{X,Y} \in \mathsf{p}^{\mathtt{w}}(A)$ or $\mathbf{n}_{X,Y} \in \mathsf{n}^{\mathtt{w}}(A)$. Symmetrically, a type A is said to be (X,Y)-*negative* if $\mathbf{p}_{X,Y} \in \mathsf{n}^{\mathtt{w}}(A)$ or $\mathbf{n}_{X,Y} \in \mathsf{p}^{\mathtt{w}}(A)$. It is straightforward to prove the following **invariant** for the equivalence $A \equiv B$ between types induced by the recursive type constraints, by induction on the derivation of $A \equiv B$.

1. If $A \equiv B$, then A is (X,Y)-positive if and only if B is (X,Y)-positive.
2. If $A \equiv B$, then A is (X,Y)-negative if and only if B is (X,Y)-negative.

To prove Mendler's positivity condition, we must check that given any type variable α of the form $\mathbf{p}_{A,B}$ or of the form $\mathbf{n}_{A,B}$, then whenever $\alpha \equiv C$ we have that α does not occur negatively in C. We consider two cases, depending on whether $\alpha = \mathbf{p}_{A,B}$ or $\alpha = \mathbf{n}_{A,B}$:

1. Let $\mathbf{p}_{A,B} \equiv C$ and suppose that $\mathbf{p}_{A,B} \in \mathsf{n}(C)$. Then we have that $\mathbf{p}_{A,B} \in \mathsf{n}^{\mathtt{w}}(C)$, so C is (A,B)-negative. By the invariant, $\mathbf{p}_{A,B}$ is also (A,B)-negative, so either $\mathbf{p}_{A,B} \in \mathsf{n}^{\mathtt{w}}(\mathbf{p}_{A,B})$ or $\mathbf{n}_{A,B} \in \mathsf{p}^{\mathtt{w}}(\mathbf{p}_{A,B})$. Both conditions are impossible, indeed:
   - 1.1 Suppose that $\mathbf{p}_{A,B} \in \mathsf{n}^{\mathtt{w}}(\mathbf{p}_{A,B})$. Then, given that $\mathbf{p}_{A,B}$ does not occur weakly negatively at the root of $\mathbf{p}_{A,B}$, so it must occur either inside A or inside B, so $||\mathbf{p}_{A,B}|| < ||\mathbf{p}_{A,B}||$, which is a contradiction.
   - 1.2 Suppose that $\mathbf{n}_{A,B} \in \mathsf{n}^{\mathtt{w}}(\mathbf{p}_{A,B})$. Then, again, $\mathbf{n}_{A,B}$ must occur either inside A or inside B, so $||\mathbf{n}_{A,B}|| < ||\mathbf{p}_{A,B}||$, which is a contradiction.
2. If $\mathbf{n}_{A,B} \equiv C$ then, symmetrically as above, we have that $\mathbf{n}_{A,B} \notin \mathsf{n}(C)$.

□

A.7. Proof of the Simulation Lemma for the Translation from PRK to the Extended System F

Lemma 55. *If $t \to s$ in λ^{PRK} then $[\![t]\!] \to^+ [\![s]\!]$ in System F extended with $\mathcal{C}_{\mathbf{pn}}$.*

Proof. By case analysis on the rewriting rule used to derive the step $t \to s$. Note that showing contextual closure is immediate, so we only study the cases in which the rewriting rule is applied at the root:

• proj: $[\![\pi_i^{\pm}(\langle t_1, t_2\rangle^{\pm})]\!] = \pi_i(\langle [\![t_1]\!], [\![t_2]\!]\rangle) \to [\![t_i]\!]$.

• case:

$$\begin{array}{cl}
 & [\![\delta^{\pm}\mathsf{in}_i^{\pm}(t)\,[_{(x:P)}.s_1][_{(x:Q)}.s_2]]\!] \\
= & \delta\mathsf{in}_i([\![t]\!])\,[_{(x:[\![P]\!])}.[\![s_1]\!]][_{(x:[\![Q]\!])}.[\![s_2]\!]] \\
\to & [\![s_i]\!][x:=[\![t]\!]] \\
= & [\![s_i[x:=[\![t]\!]]]\!] \\
 & \quad \text{by Lem. 30}
\end{array}$$

• neg: $[\![\mu^{\pm}\nu^{\pm}t]\!] = (\lambda x^{\mathbf{1}}.\,[\![t]\!])\,\star \to [\![t]\!][x := \star] = [\![t]\!]$ by Lem. 30, since $x \notin \mathsf{fv}(t)$ by definition of $[\![\nu^{\pm}t]\!]$.

• beta: $[\![(\mathsf{IC}^{\pm}_{(x:P)}.\,t) \bullet^{\pm} s]\!] = (\lambda x^{[\![P]\!]}.\,[\![t]\!])\,[\![s]\!] \to [\![t]\!][x := [\![s]\!]] = [\![t[x:=s]]\!]$ by Lem. 30.

• absPairInj: we consider two subcases, depending on the signs:

1. Let $\vdash t_1 : {A_1}^{\oplus}$, $\vdash t_2 : {A_2}^{\oplus}$, and $\vdash s : {A_i}^{\ominus}$ for some $i \in \{1,2\}$. Then:

$$\begin{array}{cl}
 & [\![\langle t_1, t_2\rangle^{+} \bowtie_P \mathsf{in}_i^{-}(s)]\!] \\
= & \mathsf{abs}_P^{(A_1\wedge A_2)^+}\,\langle [\![t_1]\!], [\![t_2]\!]\rangle\,\mathsf{in}_i([\![s]\!]) \\
\to^+ & \quad \delta\mathsf{in}_i([\![s]\!]) \\
 & \qquad [_{(z:[\![{A_1}^{\ominus}]\!])}.\mathsf{abs}_P^{{A_1}^{\oplus}}\ \pi_1(\langle [\![t_1]\!], [\![t_2]\!]\rangle)\,z] \\
 & \qquad [_{(z:[\![{A_2}^{\ominus}]\!])}.\mathsf{abs}_P^{{A_2}^{\oplus}}\ \pi_2(\langle [\![t_1]\!], [\![t_2]\!]\rangle)\,z] \\
 & \quad \text{by definition of } \mathsf{abs}_P^{(A_1\wedge A_2)^+} \\
\to & \mathsf{abs}_P^{{A_i}^{\oplus}}\ \pi_i(\langle [\![t_1]\!], [\![t_2]\!]\rangle)\,[\![s]\!] \\
\to & \mathsf{abs}_P^{{A_i}^{\oplus}}\ [\![t_i]\!]\,[\![s]\!] \\
\to^+ & \mathsf{abs}_P^{{A_i}^{+}}\ ([\![t_i]\!]\,[\![s]\!])([\![s]\!]\,[\![t_i]\!]) \\
 & \quad \text{by definition of } \mathsf{abs}_P^{{A_i}^{\oplus}} \\
= & [\![(t_i \bullet^{+} s) \bowtie_P (t_i \bullet^{-} s)]\!] \\
= & [\![t_i \bowtie_P s]\!]
\end{array}$$

2. Let $\Gamma \vdash t_1 : {A_1}^{\ominus}$, $\Gamma \vdash t_2 : {A_2}^{\ominus}$, and $\Gamma \vdash s : {A_i}^{\oplus}$ for some $i \in \{1,2\}$. Then, symmetrically as for the previous case, $[\![\langle t_1, t_2\rangle^{-} \bowtie_P \mathsf{in}_i^{+}(s)]\!] \to^+ [\![t_i \bowtie_P s]\!]$.

• absInjPair: symmetric to the previous case.

• absNeg: we consider two subcases, depending on the signs:

1. Let $\Gamma \vdash t : A^{\ominus}$ and $\Gamma \vdash s : A^{\oplus}$. Then:

$$\begin{array}{cl}
 & [\![(\nu^{+}t) \bowtie_P (\nu^{-}s)]\!] \\
= & \mathsf{abs}_P^{(\neg A)^+}(\lambda x^{\mathbf{1}}.\,[\![t]\!])(\lambda y^{\mathbf{1}}.\,[\![s]\!]) \\
 & \quad \text{where } x \notin \mathsf{fv}(t),\ y \notin \mathsf{fv}(s) \\
\to^+ & \mathsf{abs}_P^{A^{\ominus}}((\lambda x^{\mathbf{1}}.\,[\![t]\!])\,\star)((\lambda y^{\mathbf{1}}.\,[\![s]\!])\,\star) \\
 & \quad \text{by definition of } \mathsf{abs}_P^{(\neg A)^+} \\
\to^+ & \mathsf{abs}_P^{A^{\ominus}}\ [\![t]\!]\,[\![s]\!] \\
\to^+ & \mathsf{abs}_P^{A^{-}}\ ([\![t]\!]\,[\![s]\!])\,([\![s]\!]\,[\![t]\!]) \\
 & \quad \text{by definition of } \mathsf{abs}_P^{A^{\ominus}} \\
= & [\![(t \bullet^{-} s) \bowtie_P (s \bullet^{+} t)]\!] \\
= & [\![t \bowtie_P s]\!]
\end{array}$$

2. Let $\Gamma \vdash t : A^{\oplus}$ and $\Gamma \vdash s : A^{\ominus}$. Then, symmetrically as for the previous case: $[\![(\nu^{-}t) \bowtie_P (\nu^{+}s)]\!] \to^{+} [\![t \bowtie_P s]\!]$.

$\square$

A.8. Proof of Characterization of Normal Forms (Prop. 34)

Proposition 56. *A term is normal if and only if it does not reduce in λ^{PRK}.*

Proof. ($\Rightarrow$) Let t be a normal term, and let us check that it is a $\to$-normal form. We proceed by induction on the derivation that t is a normal term.

The cases corresponding to introduction rules are straightforward by IH. For example, if $t = \langle N_1, N_2\rangle^{\pm}$, then by IH N_1 and N_2 have no $\to$-redexes. Moreover, there are no rules involving a pair $\langle -, -\rangle^{\pm}$ at the root, so $\langle N_1, N_2\rangle^{\pm}$ is in $\to$-normal form.

The cases corresponding to elimination rules and the absurdity rule are also straightforward by IH, observing that there cannot be a redex at the root. For example, if $t = \pi_i^{\pm}(S)$, then by IH S has no $\to$-redexes. Moreover, the only rule involving a projection $\pi_i^{\pm}(-)$ at the root is proj, which would require that $S = \langle t_1, t_2\rangle^{\pm}$. But this is impossible — as can be checked by exhaustive case analysis on S—, so t is in $\to$-normal form.

($\Leftarrow$) Let t be a $\to$-normal form, let us check that it is a normal term. We proceed by induction on the structure of the term t:

1. *Variable, x:* it is a neutral term.
2. *Absurdity, $t \bowtie_P s$:* by IH, t and s are normal terms. If either t or s is a neutral term, we are done. We are left to analyze the case in which they are not neutral terms, *i.e.* both t and s are built using introduction rules. Note that the types of t and s are Q and $Q^{\sim}$ respectively, for some strong type Q. We proceed by case analysis on the form of the proposition Q. There are four cases:
   - 2.1 *Proof/refutation of a propositional variable, $Q = \alpha^{\pm}$.* This case is impossible, since t only may be of one of the following forms: $\langle N, N\rangle^{\pm}$, $\mathsf{in}_i^{\pm}(N)$, $\mathsf{IC}^{\pm}_{x:P}.\,N$, or $\nu^{\pm}N$, none of which are of type $\alpha^{\pm}$.
   - 2.2 *Proof of a conjunction, $Q = (A \wedge B)^{+}$ or refutation of a disjunction $Q = (A \vee B)^{-}$.* Then t is of the form $\langle t_1, t_2\rangle^{\pm}$ and s is of the form $\mathsf{in}_i^{\mp}(s')$ for some $i \in \{1, 2\}$, so the rule $\mathsf{absPairInj}$ may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
   - 2.3 *Disjunction, $Q = (A \wedge B)^{\pm}$.* Then t is of the form $\mathsf{in}_i^{\pm}(s')$ for some $i \in \{1, 2\}$ and s is of the form $\langle t_1, t_2\rangle^{\mp}$, so the rule $\mathsf{absInjPair}$ may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
   - 2.4 *Negation, $Q = (\neg A)^{\pm}$.* Then t is of the form $\nu^{\pm}t'$ and s is of the form $\nu^{\mp}s'$, so the rule absNeg may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
3. *Pair, $\langle t, s\rangle^{\pm}$:* by IH, t and s are normal terms, so $\langle t, s\rangle^{\pm}$ is also a normal term.
4. *Projection, $\pi_i^{\pm}(t)$:* by IH, t is a normal term. It suffices to show that t is neutral. Indeed, if t is a normal but not neutral term, then since the type of t may be either of the form $(A \wedge B)^{+}$ or of the form $(A \vee B)^{-}$, we have that t is of the form $\langle s, u\rangle^{\pm}$. Then the rule proj may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
5. *Injection, $\mathsf{in}_i^{\pm}(t)$:* by IH, t is a normal term, so $\mathsf{in}_i^{\pm}(t)$ is also normal.
6. *Case, $\delta^{\pm}t\,[_x.s][_x.u]$:* by IH t, s and u are normal terms. It suffices to show that t is neutral. Indeed, if t is a normal but not neutral term, then since the type of t may be either of the form $(A \vee B)^{+}$ or of the form $(A \wedge B)^{-}$, we have that t is of the form $\mathsf{in}_i^{\pm}(t')$ for some $i \in \{1, 2\}$. Then the rule case may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
7. *Negation introduction, $\nu^{\pm}t$:* by IH, t is a normal term. Then $\nu^{\pm}t$ is also normal.
8. *Negation elimination, $\mu^{\pm}t$:* by IH, t is a normal term. It suffices to show that t is neutral. Indeed, if t is a normal but not neutral term, then since the type of t is of the form $(\neg A)^{\pm}$, then t is of the form $\nu^{\pm}t'$. Then the rule neg may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.
9. *Classical introduction, $\mathsf{IC}^{\pm}_{x:P}.\,t$:* by IH, t is a normal term, so $\mathsf{IC}^{\pm}_{x:P}.\,t$ is also normal.
10. *Classical elimination, $t \bullet^{\pm} s$:* by IH, t and s are normal terms. It suffices to show that t is neutral. Indeed, if t is a normal but not neutral term, then since the type of t may be either of the form $A^{\oplus}$ or of the form $A^{\ominus}$, we have that t is of the form $\mathsf{IC}^{\pm}_{x}.\,t'$. Then the rule beta may be applied at the root, contradicting the hypothesis that the term is $\to$ -normal.

$\square$

A.9. Proof of Canonicity (Thm. 35)

We give a slightly different statement of Canonicity, adding the additional hypothesis that t is already a normal form. This addition comes at no loss of generality, given that λ^{PRK} enjoys subject reduction (Prop. 24) and strong normalization (Thm. 32).

Theorem 57 (Canonicity)**.**

1. *Let $\vdash t : P$ where t is a normal form. Then t is canonical.*

2. *Let $\Gamma \vdash t : A^{\pm}$ where Γ is classical and t is a normal form. Then either t is canonical or t is of the form $\mathtt{K}\langle t' \rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion.*
3. *Let $\Gamma \vdash t : A^{\oplus}$ or $\Gamma \vdash t : A^{\ominus}$, where Γ is classical and t is a normal form. Then either $t = \mathsf{IC}^{\pm}_{x}.\, t'$ or $t = \mathtt{E}\langle t' \rangle$, where $\mathtt{E}$ is an eliminative context and t' is a variable or an open explosion.*

Proof.

1. Let $\vdash t : P$ where t is a normal form. Note, by induction on the formation rules for neutral terms (Def. 33) that a neutral term must have at least one free variable. But t is typed in the empty typing context, so it must be closed. Hence t is not a neutral term, so by Prop. 34, it must be canonical.
2. Let $\Gamma \vdash t : P$ where Γ is classical and t is a normal form. By Prop. 34 either t is canonical or it is a neutral term. If t is canonical we are done. If t is a neutral term it suffices to show the following claim, namely that if $\Gamma \vdash t : B^{\pm}$ is a derivable judgment such that Γ is classical and t is a neutral term, then t is of the form $t = \mathtt{K}\langle t' \rangle$, where $\mathtt{K}$ is a case-context and t' is an open explosion. We proceed by induction on the formation rules for neutral terms (Def. 33):

   2.1 *Variable,* $t = x$. this case is impossible, given that Γ is assumed to be classical, so $\Gamma \vdash x : P$ where P must be of the form $C^{\oplus}$ or $C^{\ominus}$, hence P cannot be of the form $B^{\pm}$.

   2.2 *Projection,* $\pi^{\pm}_{i}(S)$*:* this case is impossible, as $\Gamma \vdash \pi^{\pm}_{i}(S) : P$ where P must be of the form $C^{\oplus}$ or $C^{\ominus}$, hence P cannot be of the form $B^{\pm}$.

   2.3 *Case,* $\delta^{\pm} S\,[_{x}.N_1][_{x}.N_2]$*:* by inversion of the typing rules we have that either $\Gamma \vdash S : (A \vee B)^{+}$ or $\Gamma \vdash S : (A \wedge B)^{-}$. In both cases we may apply the IH to conclude that S is of the form $S = \mathtt{K}\langle t' \rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion. Therefore $t = \delta^{\pm}(\mathtt{K}\langle t' \rangle)\,[_{x}.N_1][_{x}.N_2]$ where now $\delta^{\pm}(\mathtt{K})\,[_{x}.N_1][_{x}.N_2]$ is a case-context.

   2.4 *Classical elimination,* $S \bullet^{\pm} N$*:* then t is an explosion under the empty case-context. Moreover, S must have at least one free variable so t is indeed an open explosion.

   2.5 *Negation elimination,* $\mu^{\pm} S$*:* this case is impossible, as $\Gamma \vdash \mu^{\pm} S : P$ where P must be of the form $C^{\oplus}$ or $C^{\ominus}$, hence P cannot be of the form $B^{\pm}$.

   2.6 *Absurdity,* $S \bowtie N$ *or* $N \bowtie S$*:* then t is an explosion under the empty case-context. Moreover, S must have at least one free variable so t is indeed an open explosion.
3. Let $\Gamma \vdash t : A^{\oplus}$ or $\Gamma \vdash t : A^{\ominus}$, where Γ is classical and t is a normal form. By Prop. 34 either t is canonical or it is a neutral term. If t is canonical, then by the constraints on its type it must be of the form $t = \mathsf{IC}^{\pm}_{x}.\, t'$, so we are done. If t is neutral, it suffices to show the following claim namely that if $\Gamma \vdash t : P$ is a derivable judgment, with $P \in \{B^{\oplus}, B^{\ominus}\}$, such that Γ is classical and t is a neutral term, then t is of the form $t = \mathtt{E}\langle t' \rangle$, where $\mathtt{E}$ is an eliminative context and t' is a variable or an open explosion. We proceed by induction on the formation rules for neutral terms (Def. 33):

   3.1 *Variable,* $t = x$. immediate, as t is a variable under the empty eliminative context.

   3.2 *Projection,* $\pi^{\pm}_{i}(S)$*:* by inversion of the typing rules, we have that either $\Gamma \vdash S : (A \wedge B)^{+}$ or $\Gamma \vdash S : (A \vee B)^{-}$. In both cases we may apply the second item of this lemma to conclude that S is of the form $S = \mathtt{K}\langle t' \rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion. Therefore $t = \pi^{\pm}_{i}(\mathtt{K}\langle t' \rangle)$, where now $\pi^{\pm}_{i}(\mathtt{K})$ is an eliminative context.

   3.3 *Case,* $\delta^{\pm} S\,[_{x}.N_1][_{x}.N_2]$*:* by inversion of the typing rules, we have that either $\Gamma \vdash S : (A \vee B)^{+}$ or $\Gamma \vdash S : (A \wedge B)^{-}$. In both cases we may apply the second item of this lemma to conclude that S is of the form $S = \mathtt{K}\langle t' \rangle$ where $\mathtt{K}$ is an eliminative context and t' is an open explosion. Therefore $t = \delta^{\pm}(\mathtt{K}\langle t' \rangle)\,[_{x}.N_1][_{x}.N_2]$, where now $\delta^{\pm}(\mathtt{K})\,[_{x}.N_1][_{x}.N_2]$ is an eliminative context.

   3.4 *Classical elimination,* $S \bullet^{\pm} N$*:* then t is an explosion under the empty eliminative context. Moreover, S must have at least one free variable so t is indeed an open explosion.

   3.5 *Negation elimination,* $\mu^{\pm} S$*:* by inversion of the typing rules, we have that $\Gamma \vdash S : (\neg A)^{\pm}$. By the second item of this lemma, S is of the form $S = \mathtt{K}\langle t' \rangle$ where $\mathtt{K}$ is a case-context and t' is an open explosion. Therefore $t = \mu^{\pm}\mathtt{K}\langle t' \rangle$, where now $\mu^{\pm}\mathtt{K}$ is an eliminative context.

   3.6 *Absurdity,* $S \bowtie N$ *or* $N \bowtie S$*:* then t is an explosion under the empty eliminative context. Moreover, S must have at least one free variable so t is indeed an open explosion.

□

A.10. Proof that $\lambda^{\text{PRK}}_{\eta}$ is Strongly Normalizing and Confluent (Thm. 37)

Lemma 58 (Local confluence)**.** *The $\lambda^{\text{PRK}}_{\eta}$-calculus has the weak Church–Rosser property.*

Proof. Let $t_0 \to t_1$ and $t_0 \to t_2$, and let us show that the diagram can be closed, *i.e.* that there is a term t_3 such that

$t_1 \to^* t_3$ and $t_2 \to^* t_3$. The proof is by induction on t_0 and by case analysis on the relative positions of the steps $t_0 \to t_1$ and $t_0 \to t_2$. Most cases are straightforward by resorting to the IH. We study only the interesting cases, when the patterns of the redexes overlap. There are two such cases:

1. beta/eta: Let $x \notin \mathsf{fv}(t)$. The overlap involves a step $(\mathsf{IC}^{\pm}_{x}.\, t \bullet^{\pm} x) \bullet^{\pm} s \xrightarrow{\mathsf{beta}} t \bullet^{\pm} s$ and a step $(\mathsf{IC}^{\pm}_{x}.\, t \bullet^{\pm} x) \bullet^{\pm} s \xrightarrow{\mathsf{eta}} t \bullet^{\pm} s$, so the diagram is trivially closed in zero rewriting steps.
2. eta/beta: Let $x \notin \mathsf{fv}(t)$. The overlap involves a step $\mathsf{IC}^{\pm}_{x}.\, (\mathsf{IC}^{\pm}_{y}.\, t) \bullet^{\pm} x \xrightarrow{\mathsf{eta}} \mathsf{IC}^{\pm}_{y}.\, t$ and a step $\mathsf{IC}^{\pm}_{x}.\, (\mathsf{IC}^{\pm}_{y}.\, t) \bullet^{\pm} x \xrightarrow{\mathsf{beta}} \mathsf{IC}^{\pm}_{x}.\, t[y:=x]$. Note that the targets of the steps are α-equivalent, so the diagram is trivially closed in zero rewriting steps.

□

Lemma 59 (Properties of reduction in $\lambda^{\text{PRK}}_{\eta}$)**.**

1. Reduction does not create free variables. *If $t \to t'$ then $\mathsf{fv}(t) \supseteq \mathsf{fv}(t')$.*
2. Substitution (I). *Let $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. If $t \to t'$ then $t[x:=s] \to t'[x:=s]$.*
3. Substitution (II). *Let $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. If $s \to s'$ then $t[x:=s] \to^* t[x:=s']$.*
4. Substitution (III). *Let $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. If $t \to^* t'$ and $s \to^* s'$ then $t[x:=s] \to^* t'[x:=s']$.*

Proof. Items 1., 2., and 3. are by induction on t. Item 4. is by induction on the sum of the lengths of the sequences $t \to^* t'$ and $s \to^* s'$, resorting to the two previous items. □

Lemma 60 (Postponement of eta steps)**.** *Let $t \xrightarrow{\mathsf{eta}} s \xrightarrow{\mathsf{r}} u$ where r is a rewriting rule other than eta. Then there exists a term s' such that $t \xrightarrow{\mathsf{r}}{}^{+} s' \xrightarrow{\mathsf{eta}}{}^{*} u$.*

Proof. By induction on t. If the eta step and the r step are not reduction steps at the root, it is immediate to conclude, resorting to the IH when appropriate.

If the eta step is at the root, then the first step is of the form $t = \mathsf{IC}^{\pm}_{x}.\, (s \bullet^{\pm} x) \xrightarrow{\mathsf{eta}} s$, where $x \notin \mathsf{fv}(s)$. Taking $s' := \mathsf{IC}^{\pm}_{x}.\, (u \bullet^{\pm} x)$ we have that $t = \mathsf{IC}^{\pm}_{x}.\, (s \bullet^{\pm} x) \xrightarrow{\mathsf{r}} \mathsf{IC}^{\pm}_{x}.\, (u \bullet^{\pm} x) \xrightarrow{\mathsf{eta}} u$, so we are done. For the last reduction step, we use the fact that reduction does not create free variables (Lem. 59).

Otherwise, we have that the eta step is *not* at the root and the r step is at the root. Then we proceed by case analysis, depending on the kind of rule applied. We only study the positive cases (the negative cases are symmetric):

1. proj: then we have that $t \xrightarrow{\mathsf{eta}} s = \pi^{+}_{i}(\langle s_1, s_2\rangle^{+}) \xrightarrow{\mathsf{proj}} s_i$. Recall that the eta step is not at the root of t. Moreover, it cannot be the case that $t = \pi^{+}_{i}(t')$ and the eta step is at the root of t', because the type of t' must be of the form $(A \wedge B)^{+}$ but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^{\pm}_{-}.\, -$, whose type is classical. This means that t must be of the form $\pi^{+}_{i}(\langle t_1, t_2\rangle^{+})$ and that the eta step is either internal to t_1 or internal to t_2, which implies that $t_1 \xrightarrow{\mathsf{eta}}{}^{*} s_1$ and $t_2 \xrightarrow{\mathsf{eta}}{}^{*} s_2$. Taking $s' := t_i$ we have that $t = \pi^{+}_{i}(\langle t_1, t_2\rangle^{+}) \xrightarrow{\mathsf{proj}} t_i \xrightarrow{\mathsf{eta}}{}^{*} s_i$, as required.
2. case: then we have that $t \xrightarrow{\mathsf{eta}} s = \delta^{+}\mathsf{in}^{+}_{i}(s_0)\, [_{y}.s_1][_{y}.s_2] \xrightarrow{\mathsf{case}} s_i[y:=s_0]$. Recall that the eta step is not at the root of t. Moreover, it cannot be the case that $t = \delta^{+}t'\, [_{y}.s_1][_{y}.s_2]$ and the eta step is at the root of t', because the type of t' must be of the form $(A \vee B)^{+}$, but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^{\pm}_{-}.\, -$, whose type is classical. This means that t must be of the form $\delta^{+}\mathsf{in}^{+}_{i}(t_0)\, [_{y}.t_1][_{y}.t_2]$ and that the eta-step is either internal to t_0, or internal to t_1, or internal to t_2, which implies that $t_0 \xrightarrow{\mathsf{eta}}{}^{*} s_0$ and $t_1 \xrightarrow{\mathsf{eta}}{}^{*} s_1$ and $t_2 \xrightarrow{\mathsf{eta}}{}^{*} s_2$. Taking $s' := t_i[y:=t_0]$ we have that $t = \delta^{+}\mathsf{in}^{+}_{i}(t_0)\, [_{y}.t_1][_{y}.t_2] \xrightarrow{\mathsf{case}} t_i[y:=t_0] \xrightarrow{\mathsf{eta}}{}^{*} s_i[y:=s_0]$ resorting to Lem. 59 for the last step.
3. neg: then we have that $t \xrightarrow{\mathsf{eta}} \mu^{+}(\nu^{+}s_1) \xrightarrow{\mathsf{neg}} s_1$. Recall that the eta-reduction step is not at the root of t. Moreover, it cannot be the case that $t = \mu^{+}t'$ and the eta-reduction step is at the root of t', because the type of t' must be of the form $(\neg A)^{+}$ but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^{\pm}_{-}.\, -$, whose type is classical. This means that t must be of the form $\mu^{+}(\nu^{+}t_1)$ and that the eta step is internal to t_1, *i.e.* $t_1 \xrightarrow{\mathsf{eta}} s_1$. Then taking $s' := t_1$ we have that $t = \mu^{+}(\nu^{+}t_1) \xrightarrow{\mathsf{neg}} t_1 \xrightarrow{\mathsf{eta}} s_1$ as required.
4. beta: then we have that $t \xrightarrow{\mathsf{eta}} (\mathsf{IC}^{+}_{y}.\, s_1) \bullet^{+} s_2 \xrightarrow{\mathsf{beta}} s_1[y := s_2]$. Recall that the eta step is not at the root of t. There are three cases, depending on the position of the eta-step:

   4.1 *Immediately to the left of the application.* That is, $t = t' \bullet^{+} s_2$ and the eta step is at the root of t', *i.e.* $t' \xrightarrow{\mathsf{eta}} \mathsf{IC}^{+}_{y}.\, s_1$ is a reduction step at the root. Then $t' = \mathsf{IC}^{+}_{x}.\, ((\mathsf{IC}^{+}_{y}.\, s_1) \bullet^{+} x)$. Hence taking $s' := s_1[y := s_2]$ we have that
   $$\begin{array}{rl} & t = (\mathsf{IC}^{+}_{x}.\, ((\mathsf{IC}^{+}_{y}.\, s_1) \bullet^{+} x)) \bullet^{+} s_2 \\ \xrightarrow{\mathsf{beta}} & (\mathsf{IC}^{+}_{y}.\, s_1) \bullet^{+} s_2 \\ \xrightarrow{\mathsf{beta}} & s_1[y:=s_2] \end{array}$$
   using two beta steps and no eta steps.

   4.2 *Inside the abstraction.* That is, $t = (\mathsf{IC}^{+}_{y}.\, t_1) \bullet^{+} s_2$ with $t_1 \xrightarrow{\mathsf{eta}} s_1$. Then taking $s' := t_1[y := s_2]$ we have that $t = (\mathsf{IC}^{+}_{y}.\, t_1) \bullet^{+} s_2 \xrightarrow{\mathsf{beta}} t_1[y:=s_2] \xrightarrow{\mathsf{eta}} s_1[y:=s_2]$ resorting to Lem. 59 for the last step.

   4.3 *To the right of the application.* That is, $t = (\mathsf{IC}^{+}_{y}.\, s_1) \bullet^{+} t_2$ with $t_2 \xrightarrow{\mathsf{eta}} s_2$. Then taking $s' := s_1[y := t_2]$ we have that $t = (\mathsf{IC}^{+}_{y}.\, s_1) \bullet^{+} t_2 \xrightarrow{\mathsf{beta}} s_1[y := t_2] \xrightarrow{\mathsf{eta}}{}^{*} s_1[y := s_2]$ resorting to Lem. 59 for the last step.

5. $\mathsf{absPairInj}$: then we have that $t \xrightarrow{\mathsf{eta}} \langle s_1, s_2\rangle^{+} \mathbin{\blacktriangleright\!\!\blacktriangleleft} \mathsf{in}^{-}_{i}(s_3) \xrightarrow{\mathsf{absPairInj}} s_i \bowtie s_3$. Recall that the eta step is not at the root of t. Moreover, it cannot be the case that $t = t' \mathbin{\blacktriangleright\!\!\blacktriangleleft} \mathsf{in}^{-}_{i}(s_3)$ and the eta step is at the root of t', because the type of t' must be of the form $(A \wedge B)^{+}$, but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^{\pm}_{-}.\,-$, whose type is classical. For similar reasons, it cannot be the case that $t = \langle s_1, s_2\rangle^{+} \mathbin{\blacktriangleright\!\!\blacktriangleleft} t'$ with the eta step is at the root of t', because then the type of t' must be of the form $(A \wedge B)^{-}$. This means that t must be of the form $\langle t_1, t_2\rangle^{+} \mathbin{\blacktriangleright\!\!\blacktriangleleft} \mathsf{in}^{-}_{i}(t_3)$ and that the eta step is either internal to t_1, or internal to t_2, or internal to t_3. This implies that $t_1 \xrightarrow{\mathsf{eta}}{}^{*} s_1$ and $t_2 \xrightarrow{\mathsf{eta}}{}^{*} s_2$ and $t_3 \xrightarrow{\mathsf{eta}}{}^{*} s_3$. Taking $s' := t_i \bowtie t_3$ we have that $t = \langle t_1, t_2\rangle^{+} \mathbin{\blacktriangleright\!\!\blacktriangleleft} \mathsf{in}^{-}_{i}(t_3) \xrightarrow{\mathsf{absPairInj}} t_i \bowtie t_3 = (t_i \bullet^{+} t_3) \mathbin{\blacktriangleright\!\!\blacktriangleleft} (t_3 \bullet^{-} t_i) \xrightarrow{\mathsf{eta}}{}^{*} (s_i \bullet^{+} s_3) \mathbin{\blacktriangleright\!\!\blacktriangleleft} (s_3 \bullet^{-} s_i) = s_i \bowtie s_3$.
6. $\mathsf{absInjPair}$: Symmetric to the previous case.
7. absNeg: then we have that $t \xrightarrow{\mathsf{eta}} (\nu^{+} s_1) \mathbin{\blacktriangleright\!\!\blacktriangleleft} (\nu^{-} s_2) \xrightarrow{\mathsf{absNeg}} s_1 \bowtie s_2$. Recall that the eta step is not at the root of t. Moreover, it cannot be the case that $t = t' \mathbin{\blacktriangleright\!\!\blacktriangleleft} (\nu^{-} s_2)$ and the eta step is at the root of t', because the type of t' must be of the form $(\neg A)^{+}$, but the eta rule can only be applied on a term constructed with a $\mathsf{IC}^{\pm}_{-}.\,-$, whose type is classical. For similar reasons, it cannot be the case that $t = \nu^{+} s_1 \mathbin{\blacktriangleright\!\!\blacktriangleleft} t'$ with the eta step is at the root of t', because then the type of t' must be of the form $(\neg A)^{-}$. This means that t must be of the form $(\nu^{+} t_1) \mathbin{\blacktriangleright\!\!\blacktriangleleft} (\nu^{-} t_2)$ and that the eta step is either internal to t_1 or internal to t_2. This implies that $t_1 \xrightarrow{\mathsf{eta}}{}^{*} s_1$ and $t_2 \xrightarrow{\mathsf{eta}}{}^{*} s_2$. Taking $s' := t_1 \bowtie t_2$ we have that $t = (\nu^{+} t_1) \mathbin{\blacktriangleright\!\!\blacktriangleleft} (\nu^{-} t_2) \xrightarrow{\mathsf{absNeg}} t_1 \bowtie t_2 = (t_1 \bullet^{+} t_2) \mathbin{\blacktriangleright\!\!\blacktriangleleft} (t_2 \bullet^{+} t_1) \xrightarrow{\mathsf{eta}}{}^{*} (s_1 \bullet^{+} s_2) \mathbin{\blacktriangleright\!\!\blacktriangleleft} (s_2 \bullet^{+} s_1) = s_1 \bowtie s_2$.

□

Theorem 61. *The $\lambda^{\text{PRK}}_{\eta}$-calculus is strongly normalizing and confluent.*

Proof. Strong normalization follows from postponement of the eta rule (Lem. 60) and strong normalization of the calculus without eta (Thm. 32) by the usual rewriting techniques.

More precisely, let us write $\xrightarrow{\neg\mathsf{eta}}$ for reduction not using eta, that is, $\xrightarrow{\neg\mathsf{eta}} \stackrel{\text{def}}{=} (\to \setminus \xrightarrow{\mathsf{eta}})$. Suppose there is an infinite reduction sequence $t_1 \to t_2 \to t_3 \ldots$ in $\lambda^{\text{PRK}}_{\eta}$. Let $t_1 \xrightarrow{\neg\mathsf{eta}}{}^{*} t_i$ be the longest prefix of the sequence whose steps are not eta steps. This prefix cannot be infinite given that λ^{PRK} is strongly normalizing. Let $t_i \xrightarrow{\mathsf{eta}}{}^{*} t_{i+n}$ be the longest sequence of eta steps starting on t_i. This sequence cannot be infinite given that an eta step decreases the size of the term. Now there must be a step $t_{i+n} \xrightarrow{\neg\mathsf{eta}} t_{i+n+1}$. Applying the postponement lemma (Lem. 60) n times, we obtain an infinite sequence of the form $t_1 \xrightarrow{\neg\mathsf{eta}}{}^{*} t_i \xrightarrow{\neg\mathsf{eta}} t'_{i+1} \ldots$. By repeatedly applying this argument, we may build an infinite sequence of $\xrightarrow{\neg\mathsf{eta}}$ steps, contradicting the fact that λ^{PRK} is strongly normalizing.

Confluence of $\lambda^{\text{PRK}}_{\eta}$ follows from the fact that it is strongly normalizing and locally confluent (Lem. 58), resorting to Newman's Lemma [36, Theorem 1.2.1]. □

A.11. Computation Rules for the Embedding of Classical Logic into PRK

The statements of all of the following lemmas are in $\lambda^{\text{PRK}}_{\eta}$ (with eta reduction).

A.11.1. Simulation of conjunction.

Definition 62 (Conjunction introduction)**.** Let $\Gamma \vdash t : A^{\oplus}$ and $\Gamma \vdash s : B^{\oplus}$. Then $\Gamma \vdash \langle t, s\rangle^{\mathcal{C}} : (A \wedge B)^{\oplus}$ where:

$$\langle t, s\rangle^{\mathcal{C}} \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{(\_:(A\wedge B)^{\ominus})}.\, \langle t, s\rangle^{+}$$

Definition 63 (Conjunction elimination)**.** Let $\Gamma \vdash t : (A_1 \wedge A_2)^{\oplus}$. Then $\Gamma \vdash \pi^{\mathcal{C}}_{i}(t) : A_i{}^{\oplus}$ where:

$$\pi^{\mathcal{C}}_{i}(t) \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{(x:A_i{}^{\ominus})}.\, \pi^{+}_{i}(t \bullet^{+} \mathsf{IC}^{-}_{(\_:(A_1\wedge A_2)^{\oplus})}.\, \mathsf{in}^{-}_{i}(x)) \bullet^{+} x$$

Lemma 64. $\pi^{\mathcal{C}}_{i}(\langle t_1, t_2\rangle^{\mathcal{C}}) \to^{*} t_i$

Proof.

$$\begin{array}{cl}
 & \pi^{\mathcal{C}}_{i}(\langle t_1, t_2\rangle^{\mathcal{C}}) \\
= & \mathsf{IC}^{+}_{x:A_i{}^{\ominus}}.\, \pi^{+}_{i}((\mathsf{IC}^{+}_{_}.\, \langle t_1, t_2\rangle^{+}) \bullet^{+} \mathsf{IC}^{-}_{_}.\, \mathsf{in}^{-}_{i}(x)) \bullet^{+} x \\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}^{+}_{x:A_i{}^{\ominus}}.\, \pi^{+}_{i}(\langle t_1, t_2\rangle^{+}) \bullet^{+} x \\
\xrightarrow{\mathsf{proj}} & \mathsf{IC}^{+}_{x:A_i{}^{\ominus}}.\, t_i \bullet^{+} x \\
\xrightarrow{\mathsf{eta}} & t_i
\end{array}$$

□

A.11.2. Simulation of disjunction.

Definition 65 (Disjunction introduction)**.** Let $\Gamma \vdash t : A_i{}^{\oplus}$. Then $\Gamma \vdash \mathsf{in}^{\mathcal{C}}_{i}(t) : (A_1 \vee A_2)^{\oplus}$ where:

$$\mathsf{in}^{\mathcal{C}}_{i}(t) \stackrel{\text{def}}{=} \mathsf{IC}^{+}_{(\_:(A_1\vee A_2)^{\ominus})}.\, \mathsf{in}^{+}_{i}(t)$$

Definition 66 (Disjunction elimination)**.** Let $\Gamma \vdash t : (A \vee B)^{\oplus}$ and $\Gamma, x : A^{\oplus} \vdash s : C^{\oplus}$ and $\Gamma, x : B^{\oplus} \vdash u : C^{\oplus}$. Then $\Gamma \vdash \delta^{\mathcal{C}} t\, [_{(x:A^{\oplus})}.s][_{(x:B^{\oplus})}.u] : C^{\oplus}$, where:

$$\begin{array}{ll}
\mathsf{IC}^{+}_{(y:C^{\ominus})}. & \delta^{+}\ (t \bullet^{+} \mathsf{IC}^{-}_{(_:(A\vee B)^{\oplus})}.\, \langle \updownarrow^{y}_{x}(s), \updownarrow^{y}_{x}(u)\rangle^{-}) \\
 & \quad [_{(x:A^{\oplus})}.s \bullet^{+} y] \\
 & \quad [_{(x:B^{\oplus})}.u \bullet^{+} y]
\end{array}$$

Lemma 67. $\delta^{\mathcal{C}} \mathsf{in}^{\mathcal{C}}_{i}(t_i)\, [_{x}.s_1][_{x}.s_2] \to^{*} s_i[x := t]$

Proof.

$$
\begin{array}{ll}
 & \delta^{\mathcal{C}}\mathsf{in}_i^{\mathcal{C}}(t_i)\,[_x.s_1][_x.s_2] \\
= & \mathsf{IC}^+_{(y:C\ominus)}.\\
 & \delta^+\left(\begin{array}{l}\mathsf{IC}^+_{(\_:(A_1\vee A_2)\ominus)}.\,\mathsf{in}_i^+(t)\\ \bullet^+\,\mathsf{IC}^-_{(\_:(A\vee B)\oplus)}.\,\langle \updownarrow^y_x(s_1), \updownarrow^y_x(s_2)\rangle^-\end{array}\right)\\
 & \quad [_{(x:A\oplus)}.s_1 \bullet^+ y]\\
 & \quad [_{(x:B\oplus)}.s_2 \bullet^+ y]\\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}^+_{(y:C\ominus)}.\,\delta^+\mathsf{in}_i^+(t)\,[_{(x:A\oplus)}.s_1 \bullet^+ y][_{(x:B\oplus)}.s_2 \bullet^+ y]\\
\xrightarrow{\mathsf{case}} & \mathsf{IC}^+_{(y:C\ominus)}.\,s_i[x:=t] \bullet^+ y\\
\xrightarrow{\mathsf{eta}} & s_i[x:=t]
\end{array}
$$

□

A.11.3. Simulation of negation.

Definition 68 (Negation introduction)**.** By Lem. 20 we have that $\Gamma \vdash \pitchfork^-_{\alpha_0} : (\alpha_0 \wedge \neg\alpha_0)^\ominus$, that is $\Gamma \vdash \pitchfork^-_{\alpha_0} : \bot^\ominus$. Moreover, suppose that $\Gamma, x : A^\oplus \vdash t : \bot^\oplus$. Then $\Gamma \vdash \Lambda^{\mathcal{C}}_{(x:A\oplus)}.\,t : (\neg A)^\oplus$, where:

$$\Lambda^{\mathcal{C}}_{(x:A\oplus)}.\,t \stackrel{\text{def}}{=} \mathsf{IC}^+_{(\_:(\neg A)\ominus)}.\,\nu^+\mathsf{IC}^-_{(x:A\oplus)}.\,(t \bowtie_{A^-} \pitchfork^-_{\alpha_0})$$

Definition 69 (Negation elimination)**.** Let $\Gamma \vdash t : (\neg A)^\oplus$ and $\Gamma \vdash s : A^\oplus$. Then $\Gamma \vdash t \#^{\mathcal{C}} s : \bot^\oplus$, where:

$$t \#^{\mathcal{C}} s \stackrel{\text{def}}{=} t \bowtie_{\bot^\oplus} \mathsf{IC}^-_{(\_:(\neg A)\oplus)}.\,\nu^- s$$

Lemma 70. $(\Lambda^{\mathcal{C}}_x.\,t)\#^{\mathcal{C}} s \to^* (t[x := s] \bowtie \pitchfork^-_{\alpha_0}) \blacktriangleright\!\!\blacktriangleleft (s \bullet^- (\mathsf{IC}^-_x.\,(t \bowtie \pitchfork^-_{\alpha_0})))$

Proof.

$$
\begin{array}{ll}
 & (\Lambda^{\mathcal{C}}_x.\,t)\#^{\mathcal{C}} s\\
= & (\mathsf{IC}^+_\_.\,\nu^+\mathsf{IC}^-_x.\,(t \bowtie \pitchfork^-_{\alpha_0})) \bowtie_{\bot^\oplus} \mathsf{IC}^-_\_.\,\nu^- s\\
= & \quad ((\mathsf{IC}^+_\_.\,\nu^+\mathsf{IC}^-_x.\,(t \bowtie \pitchfork^-_{\alpha_0})) \bullet^+ \mathsf{IC}^-_\_.\,\nu^- s)\\
 & \blacktriangleright\!\!\blacktriangleleft(\mathsf{IC}^-_\_.\,\nu^- s \bullet^- (\mathsf{IC}^+_\_.\,\nu^+\mathsf{IC}^-_x.\,(t \bowtie \pitchfork^-_{\alpha_0})))\\
\xrightarrow{\mathsf{beta}}(2) & (\nu^+\mathsf{IC}^-_x.\,(t \bowtie \pitchfork^-_{\alpha_0})) \blacktriangleright\!\!\blacktriangleleft (\nu^- s)\\
\xrightarrow{\mathsf{absNeg}} & ((\mathsf{IC}^-_x.\,(t \bowtie \pitchfork^-_{\alpha_0})) \bullet^+ s) \blacktriangleright\!\!\blacktriangleleft (s \bullet^- (\mathsf{IC}^-_x.\,(t \bowtie \pitchfork^-_{\alpha_0})))\\
\xrightarrow{\mathsf{beta}} & (t[x:=s] \bowtie \pitchfork^-_{\alpha_0}) \blacktriangleright\!\!\blacktriangleleft (s \bullet^- (\mathsf{IC}^-_x.\,(t \bowtie \pitchfork^-_{\alpha_0})))
\end{array}
$$

□

A.11.4. Simulation of implication.

Define implication $A \Rightarrow B$ as an abbreviation of $\neg A \vee B$.

Definition 71 (Implication introduction)**.** If $\Gamma, x : A^\oplus \vdash t : B^\oplus$ then $\Gamma \vdash \lambda^{\mathcal{C}}_{(x:A)}.\,t : (A \Rightarrow B)^\oplus$ where:

$$
\begin{array}{rcl}
\lambda^{\mathcal{C}}_x.\,t & \stackrel{\text{def}}{=} & \mathsf{IC}^+_{(y:(A\Rightarrow B)\ominus)}.\,\mathsf{in}_2^+(t[x:=\mathbf{X}_y])\\
\mathbf{X}_y & \stackrel{\text{def}}{=} & \mathsf{IC}^+_{(z:A\ominus)}.\,(\mu^-(\mathbf{X'}_{y,z} \bullet^- \mathsf{IC}^+_{(\_:(\neg A)\ominus)}.\,\nu^+ z)) \bullet^+ z\\
\mathbf{X'}_{y,z} & \stackrel{\text{def}}{=} & \pi_1^+(y \bullet^- \mathsf{IC}^+_{(\_:(A\Rightarrow B)\ominus)}.\,\mathsf{in}_1^+(\mathsf{IC}^+_{(\_:(\neg A)\ominus)}.\,\nu^+ z))
\end{array}
$$

Definition 72 (Implication elimination)**.** If $\Gamma \vdash t : (A \Rightarrow B)^\oplus$ and $\Gamma \vdash s : A^\oplus$, then $\Gamma \vdash t \mathbin{@^{\mathcal{C}}} s : B^\oplus$, where:

$$
\begin{array}{l}
t \mathbin{@^{\mathcal{C}}} s \stackrel{\text{def}}{=} \mathsf{IC}^+_{(x:B\ominus)}.\\
\quad \delta^+\,(t \bullet^+ \mathsf{IC}^-_{(\_:(A\Rightarrow B)\oplus)}.\,\langle(\mathsf{IC}^-_{(\_:(\neg A)\oplus)}.\,\nu^- s), x\rangle^-)\\
\qquad [_{(y:(\neg A)\oplus)}.s \bowtie_{B^+} \mu^-(y \bullet^+ \mathsf{IC}^-_{(\_:(\neg A)\oplus)}.\,\nu^- x)]\\
\qquad [_{(z:B\oplus)}.z \bullet^+ x]
\end{array}
$$

The following lemma is the computational rule for implication:

Lemma 73. $(\lambda^{\mathcal{C}}_x.\,t) \mathbin{@^{\mathcal{C}}} s \to^* t[x:=s]$

Proof. First let $u = \mathsf{IC}^-_\_.\,\langle(\mathsf{IC}^-_\_.\,\nu^- s), x'\rangle^-$ and note that:

$$
\begin{array}{ll}
 & \mathbf{X'}_{u,z}\\
= & \pi_1^+(u \bullet^- \mathsf{IC}^+_{(\_:(A\Rightarrow B)\ominus)}.\,\mathsf{in}_1^+(\mathsf{IC}^+_{(\_:(\neg A)\ominus)}.\,\nu^+ z))\\
\xrightarrow{\mathsf{beta}} & \pi_1^+(\langle(\mathsf{IC}^-_\_.\,\nu^- s), x'\rangle^-)\\
\xrightarrow{\mathsf{proj}} & \mathsf{IC}^-_\_.\,\nu^- s
\end{array}
$$

Hence:

$$
\begin{array}{ll}
 & \mathbf{X}_u\\
= & \mathsf{IC}^+_{(z:A\ominus)}.\,(\mu^-(\mathbf{X'}_{u,z} \bullet^- \mathsf{IC}^+_{(\_:(\neg A)\ominus)}.\,\nu^+ z)) \bullet^+ z\\
\to^* & \mathsf{IC}^+_{(z:A\ominus)}.\,(\mu^-((\mathsf{IC}^-_\_.\,\nu^- s) \bullet^- \mathsf{IC}^+_{(\_:(\neg A)\ominus)}.\,\nu^+ z)) \bullet^+ z\\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}^+_{(z:A\ominus)}.\,(\mu^-(\nu^- s)) \bullet^+ z\\
\xrightarrow{\mathsf{neg}} & \mathsf{IC}^+_{(z:A\ominus)}.\,s \bullet^+ z\\
\xrightarrow{\mathsf{eta}} & s
\end{array}
$$

Hence:

$$
\begin{array}{ll}
 & (\lambda^{\mathcal{C}}_x.\,t) \mathbin{@^{\mathcal{C}}} s\\
= & \mathsf{IC}^+_{x'}.\\
 & \delta^+\left(\begin{array}{l}(\mathsf{IC}^+_y.\,\mathsf{in}_2^+(t[x:=\mathbf{X}_y]))\\ \bullet^+\,\mathsf{IC}^-_\_.\,\langle(\mathsf{IC}^-_\_.\,\nu^- s), x'\rangle^-\end{array}\right)\\
 & \quad [_{y'}.s \bowtie_{B^+} \mu^-(y' \bullet^+ \mathsf{IC}^-_\_.\,\nu^- x')]\\
 & \quad [_{z'}.z' \bullet^+ x']\\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}^+_{x'}.\\
 & \delta^+\,\mathsf{in}_2^+(t[x:=\mathbf{X}_u])\\
 & \quad [_{y'}.s \bowtie_{B^+} \mu^-(y' \bullet^+ \mathsf{IC}^-_\_.\,\nu^- x')]\\
 & \quad [_{z'}.z' \bullet^+ x']\\
\xrightarrow{\mathsf{case}} & \mathsf{IC}^+_{x'}.\,t[x:=\mathbf{X}_u] \bullet^+ x'\\
\to^* & \mathsf{IC}^+_{x'}.\,t[x:=s] \bullet^+ x'\\
\xrightarrow{\mathsf{eta}} & t[x:=s]
\end{array}
$$

□

A.11.5. Computational content of the law of excluded middle.

Lemma 74.

$$
\begin{array}{ll}
 & \delta^{\mathcal{C}}\,\pitchfork^{\mathcal{C}}_A\;[_x.s_1][_x.s_2]\\
\to^* & \mathsf{IC}^+_y.\,s_2[x:=\mathsf{IC}^+_\_.\,\nu^+(\mathsf{IC}^-_x.\,s_1 \bowtie y)] \bullet^+ y
\end{array}
$$

Proof. Recall that $\pitchfork^{\mathcal{C}}_A = \pitchfork^+_A$, where:

$$
\begin{array}{l}
\pitchfork^+_A \stackrel{\text{def}}{=} \mathsf{IC}^+_{(x:(A\vee\neg A)\ominus)}.\,\mathsf{in}_2^+(\mathsf{IC}^+_{(y:\neg A\ominus)}.\,\nu^+\pi_1^-(x \bullet^- \Delta^+_{y,A}))\\
\Delta^+_{y,A} \stackrel{\text{def}}{=} \mathsf{IC}^+_{(\_:(A\vee\neg A)\ominus)}.\,\mathsf{in}_1^+(\mathsf{IC}^+_{(z:A\ominus)}.\,(y \bowtie_{A^+} \mathsf{IC}^+_{(\_:\neg A\ominus)}.\,\nu^+ z))
\end{array}
$$

Let $u = \mathsf{IC}^-_{\_}.\, \langle \updownarrow^{y'}_{x'} (s_1), \updownarrow^{y'}_{x'} (s_2) \rangle^-$. Then:

$$
\begin{array}{rl}
& \delta^{\mathcal{C}}\ \pitchfork^{\mathcal{C}}_{A}\ [_{x'}.s_1][_{x'}.s_2] \\
= & \mathsf{IC}^+_{y'}. \\
& \quad \delta^+ \left(\begin{array}{l} \quad \mathsf{IC}^+_{x}.\, \mathsf{in}^+_2(\mathsf{IC}^+_{y}.\, \nu^+ \pi^-_1 (x \bullet^- \Delta^+_{y,A})) \\ \bullet^+ u \end{array} \right) \\
& \quad\quad [_{x'}.s_1 \bullet^+ y'] \\
& \quad\quad [_{x'}.s_2 \bullet^+ y'] \\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}^+_{y'}. \\
& \quad \delta^+\ \mathsf{in}^+_2(\mathsf{IC}^+_{y}.\, \nu^+ \pi^-_1 (u \bullet^- \Delta^+_{y,A})) \\
& \quad\quad [_{x'}.s_1 \bullet^+ y'] \\
& \quad\quad [_{x'}.s_2 \bullet^+ y'] \\
\xrightarrow{\mathsf{case}} & \mathsf{IC}^+_{y'}.\, s_2[x' := \mathsf{IC}^+_{y}.\, \nu^+ \pi^-_1 (u \bullet^- \Delta^+_{y,A})] \bullet^+ y' \\
\xrightarrow{\mathsf{beta}} & \mathsf{IC}^+_{y'}.\, s_2[x' := \mathsf{IC}^+_{\_}.\, \nu^+ \pi^-_1 (\langle \updownarrow^{y'}_{x'} (s_1), \updownarrow^{y'}_{x'} (s_2) \rangle^-)] \bullet^+ y' \\
\xrightarrow{\mathsf{proj}} & \mathsf{IC}^+_{y'}.\, s_2[x' := \mathsf{IC}^+_{\_}.\, \nu^+ \updownarrow^{y'}_{x'} (s_1)] \bullet^+ y' \\
= & \mathsf{IC}^+_{y'}.\, s_2[x' := \mathsf{IC}^+_{\_}.\, \nu^+ (\mathsf{IC}^-_{x'}.\, s_1 \bowtie y')] \bullet^+ y'
\end{array}
$$

□

B. Formal Systems

B.1. System F Extended with Recursive Type Constraints

Definition 75 (System F$\langle\mathcal{C}\rangle$)**.** The set of *types* is given by:

$$A, B, \ldots ::= \alpha \mid A \to B \mid \forall \alpha. A$$

The set of *terms* is given by:

$$t, s, \ldots ::= x \mid \lambda x^A.\, t \mid t\, s \mid \lambda \alpha.\, t \mid t\, A$$

we omit type annotations over variables when clear from the context. A *type constraint* is an equation of the form $\alpha \equiv A$. Each set $\mathcal{C}$ of type constraints induces a notion of equivalence between types, written $A \equiv B$ and defined as the congruence generated by $\mathcal{C}$. More precisely:

$$\frac{(A \equiv B) \in \mathcal{C}}{A \equiv B}\ \textsc{constr} \qquad \frac{}{A \equiv A}\ \textsc{refl} \qquad \frac{A \equiv B}{B \equiv A}\ \textsc{sym}$$

$$\frac{A \equiv B \qquad B \equiv C}{A \equiv C}\ \textsc{trans} \qquad \frac{A \equiv B}{C[\alpha := A] \equiv C[\alpha := B]}\ \textsc{cong}$$

We suppose that $\mathcal{C}$ is fixed. Typing judgments are of the form $\Gamma \vdash t : A$.

$$\frac{}{\Gamma, x : A \vdash x : A}\ \textsc{Ax} \qquad \frac{\Gamma \vdash t : A \qquad A \equiv B}{\Gamma \vdash t : B}\ \textsc{Conv}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.\, t : A \to B}\ \mathrm{I}{\to} \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash s : A}{\Gamma \vdash t\, s : B}\ \mathrm{E}{\to}$$

$$\frac{\Gamma \vdash t : A \qquad \alpha \notin \mathsf{fv}(\Gamma)}{\Gamma \vdash \lambda \alpha.\, t : \forall \alpha. A}\ \mathrm{I}\forall \qquad \frac{\Gamma \vdash t : \forall \alpha. A}{\Gamma \vdash t\, B : A[\alpha := B]}\ \mathrm{E}\forall$$

Reduction is defined as the closure by arbitrary contexts of the following rewriting rules:

$$\begin{array}{rcl} (\lambda x.\, t)\, s & \to & t[x := s] \\ (\lambda \alpha.\, t)\, A & \to & t[\alpha := A] \end{array}$$

Definition 76 (Positive/negative occurrences)**.** The set of type variables occurring positively (resp. negatively) in a type A are written $\mathsf{p}(A)$ (resp. $\mathsf{n}(A)$) and defined by:

$$\begin{array}{rcl@{\qquad}rcl} \mathsf{p}(\alpha) & \stackrel{\text{def}}{=} & \{\alpha\} & \mathsf{n}(\alpha) & \stackrel{\text{def}}{=} & \varnothing \\ \mathsf{p}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{n}(A) \cup \mathsf{p}(B) & \mathsf{n}(A \to B) & \stackrel{\text{def}}{=} & \mathsf{p}(A) \cup \mathsf{n}(B) \\ \mathsf{p}(\forall \alpha. A) & \stackrel{\text{def}}{=} & \mathsf{p}(A) \setminus \{\alpha\} & \mathsf{n}(\forall \alpha. A) & \stackrel{\text{def}}{=} & \mathsf{n}(A) \setminus \{\alpha\} \end{array}$$

Definition 77 (Positivity condition)**.** A set of type constraints $\mathcal{C}$ verifies the *positivity condition* if for every type constraint $(\alpha \equiv A) \in \mathcal{C}$ and every type B such that $\alpha \equiv B$ one has that $\alpha \notin \mathsf{n}(B)$.

Theorem 78 (Mendler)**.** *If $\mathcal{C}$ verifies the positivity condition, then System F$\langle\mathcal{C}\rangle$ is strongly normalizing.*

Proof. See [16, Theorem 13]. □

Abbreviations. We define the following standard abbreviations for types:

$$
\begin{array}{rcl}
\mathbf{1} & \stackrel{\text{def}}{=} & \forall\alpha.(\alpha \to \alpha) \\
\mathbf{0} & \stackrel{\text{def}}{=} & \forall\alpha.\alpha \\
\neg A & \stackrel{\text{def}}{=} & A \to \mathbf{0} \\
A \times B & \stackrel{\text{def}}{=} & \forall\alpha.((A \to B \to \alpha) \to \alpha) \\
A + B & \stackrel{\text{def}}{=} & \forall\alpha.((A \to \alpha) \to (B \to \alpha) \to \alpha)
\end{array}
$$

And the following terms. We omit the typing contexts for succintness:

$$
\begin{array}{rcl}
\star & \stackrel{\text{def}}{=} & \lambda\alpha.\, \lambda x^{\alpha}.\, x \\
 & : & \mathbf{1} \\
 & & \\
\mathcal{E}_A(t) & \stackrel{\text{def}}{=} & t\, A \\
 & : & A \\
 & & \text{if } t : \mathbf{0} \\
 & & \\
\langle t, s \rangle & \stackrel{\text{def}}{=} & \lambda\alpha.\, \lambda f^{A \to B \to \alpha}.\, f\, t\, s \\
 & : & A \times B \\
 & & \text{if } t : A \text{ and } s : B \\
 & & \\
\pi_i(t) & \stackrel{\text{def}}{=} & t\, A_i\, (\lambda x_1^{A_1}.\, \lambda x_2^{A_2}.\, x_i) \\
 & : & A_i \\
 & & \text{if } t : A_1 \times A_2 \\
 & & \\
\mathsf{in}_i(t) & \stackrel{\text{def}}{=} & \lambda\alpha.\, \lambda f_1^{A_1 \to \alpha}.\, \lambda f_2^{A_2 \to \alpha}.\, f_i\, t \\
 & : & A_1 + A_2 \\
 & & \text{if } t : A_i \text{ and } i \in \{1, 2\} \\
 & & \\
\delta t\, [_{x:A_1}.s_1][_{x:A_2}.s_2] & \stackrel{\text{def}}{=} & t\, B\, (\lambda x^{A_1}.\, s_1)\, (\lambda x^{A_2}.\, s_2) \\
 & : & B \\
 & & \text{if } t : A_1 + A_2 \text{ and } s_i : B \\
 & & \text{for each } i \in \{1, 2\}
\end{array}
$$